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FINITE DIFFERENCE APPROXIMATION OF A PARABOLIC HEMIVARIATIONAL INEQUALITIES ARISING FROM TEMPERATURE CONTROL PROBLEM

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Abstract. In this paper we study the finite difference approximation of a *hemivariational* inequality of parabolic type arising from temperature control problem. Stability and convergence of the proposed method are analyzed. Numerical results are also presented to show the effectiveness and usefulness of the discretization scheme.

Key words. temperature control problem, *hemivariational* inequality, existence, stability, convergence.

1. Introduction

The theory of inequalities has received remarkable development in both pure and applied mathematics as well as in mechanics, engineering sciences and economics. This theory has been a key feature in the understanding and solution of many practical problem such as market price equilibria, heat control, elastic contact and so on (cf.[6],[10],[15]). The constitutional law of these problems is usually given by a non-monotone, possibly multi-valued mapping. Such problems is described by the so-called *hemivariational* inequality, which can been viewed to be the week formulation of a certain differential inclusion. The concept of a *hemivariational* inequality is introduced by Panagiotopoulos in [12]. In the static case the *hemivariational* inequality is often equivalent to the problem of finding all sub-stationary points of a super-potential Φ which is non-convex and non-smooth, in general, provided our problem is of potential type. There is a number of results on the existence and the approximation of elliptic *hemivariational* inequalities (cf.[1],[7],[12]), however, much fewer results on the existence and the approximation of the solution of the dynamic *hemivariational* inequalities.

In this paper we shall consider a discontinuous non-linear non-monotone parabolic initial boundary value problem, i.e., a parabolic *hemivariational* inequality.

$$\begin{cases} u'(t) - Au(t) + \Xi(t) = g(t) \\ u(0) = u_0 \ \text{ and } u(t) = 0 \text{ on } \partial\Omega \ for\ a.a\ t \in (0,T) \\ \Xi(x,t) \in \partial j(u(x,t)) \quad a.e.(x,t) \in \Omega_T. \end{cases} \tag{1.1}$$

The non-linearity and the discontinuity only lie in the lower order term $\partial j(u(x,t))$, and the operator A is linear and continuous. This kind of stationary problems have been studied, for example, in (cf. [1],[7],[12]), and dynamic problems in (see [3], [5],[8],[10]-[15]). As an important application of (1.1), We shall discuss the finite difference approximation [6] and numerical modeling of temperature control problem. To the best of our knowledge, there are relatively few papers in

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which numerical methods and modeling were studied for parabolic *hemivariational* inequalities(cf.[1],[8],[11],[14]) . These papers mainly discussed the finite element numerical methods and the proof of the existence, stability and convergence of the solution of these methods, but there are nearly not papers to deal with the numerical implementation of these methods. The main difficulties in numerical modeling include: (1) the solution of the parabolic *hemivariational* inequalities is not unique, i.e., there exist more than one solution under certain conditions; (2) for numerical implementation of parabolic *hemivariational* inequalities using the finite element method, we must first transform parabolic *hemivariational* inequalities into a sub-stationary point type problem, and then solve a non-smooth and non-convex optimization problem. These need much computation. To bypass these difficulties, we adopt the finite difference method based on Galerkin variational principle to approximate the parabolic *hemivariational* inequalities. We have not found similar work in published papers. Our method is an exterior approximate method and its finite dimensional space is generated by characteristic functions. In this paper, we first analyzed the existence of solution, stability and convergence of the finite difference scheme based on the finite dimensional space, and then discussed the numerical implementation of this method. Finally, as numerical examples, figures of solutions generated by multi-value functions are presented for several cases. In contrast to finite element method, finite difference method is simple and effective for solving numerically the parabolic *hemivariational* inequalities.

The outline of this paper is as follows. In section 2, we formulate the problem, and state the main assumptions of this paper. In section 3, we construct the finite difference scheme to approximate *hemivariational* inequality arising from temperature control problem. The existence of the solution, stability and convergence of the finite difference scheme are proven in Section 4. Numerical results are reported in Section 5. Finally, we give concluding remarks.

2. The description of problem

We consider a heat conduction problem with a non-monotone relation (a temperature control problem without assuming any monotonicity for the control device). Let $\Omega \subset R^2$ be a bounded domain with the Lipschitz boundary $\partial\Omega$, representing a body, in which the temperature distribution is governed by the time dependent heat equation (cf. [2],[7])

$$u'(t) - \Delta u(t) = g(t), \qquad \text{in } \Omega, \text{ for } a.a.\ t \in (0,T)$$

with g decomposed as follows:

$$\begin{cases} g = f - \Xi, \\ f \text{ is given and } \Xi(x,t) \in \partial j(u(x,t)) \text{ for } a.e.\ (x,t) \in Q_T = \Omega \times (0,T). \end{cases}$$

Where

$$j(u) = \begin{cases} g_1(u - s_1), & \text{if } u < s_1, \\ 0, & \text{if } s_1 \le u \le s_2, \\ g_2(u - s_2), & \text{if } u \ge s_2. \end{cases}$$

then

$$\partial j(u) = \begin{cases} g_1, & \text{if } u \le s_1, \\ [g_1, 0], & \text{if } u = s_1, \\ 0, & \text{if } s_1 < u < s_2, \\ [0, g_2], & \text{if } u = s_2, \\ g_2, & \text{if } u \ge s_2, \end{cases}$$