Exponentially-Convergent Strategies for Defeating the Runge Phenomenon for the Approximation of Non-Periodic Functions, Part I: Single-Interval Schemes

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Abstract. Approximating a function from its values $f(x_i)$ at a set of evenly spaced points x_i through $(N+1)$-point polynomial interpolation often fails because of divergence near the endpoints, the "Runge Phenomenon". Here we briefly describe seven strategies, each employing a single polynomial over the entire interval, to wholly or partially defeat the Runge Phenomenon such that the error decreases *exponentially* fast with N. Each is successful in obtaining high accuracy for Runge's original example. Unfortunately, each of these single-interval strategies also has liabilities including, depending on the method, various permutations of inefficiency, ill-conditioning and a lack of theory. Even so, the Fourier Extension and Gaussian RBF methods are worthy of further development.

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1 Introduction

More than a century ago, Carl Runge, C. Meray and Emilie Borel independently made an astonishing discovery: polynomial interpolation on an equispaced grid was unreliable [11, 27–29, 33]. Borel gave an example of non-convergent interpolation at the Heidelberg Mathematical Congress in 1904, but apparently did not publish it. Even if $f(x)$ is analytic for all real x, its interpolants $f_N(x)$ will diverge as $N \to \infty$ near the endpoints $x = \pm 1$ if $f(x)$ has singularities within the "Runge Zone" in the complex x-plane illustrated in Fig. 1.

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Runge's own example was $f(x) = 1/(1+25x^2)$, which is analytic for all real x, but has a divergent equispaced polynomial interpolant sequence because of poles at $x = \pm \pi i/5$. Exponential convergence can be recovered by using the highly nonuniform Chebyshev grid [3], but what is one to do with experimental data collected at evenly spaced levels?
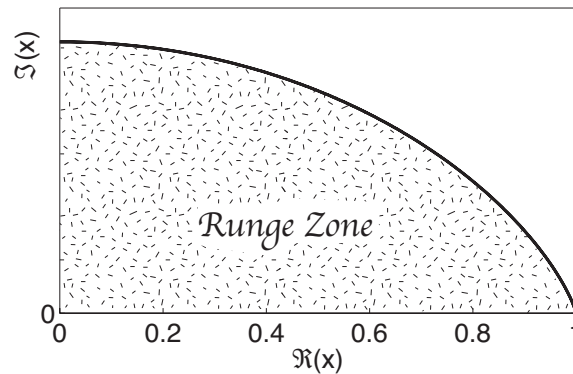

Figure 1: The Runge Zone in the complex x-plane for polynomial interpolation with a uniformly spaced grid on $x \in [-1,1]$. Because the boundary curve is symmetric under reflection with respect to the both the real and imaginary axes, only the portion in the upper right quadrant of the complex plane is illustrated. If $f(x)$ has any singularities in the sense of complex variable theory anywhere within the shaded region (or its reflections about either axis), then interpolation diverges as $N \to \infty$. If $f(x)$ has singularities only outside the shaded "Runge Zone", then interpolation will converge everywhere on $x \in [-1,1]$.

As explained in the reviews [5, 6, 16], defeating Gibbs Phenomenon in Fourier also requires reconstructing a function $f(x)$ everywhere on $x \in [-1,1]$ with exponential accuracy from knowledge only of its analyticity on the interval and its samples on an evenly spaced grid of $(N+1)$ points on the interval. Symbolically,

$$\text{anti} - \text{Gibbs} = \text{EdgeDetection} + \text{Anti} - \text{Runge}, \tag{1.1}$$

where edge detection identifies the boundaries of regions that are free of discontinuities ("edges", alias "shocks" and "fronts" in fluids), and then an anti-Runge procedure is applied on each smooth sub-interval to approximate $f(x)$.

There is a wide variety of *finite order* strategies to defeat the Runge Phenomenon whereby "finite order" denotes an approximation scheme whose error decreases as $\mathcal{O}(1/P^K)$ where P is the number of sample points and $K > 0$ is the "algebraic order of convergence". The simplest is piecewise polynomial interpolation: the "connect-the-dots" diagrams of coloring books allow a preschooler to draw a butterfly or a fish by drawing linear polynomials from dot to dot with a crayon. This is only first order, but cubic splines provide a higher order "Old Reliable".

Our goal is more ambitious, which is to develop schemes with an *exponential* rate of convergence as $P \to \infty$. Curiously, although this problem is over a century old, it is only in recent times that exponentially-convergent Runge-defeating methods have been developed. Now, there are so many strategies that it is impossible to describe them in a single article. In this work, we shall specialize to *single-interval* schemes: