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PARABOLIC SINGULARLY PERTURBED PROBLEMS WITH EXPONENTIAL LAYERS: ROBUST DISCRETIZATIONS USING FINITE ELEMENTS IN SPACE ON SHISHKIN MESHES

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Dedicated to G. I. Shishkin on the occasion of his 70th birthday

Abstract. A parabolic initial-boundary value problem with solutions displaying exponential layers is solved using layer-adapted meshes. The paper combines finite elements in space, i.e., a pure Galerkin technique on a Shishkin mesh, with some standard discretizations in time. We prove error estimates as well for the θ-scheme as for discontinuous Galerkin in time.

Key Words. convection-diffusion, transient, finite element, Shishkin mesh, time discretization.

1. Introduction

We consider 1D unsteady convection-diffusion problems of the type

$$u_t + Lu = f \qquad \text{in} \quad Q = (0,1) \times (0,T], \tag{1a}$$

$$u(x,0) = u_0(x) \qquad \text{for } x \in [0,1], \tag{1b}$$

$$u(0,t) = u(1,t) = 0 \qquad \text{for } t \in (0,T], \tag{1c}$$

with $f : (0,1) \times (0,T] \to \mathbb{R}$. Here the differential operator L is given by,

$$Lu := -\varepsilon u_{xx} + bu_x + cu, \tag{2}$$

$0 < \varepsilon << 1$ is a small parameter and $b, c : (0,1) \to \mathbb{R}$ are sufficiently smooth with

$$b(x) > \beta > 0 \qquad \text{for } x \in (0,1). \tag{3}$$

By changing the dependent variable we may also assume that

$$c - \frac{1}{2} b_x \geq c_0 > 0 \qquad \text{for } x \in (0,1). \tag{4}$$

Here β and c_0 are constants. The exact solution of (1) has, in general, an exponential boundary layer at $x = 1$. Additionally, a discontinuity in the initial-boundary data at the point $x = 0$, $t = 0$ would lead to an interior layer along the subcharacteristics through that point. We assume sufficient compatibility of the data to exclude the existence of an interior layer, see [9].
In recent years many numerical methods have been developed to solve the corresponding stationary problem on layer-adapted meshes, resulting in error estimates

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that are uniform with respect to the parameter ε, see [9]. For unsteady problems, however, the situation is different.
Most existing papers deal with low order finite difference schemes, beginning with [10] and the error estimate

$$| u(x_i, t_j) - u_{i,j} | \leq C(N^{-1} \ln^2 N + \tau) \tag{5}$$

for backward differencing in time and upwind differencing in space on a Shishkin mesh. This result was extended in [5], [1] and [4]; in the last paper defect correction in both space and time is applied to enhance the accuracy of the computed solution.
Concerning finite elements in space on a Shishkin mesh, we only know the pointwise error estimates of [3] using space-time finite elements that are linear and continuous in space but discontinuous in time, while additionally the streamline diffusion stabilization in space-time is applied.
It is the aim of this paper to combine systematically finite elements in space (based on a Galerkin technique or stabilization on a Shishkin mesh) with some standard discretizations in time. First we shall study the θ-scheme which gives maximal order 2 with respect to time. As a higher order scheme we decided to choose and to analyze discontinuous Galerkin, because the analysis of higher order methods is similar to lower order versions and discontinuous Galerkin offers the possibility to investigate a posteriori error estimates based on standard ideas for Galerkin techniques. In the numerical experiments we restricted ourselves to low order methods, a careful numerical study of higher order methods is a task for subsequent studies. For simplicity, we present the results for problems one-dimensional in space but we apply only techniques which can be used in several dimensions as well.

2. The continuous problem

It is well known that for $f \in L_2(Q)$ and $u_0 \in L_2(\Omega)$ problem (1) has a unique solution $u \in L_2(0,T;H_0^1(\Omega))$ with $u' \in L_2(0,T;H^{-1}(\Omega))$ (in our case we have $\Omega = (0,1)$).
If we introduce the ε-weighted H^1-norm defined by

$$\|v\|_\varepsilon^2 := \varepsilon |v|_1^2 + \|v\|_0^2 \qquad \text{for } v \in H^1(\Omega), \tag{6}$$

where $\|\cdot\|_0$ defines the standard L_2-norm and $|\cdot|_1$ the H^1-seminorm respectively, standard arguments lead us to the stability estimate (see [7], Theorem 11.1.1)

$$\sup_{t\epsilon(0,T)} \| u(t) \|_0 + (\int_0^T \| u(t) \|_\varepsilon^2 \, dt)^{1/2} \leq C\, ((\int_0^T \| f(t) \|_0^2 \, dt)^{1/2} + \| u_0 \|_0). \tag{7}$$

Therefore it is natural that we shall later prove error estimates in "$L_\infty(L^2)$"- and "$\sqrt{\varepsilon} L^2(H^1)$"-norms or their discrete analogues.

Remark 1. *In* [7]*, Proposition 11.1.1., we additionally can find an estimate for* $\max_{t\epsilon(0,T)} \| u(t) \|_1$*. But, in our singularly perturbed case, it seems not possible to follow the proof of Proposition 11.1.1 in such a way that the constants arising are independent of* ε *(if moreover,* $\| u(t) \|_1$ *is replaced by* $\| u(t) \|_\varepsilon$*).*$\square$

Under certain compatibility conditions [9] there exists a classical solution of problem (1).