Commun. Comput. Phys.
doi: 10.4208/cicp.150609.031209a

Vol. 8, No. 3, pp. 642-662
September 2010

The Monotone Robin-Robin Domain Decomposition Methods for the Elliptic Problems with Stefan-Boltzmann Conditions

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Received 16 June 2009; Accepted (in revised version) 3 December 2009

Available online 15 April 2010

Abstract. This paper is concerned with the elliptic problems with nonlinear Stefan-Boltzmann boundary condition. By combining with the monotone method, the Robin-Robin domain decomposition methods are proposed to decouple the nonlinear interface and boundary condition. The monotone properties are verified for both the multiplicative and the additive domain decomposition methods. The numerical results confirm the theoretical analysis.

AMS subject classifications: 35J65, 65N55

Key words: Nonlinear Stefan-Boltzmann condition, monotone methods, Robin-Robin domain decomposition method.

1 Model problems

Let u be the solution of Laplace equation with nonlinear Stefan-Boltzmann boundary condition arising from the steel-making industry:

$$-\Delta u = 0 \quad \text{in } \Omega, \tag{1.1}$$

$$\frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on } \Gamma_N, \tag{1.2}$$

$$\lambda \frac{\partial u}{\partial \mathbf{n}} = -\sigma(u^4 - u_e^4) \quad \text{on } \Gamma_e, \tag{1.3}$$

$$u = u_s \quad \text{on } \Gamma_s, \tag{1.4}$$

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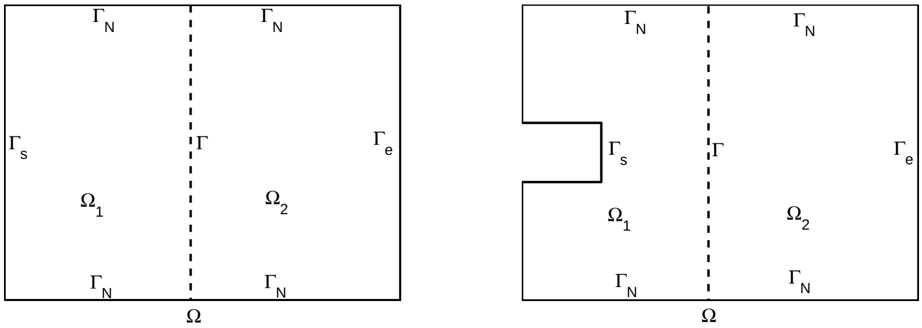

Figure 1: Domain with composite heat-resistant materials and partial corroded domain.

where u represents the temperature of the heat-resistant materials, $u_s>0$ is the temperature of the melting-steel, $u_e>0$ is the temperature of the exterior air , and the temperature of the steel is higher than the temperature of the exterior air. $\Omega=\bigcup\Omega_i$ is the domain made of composite heat-resistant materials, and the heat conduction coefficient of the material λ may be different in the every subdomain Ω_i. The Boltzmann thermal fourth power law (1.3) is imposed on the exterior boundary surrounded by air, and σ is the Boltzmann radiation coefficient. Let $u_i=u|_{\Omega_i}$ and $\lambda_i=\lambda|_{\Omega_i}$. Then

$$\lambda_1\frac{\partial u_1}{\partial \mathbf{n}_1}\Big|_\Gamma+\lambda_2\frac{\partial u_2}{\partial \mathbf{n}_2}\Big|_\Gamma=0 \tag{1.5}$$

at the interface boundary $\Gamma=\overline{\Omega}_1\cap\overline{\Omega}_2$ according to the heat transfer law, here $\mathbf{n}_1$ is the outer unit normal vector from Ω_1 to Ω_2, and $\mathbf{n}_2$ from Ω_2 to Ω_1. Specially, another relationship

$$\lambda_1\frac{\partial u_1}{\partial \mathbf{n}_1}=-\sigma(u_1^4-u_2^4) \tag{1.6}$$

is observed by the experiments which is different from the conventional condition $u_1|_\Gamma= u_2|_\Gamma$. This condition (1.6) is explained as following: the interface Γ is an approximation of very thin layer which is filled with air, and the Boltzmann thermal fourth power law is applied to the heat transfer between the high temperature materials and air, and (1.6) can be obtained by removing the variable of the air temperature.

In the steel-making procedure, the boundary Γ_0 may be corroded after long-time high temperature heat process [6,7,30], the detection of the corrosion is very important, and stable and efficient solvers for the problem (1.1)-(1.6) are the base of any corrosion detected algorithm.

Among the various techniques for the nonlinear partial differential equations, the monotone method is one powerful tool to obtain the existence, uniqueness and other properties of the solutions [5,15,19,26]. Moreover, by using the technique of upper and lower solutions, efficient algorithms can be constructed to solve the nonlinear equations,