

## On Triangular Lattice Boltzmann Schemes for Scalar Problems

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**Abstract.** We propose to extend the d’Humières version of the lattice Boltzmann scheme to triangular meshes. We use Bravais lattices or more general lattices with the property that the degree of each internal vertex is supposed to be constant. On such meshes, it is possible to define the lattice Boltzmann scheme as a discrete particle method, without need of finite volume formulation or Delaunay-Voronoi hypothesis for the lattice. We test this idea for the heat equation and perform an asymptotic analysis with the Taylor expansion method for two schemes named D2T4 and D2T7. The results show a convergence up to second order accuracy and set new questions concerning a possible super-convergence.

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## 1 Introduction

The importance of extending the lattice Boltzmann scheme from square type regular meshes to unstructured triangulations has been recognized during the last years of 20th century [5, 22, 29]. In particular the “volumetric formulation” of Chen [5] makes a link with finite volumes, using control volumes around each vertex (the “Inria cells” [41]) of a finite element type triangulation. This method is still under active development with the work of Succi, Ubertini and co-workers [30, 35, 36]. In a dual way, van der Sman [37–40] uses rectangles and triangles as control volumes with a “cell center” type approach in Roache [33] denomination. He has developed an approximation of diffusion equation with Delaunay-Voronoi meshes for a BGK variant of the lattice Boltzmann scheme.

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In a previous contribution [12], we have observed that for usual lattice Boltzmann schemes (as for example the well known D2Q9), several (two for D2Q9) families of finite volumes are naturally associated with the scheme. As a consequence, we consider now the lattice Boltzmann scheme essentially as a “particle” method on a given (*a priori* fixed) mesh with discrete velocities. Recall that the “Particle In Cell” method has been first proposed in 1964 by Harlow *et al.* [18] and has been analyzed in the eighties by Beale and Majda [2], Raviart, Cottet and Mas Gallic [8, 28, 32] among others. We remark that this particle method does not suppose *a priori* the existence of a given lattice. The surrounding cells are recomputed at each time step in order to make the particle interact. Dynamic triangulation is an alternative to the previous methodology. It has been developed recently by Cianci, Klales, Love and co-workers [23, 26] in the context of lattice gas automata.

In this contribution, instead of adopting the volumetric formulation or a Delaunay-Voronoi hypothesis, we develop the framework of lattice Boltzmann schemes as a variant of the particle method. We propose an extension of the approach of d’Humières [9] to triangular meshes and we restrict this first tentative to scalar problems like the heat equation without advection.

The outline of the contribution is the following. We first recall the classic D2T7 lattice Boltzmann scheme in the next section. At this occasion, we put in evidence a property of symmetry of Bravais lattices. It is possible to adapt the Taylor expansion analysis [10] to this triangular lattice, with a diffusive scaling. This development is presented in Section 3 and applied to the D2T7 scheme. Several simulations with the D2T7 lattice Boltzmann scheme for the heat equation are presented in Section 4. In Section 5, we set the question of defining a discrete particle method on a finite element type triangular lattice. We propose a partial answer when each vertex of the lattice has a constant number of neighbours. This framework is applied in Section 6 to define a D2T4 lattice Boltzmann scheme for the heat equation. We repeat in Section 7 with this new scheme “D2T4” the simulations presented in Section 4. This work validates the potential of applications of our proposal. The conclusion (section 8) serves also as a discussion concerning encountered difficulties.

## 2 D2T7 lattice Boltzmann scheme

We consider a Bravais lattice  $\mathcal{L}$  connecting nodes labelled by the letter  $x$  and parametrized by a typical space scale  $\Delta x$ . The neighbour vertex number  $j$  of the node  $x \in \mathcal{L}$  is denoted by  $x_j$  and we set

$$x_j = x + \xi_j \Delta x. \quad (2.1)$$

For each  $x \in \mathcal{L}$  and each direction  $\xi_j$  linking two vertices, the “opposite node” with number  $\sigma(j)$  defined according to

$$x_{\sigma(j)} \equiv x - \xi_j \Delta x, \quad \xi_j + \xi_{\sigma(j)} \equiv 0 \quad (2.2)$$