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ANALYSIS OF AN ALTERNATING DIRECTION METHOD APPLIED TO SINGULARLY PERTURBED REACTION-DIFFUSION PROBLEMS

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Abstract. We present an analysis of an *Alternating Direction Implicit* (ADI) scheme for a linear, singularly perturbed reaction-diffusion equation. By providing an expression for the error that separates the temporal and spatial components, we can use existing results for steady-state problems to give a succinct analysis for the time-dependent problem, and that generalizes for various layer-adapted meshes. We report the results of numerical experiments that support the theoretical findings. In addition, we provide a numerical comparison between the ADI and Euler techniques, as well details of the computational advantage gained by parallelizing the algorithm.

Key Words. reaction-diffusion problems, layer-adapted meshes, alternating directions, singular perturbation

1. Introduction

We consider the problem of computing a satisfactory numerical solution to a time-dependent singularly perturbed reaction-diffusion equation using an alternating direction finite difference method. The problem under consideration is

$$\partial_t u + \mathcal{L}u = f \quad \text{in} \quad Q := \Omega \times (0,T],\ \Omega = (0,1)^2, \tag{1a}$$

where $\mathcal{L}v := -\varepsilon^2 \Delta v + rv$, $f, r : \Omega \times (0,T] \to \mathbb{R}$, $r \geq \varrho^2$ on $\bar{\Omega} \times [0,T]$, $\varrho > 0$, subject to boundary and initial conditions

$$u = 0 \quad \text{on} \quad \partial\Omega \times (0,T], \quad u(\cdot,0) = 0 \quad \text{on} \quad \bar{\Omega}. \tag{1b}$$

Solutions to (1) typically exhibit layers: narrow regions in which derivatives of the solution are large.

Miller et al. [14] give a numerical analysis of a time-dependent reaction-diffusion problem that is one-dimensional (in space), and show that the solution computed on a Shishkin mesh will converge at a rate that is almost second-order. A more general analysis based on Green's functions is in [11]. See also [17, II.3.4.3] and references provided there.

The analysis of numerical techniques for the two-dimensional steady-state analogue of (1) has received recent attention: for example, Clavero et al. [2] provide an analysis for a finite difference method on a piecewise uniform Shishkin mesh, while Kellogg et al. [8] have analysed a finite difference method for various fitted meshes used to solve coupled systems for steady reaction-diffusion problems. See also [17, Remark II.2.10] and references given there.

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The studies of two-dimensional problems mentioned above all use spatial tensor product meshes. When extending these techniques to time-dependent problems, it is very natural to consider *dimension splitting*: at each time-step, one alternately solves independent one dimensional problems in the x- and y-directions. This is because it is typically much more efficient to solve, say, N tridiagonal systems of N unknowns, than a banded system of N^2 unknowns. Furthermore, the opportunities for parallelization are easy to exploit.

Such Alternating Direction Implicit (ADI) scheme for classical problems are presented in detail in, e.g., [12, 13, 18, 21]. Of primary interest to this study is the work of Clavero et al. [5] who use an alternating directions technique to semidiscretize (1) in time first. A sequence of one-dimension problems is obtained which in turn are solved approximately by central differencing on a layer-adapted piecewise uniform mesh—a so-called Shishkin mesh. The resulting scheme is shown to be uniformly convergent with respect to the perturbation parameter ε with the maximum nodal error bounded by $C\left(\tau + N^{-1}\right)$, where τ is the maximal time-step size and N the number of mesh points used in each direction of the tensor product spatial mesh, and C is a constant that is independent of ε, τ and N.

In this paper we shall present an alternative error analysis that—in a certain sense—is simpler than that in [5]. In particular, when a Shishkin mesh is used, we show that the nodal error is bounded by $C\left(\tau + N^{-2}\ln^2 N + \varepsilon N^{-1}\right)$. Furthermore, our approach makes it possible to deduce error bounds for other fitted meshes. For example, we also show that if the graded mesh of Bakhvalov [1] is used, then the error may be bounded as $C\left(\tau + N^{-2}\right)$.

ADI schemes have also been applied to other singularly perturbed problems such as convection diffusion. For example in [4] the maximum-norm errors of a first-order ADI method combined with simple upwinding in space are shown to be bounded by $C\left(\tau + \tau^{-1}N^{-1}\ln N\right)$.

Outline. In §2 we describe the discretization of (1) both by the standard implicit Euler method and the ADI approach. Bounds on derivatives of the solution are summarized in §3, which is followed by a brief discussion of the mathematical approach of [5], and then our own numerical analysis of the technique. This leads to an expression of the error that depends on the analysis of the given method for a steady-state problem. Then in §3.5 we can derive error bounds for particular meshes.

The paper concludes with a report of detailed numerical experiments. For a specially constructed test problem where the true solution is known, we investigate separately the dependence of the error on the time and spatial discretization. We also compare the accuracies of the ADI and Euler techniques. The paper concludes by highlighting the speed-up that can be gained when the algorithm is implemented on a parallel computer.

Notation. Given arbitrary meshes $\omega_x : 0 = x_0 < x_1 < \cdots < x_N = 1$ and $\omega_y : 0 = y_0 < y_1 < \cdots < y_N = 1$ in the x- and y-directions, one may construct the tensor-product mesh $\omega := \omega_x \times \omega_y$. We write the mesh in the time variable as $\omega_t : 0 = t_0 < t_1 < \cdots < t_K = T$. The mesh intervals are denoted

$$h_i := x_i - x_{i-1}, \quad k_j := y_j - y_{j-1}, \quad \tau_n := t_n - t_{n-1}, \quad \tau := \max_{n=1,\ldots,K} \tau_n.$$

To simplify the notation we set $g^n_{i,j} := g(x_i, y_j, t_n)$ for any function $g \in C(\bar{Q})$. Similarly, $g^n := g(\cdot, \cdot, t_n)$.