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An All-Speed Asymptotic-Preserving Method for the Isentropic Euler and Navier-Stokes Equations

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Abstract. The computation of compressible flows becomes more challenging when the Mach number has different orders of magnitude. When the Mach number is of order one, modern shock capturing methods are able to capture shocks and other complex structures with high numerical resolutions. However, if the Mach number is small, the acoustic waves lead to stiffness in time and excessively large numerical viscosity, thus demanding much smaller time step and mesh size than normally needed for incompressible flow simulation. In this paper, we develop an all-speed asymptotic preserving (AP) numerical scheme for the compressible isentropic Euler and Navier-Stokes equations that is uniformly stable and accurate for all Mach numbers. Our idea is to split the system into two parts: one involves a slow, nonlinear and conservative hyperbolic system adequate for the use of modern shock capturing methods and the other a linear hyperbolic system which contains the stiff acoustic dynamics, to be solved implicitly. This implicit part is reformulated into a standard pressure Poisson projection system and thus possesses sufficient structure for efficient fast Fourier transform solution techniques. In the zero Mach number limit, the scheme automatically becomes a projection method-like incompressible solver. We present numerical results in one and two dimensions in both compressible and incompressible regimes.

AMS subject classifications: 35Q35, 65M08, 65M99, 76M12, 76N99

Key words: Low Mach number limit, asymptotic preserving schemes, incompressible limit, projection scheme, isentropic Euler equation.

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1 Introduction

We are interested in the efficient numerical simulation of unsteady compressible flows with all range of Mach numbers. These flows arise in many physical applications, including atmospheric modeling, magnetohydrodynamics and combustion. When the Mach number is of order one, modern shock capturing methods provide high resolution numerical approximations to shocks and other complex flow structures. However, when the Mach number is small, near the so-called incompressible regime, there is a wide gap between the speeds of the flow and the acoustic waves, the latter of which is often unimportant in the incompressible regime. In the incompressible regime, standard explicit shock-capturing methods require the time step to scale inversely with the maximum wave speed in the system for stability, which greatly *overresolves the solution in time*. Furthermore, these shock capturing methods will introduce numerical diffusions that scale with the inverse of the wave speeds around discontinuities, which requires *overresolution in space* in order to ensure that the numerical diffusion does not dominate the solution or physical viscosity for high Reynolds number flows.

Our goal is to develop all-speed flow simulators that work in all regimes of Mach number, including both compressible and incompressible regimes and their mixture. As a first step, in this paper, we focus on the compressible isentropic Euler and Navier-Stokes equations of gas dynamics. It was shown by Klainerman and Majda [23] that solutions to these equations converge to solutions of the incompressible equations in the limit when the Mach number goes to zero. The major difference between compressible and incompressible systems lies in the pressure term. In the compressible case, the pressure is determined by the equation of state of the system and plays an important role in the flux terms of the conservation law and is the source of the acoustic waves in the system. However, in the limiting incompressible equations the pressure term acts as a Lagrange multiplier to enforce the incompressibility condition and is in fact an asymptotic perturbation of the physical pressure from the compressible equations.

The development of computational methods for nearly incompressible (small Mach number flows) has attracted great attention for many years. Much of the early literature in this area focused on preconditioning techniques for steady state problems. In fact, Chorin's artificial compressibility approach [4] sought to avoid the difficulties of the pressure term in the incompressible equations by solving a form of the compressible low Mach number system, which has much clearer boundary conditions. It was later recognized [31] that these ideas could be used to calculate steady states of incompressible flows. Later studies applied these ideas to compute solutions to low Mach number flow by introducing preconditioning matrices to symmetrize the system in terms of a set of non-conservative variables [1, 14]. However, these methods assume that the flow is already in the low Mach number regime and thus cannot accurately compute problems where the Mach number is of order unity. Guillard and Viozat [13] followed the asymptotic analysis of Klainerman and Majda [23] to show that the artificial numerical dissipation in upwind methods for the Euler equation are what causes the method to