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Vector Addition Theorem and Its Diagonalization

W. C. Chew*

Faculty of Engineering, The University of Hong Kong, Hong Kong,
and Department of Electrical and Computer Engineering, University of Illinois,
Urbana-Champaign, Urbana, IL 61801-2918, USA.

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Abstract. The conventional vector addition theorem is written in a compact notation. Then a new and succinct derivation of the vector addition theorem is presented that is as close to the derivation of the scalar addition theorem. Newly derived expressions in this new derivation are used to diagonalize the vector addition theorem. The diagonal form of the vector addition theorem is important in the design of fast algorithms for computational wave physics such as computational electromagnetics and computational acoustics.

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1 Introduction

The development of fast algorithms for integral equation solvers opens up new realms for the applications of integral equation solvers [1–5]. One of these is their use in the arena of circuits, micro-circuits and nanotechnologies [6]. Often time, the use of fast solvers in this arena calls for the combined use of fast algorithms where both wave physics and circuit physics are captured well by the solvers [7].

In the mid-frequency regime, where the wavelength is on the order of the object, or not extremely small compared to the size of the object, the vector nature of electromagnetic waves and their phases cannot be ignored. Hence, full vector wave physics is needed to describe the wave interaction with objects in this regime. This is the regime often encountered in microwave engineering [4].

*Corresponding author. *Email addresses:* `wcchew@hku.hk, w-chew@uiuc.edu` (W. C. Chew)

In the low-frequency regime, where the size of the object is much smaller than the wavelength, electromagnetic wave physics morphs into circuit physics, often not in a seamless fashion. Circuit physics is captured by quasistatic electromagnetics, but in modern circuit design, vast lengthscales ranging from sub-micrometers to centimeters are encountered concurrently. Hence, there often is a strong mixture of wave physics together with circuit physics in modern circuit design. Therefore the design of fast solvers in this regime, which has been nicknamed the "twilight zone", remains a challenge [6,8]. Moreover, in order to capture the "inductance" physics and the "capacitance" physics correctly, the vector nature of electromagnetic physics cannot be ignored [5]. This regime is encountered in micro-circuits in chip design, as well as nanotechnologies.

In this paper, the vector addition theorem for solenoidal vector wave functions is discussed, and a succinct derivation of the vector addition theorem is also presented. The new forms of the vector addition theorem facilitate their diagonalization, which is essential for developing fast algorithms in computational electromagnetics [9,10]. Previously, only the diagonalization of the scalar addition theorem has been presented [9,10]. The vector addition theorem can also be used to factorize the dyadic Green's function, which preserves the vector nature of electromagnetic field down to very long wavelength. It can be used for the development of a mixed-form fast multipole algorithm for vector electromagnetics which is valid from very low frequency to mid frequency [7]. It can also potentially result in memory savings.

2 Some fun with the vector addition theorem

Before the diagonalization of the vector addition theorem can be described, one needs to present the vector addition theorem and its expressions in compact notation. Their expressions in compact notation facilitate insight into their further diagonalization.

The vector addition theorem has been of great interest to the mathematical physics community [11–22]. The vector addition theorem for $\mathbf{r}=\mathbf{r}''+\mathbf{r}'$, for which $|\mathbf{r}'|<|\mathbf{r}''|$, can be written as [11,17,20, Appendix D of [20]]

$$\mathbf{M}_L(\mathbf{r})=\sum_{L'}[\Re g\mathbf{M}_{L'}(\mathbf{r}')A_{L',L}(\mathbf{r}'')+\Re g\mathbf{N}_{L'}(\mathbf{r}')B_{L',L}(\mathbf{r}'')], \tag{2.1}$$

$$\mathbf{N}_L(\mathbf{r})=\sum_{L'}[\Re g\mathbf{M}_{L'}(\mathbf{r}')B_{L',L}(\mathbf{r}'')+\Re g\mathbf{N}_{L'}(\mathbf{r}')A_{L',L}(\mathbf{r}'')]. \tag{2.2}$$

In the above, $\mathbf{M}$ and $\mathbf{N}$ are vector spherical harmonics expressed in terms of spherical Hankel functions [23] and spherical harmonics [24]. The subscript $L=(l,m)$ represents an ordered pair of integers where $-l\leq m\leq l$, and $l=1,\cdots,\infty$ (no monopole or $l=0$ term). Similar definition holds for L'. If the summation is truncated at $l=l_{max}$, then the number of terms involved is $P=(l_{max}+1)^2-1$. The $\Re g$ operator implies taking the regular part of the function where a spherical Hankel function (which is singular) is replaced by a spherical Bessel function (which is regular).