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A Stochastic Galerkin Method for Stochastic Control Problems

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Abstract. In an interdisciplinary field on mathematics and physics, we examine a physical problem, fluid flow in porous media, which is represented by a stochastic partial differential equation (SPDE). We first give a priori error estimates for the solutions to an optimization problem constrained by the physical model under lower regularity assumptions than the literature. We then use the concept of Galerkin finite element methods to establish a new numerical algorithm to give approximations for our stochastic optimal physical problem. Finally, we develop original computer programs based on the algorithm and use several numerical examples of various situations to see how well our solver works by comparing its outputs to the priori error estimates.

AMS subject classifications: 65M55, 65N30

Key words: Stochastic Galerkin methods, stochastic partial differential equation, distributed control, finite element methods.

1 Introduction

In the last decade, people in the scientific computing community have taken great interest in the stochastic partial differential equations (SPDEs) and its solver called the Stochastic Galerkin Method [4, 5, 21, 29, 33]. In this paper, we use the idea from the Galerkin finite element method to analyze optimal control problems constrained by SPDE and develop its numerical solver.

The stochastic Galerkin method has been created and developed to analyze a stochastic problem in the following sense. Suppose that we have a deterministic partial differential equation (PDE) that models some natural phenomenon; for instance, pollutant transportation in groundwater. To improve this deterministic mathematical model, we assume

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that we replace some deterministic quantities in the PDE with stochastic input data. For example, there may be lack of knowledge about some materials such as rocks and soils for groundwater. For these unknown properties of rocks and/or soils, we would like to use the concept of randomness in the model so that a new mathematical model with additional random terms can represent better the natural phenomenon. If there are inputs that are random, then the solution to the new model problem should also be including randomness. We then need a stochastic domain, and may need to use probability theories to analyze the solution to the new model problem. Now the remaining question is how we apply or modify a typical method such as the Galerkin Method to analyze the new stochastic problem derived from a deterministic problem. The stochastic Galerkin method actually answers this question, and it turns out to be a good method that requires less computational efforts than Monte Carlo method in computing $E[u]$ for sufficiently strict accuracy requirements (see [4]). However, in case that one use many terms in the K-L expansion of our coefficient a, the Monte Carlo method is known to be most effective (see [5]).

In this paper, we analyze the stochastic optimal control problems subject to SPDE by using a similar approach in the literature that is used to solve SPDEs. For example, we use the truncated Karhunen-Loéve (K-L) expansion as a main tool to convert the stochastic optimal control problem to a coupled optimality system of deterministic PDEs. In fact, in the last decade, based on the K-L expansion, there has been much progress in both the analysis and the finite element approximations of SPDEs; see, e.g., [2–5, 14, 21, 29, 33, 40].

Notwithstanding the many papers devoted to discrete approximations of solutions of SPDEs and optimal control problems for SPDEs, the literature seems to lack $L^2(\Gamma;H_0^1(D))$ convergence results for optimal distributed control problems based on the K-L expansion with both feasibility and efficiency of rigorous error analysis demonstrated via numerical examples. The goals of this work are to establish the convergence of the solution of a distributed optimal control problem with the K-L expansions, derive its error estimates in the norm of the solution space under minimal regularity assumptions in the y-direction, and show the practicability and effectiveness of our theories using numerical examples.

The problem we consider is the optimization problem

$$\mathcal{J}(u,f)=\mathbb{E}\left(\frac{1}{2}\int_D |u-U|^2\,dx+\frac{\beta}{2}\int_D |f|^2\,dx\right) \tag{1.1}$$

constrained by the stochastic elliptic PDE under the Dirichlet boundary condition:

$$-\nabla\cdot[a(x,\omega)\nabla u(x,\omega)]=f(x), \quad \text{in } D, \tag{1.2a}$$

$$u(x,\omega)=0, \quad \text{on } \partial D, \tag{1.2b}$$

where $\mathbb{E}$ denotes expected value, D the spatial domain, ∂D its boundary, U a target solution to the constraint, β a positive constant that says the importance between two terms in (1.1), and f a deterministic control acting in the domain. Here, our stochastic elliptic PDE generally models fluid flow in porous media. Under the homogeneous Dirichlet