High-Order Low Dissipation Conforming Finite-Element Discretization of the Maxwell Equations

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Abstract. In this paper, we study high order discretization methods for solving the Maxwell equations on hybrid triangle-quad meshes. We have developed high order finite edge element methods coupled with different high order time schemes and we compare results and efficiency for several schemes. We introduce in particular a class of simple high order low dissipation time schemes based on a modified Taylor expansion.

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1   Introduction

Our aim is to develop very precise and very efficient solvers for the Maxwell equations in time domain and in complex geometries. So, we will need high order methods for good precision and unstructured meshes to handle complex geometries.

The numerical solution of Maxwell's equations has been performed most reliably with the Finite Difference Time Domain Yee solver which has proved very robust, but comes to its limits when unstructured meshes need to be handled or when higher order is helpful. Hence, developing new efficient and reliable Maxwell solvers has been an area of intense research in the last 40 years. Different approaches have been followed to develop

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solvers that can handle unstructured meshes, the most important certainly being the Finite Element (FE) solvers including their discontinuous variants. Finite Element methods are better adapted for complex geometries as they can be based on different computational elements (hexahedra, tetrahedra) which can be used to mesh efficiently complex geometries. Finite Elements adapted to Maxwell's equations, the so-called edge elements have been introduced by Nédélec [29] in 1980. Moreover the Finite Element methodology gives easy access to higher order methods which have proved useful recently for several applications [3,33]. Let us also mention the concept of discrete differential forms (see [25] for a review) that provide new insights of the reasons why some solvers work well and other do not. One of the drawbacks of these edge finite element methods for Time Domain computations, is the need to solve a linear system at each time step, and therefore these methods are known to be expensive. For this reason mass-lumped elements for any order have been introduced on squares and cubes [7–9] and a quasi mass-lumped method has been proposed in [17]. Mass-lumped elements for first and second order edge elements have been developed on triangles [14] and tetrahedra [15]. Mass lumped nodal Finite Elements for a formulation of Maxwell's equations keeping the divergence constraint and adding a Lagrange multiplier have also been developed [1,5]. More recently a lot of effort has gone into the derivation of high-order Discontinuous Galerkin schemes [2,16,23,24,35] which completely eliminate the need for solving a global linear system. Apart from that let us also mention Finite Volume schemes which also can be made high order on unstructured grids [18]. A detailed review of high order methods by Hesthaven can also be found in [22].

Our aim in this paper is to study high-order conforming Finite Element schemes based on hybrid triangle-quad meshes for use in the time domain. In order to really keep high-order schemes, we will work not only with high-order schemes in space but also in time. We introduce in particular a very simple high-order time stepping scheme which is based on a stabilized (when needed) Taylor expansion method. Indeed after space discretization with our finite elements, we get a linear system of ordinary differential equations which conserves exactly a discrete energy and that can be written as $dU/dt = AU$ where A is a matrix with purely imaginary eigenvalues. An order p Taylor expansion scheme for this linear system reads $U^{n+1} = (I + \Delta t A + \cdots + \frac{\Delta t}{p!} A^p) U^n$. Note that this scheme is equivalent to the unique p-stage, order p Runge-Kutta method for the linear system we consider [19,20]. Such a scheme is unstable for any Δt for some orders including 1, 2, 5 and 6, and stable for others including 3 and 4. Our stabilization method consists in adding an additional term ξA^{p+1} where ξ is chosen to stabilize the method, its order being conserved. Moreover ξ can also be tuned in order to get low dissipation. This yields stable arbitrary high order schemes which are very efficient, compared to existing schemes, for orders higher than 4. Note that the Strong Stability Preserving schemes advocated in [20] do apply only for space-discretized problems for which there is a negative real part of the eigenvalues for which explicit Euler is stable. This is the case for Discontinuous Galerkin methods with upwinding, but not for conforming Finite Elements like ours where the eigenvalues are purely imaginary. On the other hand, the modified equa-