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# A Priori and a Posteriori Error Estimates for H(div)-Elliptic Problem with Interior Penalty Method

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**Abstract.** In this paper, we propose and analyze the interior penalty discontinuous Galerkin method for H(div)-elliptic problem. An optimal a priori error estimate in the energy norm is proved. In addition, a residual-based a posteriori error estimator is obtained. The estimator is proved to be both reliable and efficient in the energy norm. Some numerical testes are presented to demonstrate the effectiveness of our method.

**AMS subject classifications**: 65N15, 65N30

**Key words**: Discontinuous Galerkin method, H(div)-elliptic problem, a priori error estimate, a posteriori error estimate.

## 1 Introduction

We are concerned with solving the $\mathbf{H}$(div)-elliptic model problem

$$-\mathbf{grad}(\mathrm{div}\mathbf{u})+\mathbf{u}=\mathbf{f} \quad \text{in } \Omega, \tag{1.1a}$$

$$\mathbf{u}\cdot\mathbf{n}=0 \quad \text{on } \Gamma, \tag{1.1b}$$

where $\Omega$ is a bounded polyhedral domain in $R^d\,(d=2,3)$ with boundary $\Gamma=\partial\Omega$, $\mathbf{n}$ is its unit outward normal vector, and $\mathbf{f}\in(L^2(\Omega))^d$.

The weak formulation of (1.1) is to find $\mathbf{u}\in H_0(\mathrm{div};\Omega)$ such that

$$a(\mathbf{u},\mathbf{v}):=\int_\Omega(\mathrm{div}\mathbf{u}\mathrm{div}\mathbf{v}+\mathbf{u}\cdot\mathbf{v})dx=\int_\Omega\mathbf{f}\cdot\mathbf{v}dx, \quad \forall\mathbf{v}\in H_0(\mathrm{div};\Omega). \tag{1.2}$$

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**H**(div)-elliptic problem (1.1) is ubiquitous in solid and fluid mechanics [11, 18]. It may arises from, the first-order system least-squares formulation of $H^1$-elliptic problem [12], the implementation of the sequential regularization method for the nonstationary incompressible Navier-Stokes equations [24], the mixed methods with augmented Lagrangians [13], or the stabilized formulations of the Stokes equations [37]. For more background on **H**(div)-elliptic problem and its applications, please see [5] for details. As we know, in two dimensions, conforming finite element methods for **H**(div)-elliptic problem can be treated by Raviart-Thomas(RT) element [30] or Brezzi-Douglas-Marini (BDM) element [10]. The extensions of RT element and BDM element to three dimensions were given by Nédélec in [26] and [27], respectively. Sometimes they are referred to as the first kind **H**(div)-conforming element and the second kind **H**(div)-conforming element.

Recently, there has been increased interest in the discontinuous Galerkin(DG) method due to its suitability for *hp*-adaptive techniques. For the applications of this method to a wide variety of problems, we can see the book [17] for details. An overview and a priori error analysis of DG for elliptic problems in $H^1$ were provided in [4]. For more details of the a priori error estimates for $H^1$-elliptic problem, please refer to [31]. A posterior error estimates of conforming finite element methods have been extensively studied, and we can refer to a series of monographs [2, 6, 28, 35] for the comprehensive analysis of such methods for elliptic problems in $H^1$, see also [13] for **H**(div)-conforming finite element method and [8] for **H**(**curl**)-conforming finite element method. However, a posteriori error estimates for DG have gained interest only in recent years, see [1,7,22,23,32,34] for the analysis of elliptic problem in $H^1$, and see [21] for elliptic problem in **H**(**curl**).

In this paper, we consider the interior penalty(IP) DG method for **H**(div)-elliptic problem (1.1), and provide a priori error estimate and a posteriori error estimate of such method. The analysis for the a posteriori error estimator is largely based on the reference [21]. To the best of our knowledge, there exists no work on DG for **H**(div)-elliptic problem, here we make an initial work on this direction.

This paper is organized as follows. In Section 2, a discontinuous Galerkin method for the problem (1.1) is introduced. An optimal a priori error estimate of the DG method in the energy norm is proved in Section 3. In Section 4, we provide a residual-based a posteriori error estimator for the DG method. And both the upper bound and lower bound analysis are proved for the error estimator in the energy norm. Finally, some numerical experiments are given in Section 5.

## 2 Discontinuous Galerkin formulation

In this section, we introduce the interior penalty discontinuous Galerkin method for the problem (1.1). For convenience, we assume that the domain is in $R^3$. Before discussion, we first give some notations: for a bounded domain $\mathcal{D}$ in $R^3$, we denote by $H^s(\mathcal{D})$ the standard Sobolev space of functions with regularity exponent $s\geq 0$ and norm $||\cdot||_{s,\mathcal{D}}$ and seminorm $|\cdot|_{s,\mathcal{D}}$. For $s=0$, $H^0(\mathcal{D})$ is written by $L^2(\mathcal{D})$. When $\mathcal{D}=\Omega$, the norm $||\cdot||_{s,\Omega}$