INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 5, Number 1, Pages 73–76

INVALIDITY OF DECOUPLING A BIHARMONIC EQUATION TO TWO POISSON EQUATIONS ON NON-CONVEX POLYGONS

SHENG ZHANG AND ZHIMIN ZHANG

(Communicated by Yanping Lin)

Abstract. We clarify the validity of a method that decouples a boundary value problem of biharmonic equation to two Poisson equations on polygonal domains. The method provides a way of computing deflections of simply supported polygonal plates by using Poisson solvers. We show that such decoupling is not valid if the polygonal domain is not convex. It may fail even when the right hand side function is infinitely smooth and supported away from the reentrant corners.

Key Words. Simply supported plate, biharmonic, Poisson, decoupling.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a two-dimensional domain. The boundary value problem of biharmonic equation

$$\Delta^2 u = f \text{ in } \Omega; \quad u|_{\partial\Omega} = 0,\ \Delta u|_{\partial\Omega} = 0. \tag{1}$$

can be formally de-coupled to two Poisson equations by introducing an intermediate function v such that

$$-\Delta v = f \text{ in } \Omega,\ v|_{\partial\Omega} = 0 \quad \text{and} \quad -\Delta\tilde{u} = v \text{ in } \Omega,\ \tilde{u}|_{\partial\Omega} = 0. \tag{2}$$

When $\partial\Omega$ and f are smooth, (1) and (2) are equivalent in the sense that $u = \tilde{u}$. This is easily seen from the regularity of solutions of these equations up to the domain boundary [3, 5]. This decoupling gives rise to a numerical method of solving the biharmonic equation (1) by using Poisson solvers. Incidentally, for polygonal domains, the equation (1) also determines the transverse deflection of a simply supported plate loaded by a resultant transverse force f [1, 2, 4], and the decoupling has been used in numerical computation of the plate deflection. However, for polygonal domains, the validity of this decoupling is not obvious. Actually, we show that it is not valid when the polygon is not convex.

To see the well-posedness of these equations on polygonal domains, we write them in weak forms. In the following, even without explicit indication, a function space is composed of functions defined on Ω. For example, H^1 means $H^1(\Omega)$. The weak formulation of (1) seeks $u \in H^2 \cap H^1_0$ such that

$$(\Delta u, \Delta w)_{L^2} = \langle f, w\rangle \quad \forall\ w \in H^2 \cap H^1_0. \tag{3}$$

Here the parenthesis stands for the inner product in the Hilbert space indicated by the subscript. The right hand side is the dual product between f, viewed as an

Received by the editors October 10, 2006.

2000 *Mathematics Subject Classification.* 35J05, 35J35, 65N30.

The first author was supported by NSF grant DMS-0513559, and the second author by DMS-0311807 and DMS-0612908.

element of the duality of the $H^2 \cap H_0^1$, and w. The equations in (2) become seeking v and $\tilde{u}$ in H_0^1, respectively, such that

$$(4) \qquad \begin{aligned} (\nabla v, \nabla w)_{[L^2]^2} &= \langle f, w\rangle \quad \forall\, w \in H_0^1, \\ (\nabla \tilde{u}, \nabla w)_{[L^2]^2} &= (v, w)_{L^2} \quad \forall\, w \in H_0^1. \end{aligned}$$

Of course, when $\partial\Omega$ and f are smooth, the equations (3) and (4) are well-posed and $u = \tilde{u}$, which are also solutions of the differential equations (1) and (2). However, the weak equations make broader sense than the differential equations: Their well-posedness requires less regularity on the domain and the loading function. The question is whether or not the equivalence between (3) and (4) remains when the domain boundary and loading function are not smooth. This is an important issue that is crucial to the validity of the numerical method for the simply supported plate model (3) obtained by combining the Poisson solvers for the two equations in (4).

The weak equation (3) is well-posed for polygonal domains as long as the function f defines a linear continuous functional on $H^2 \cap H_0^1$. This well-posedness relies on the inequality that for polygonal domains one has [4]

$$(5) \qquad \|\Delta w\|_{L^2} \geq C\|w\|_{H^2} \quad \forall\, w \in H^2 \cap H_0^1.$$

Here C is a positive constant depending on Ω. The equations in (4) are well-posed if $f \in H^{-1}$. Thus, (3) requires less regularity on f than (4) does. We shall assume that $f \in H^{-1}$ such that both u and $\tilde{u}$ are uniquely determined, and consider the question that whether $u = \tilde{u}$. The answer is that if Ω is a convex polygon then $u = \tilde{u}$. However, if Ω is not convex, then (3) is not equivalent to (4) in the sense that there exists loading function f such that $u \neq \tilde{u}$. Our argument is based on the observation that the solution u of (3) lies in $H^2 \cap H_0^1$, while $\tilde{u}$ is only required to be in H_0^1. Thus if $\tilde{u} \notin H^2$, then $\tilde{u} \neq u$. In the next section, we construct an example to show that on non-convex polygons this may occur even when f is smooth and supported away from the reentrant corners. In the last section we prove that if $\tilde{u} \in H^2$ then $\tilde{u} = u$. Therefore, if the polygonal domain is convex, then (3) and (4) are equivalent as long as $f \in H^{-1}$.

2. An example for the invalidity of the decoupling

As an example, we consider a non-convex polygonal domain and put one of its reentrant corners at the origin of the Cartesian coordinate system. See Figure 1 in which the reentrant angle $\omega > \pi$. Let (r, θ) be the usual polar coordinates. With a slight abuse of notations, we use the same letter to denote a function expressed in both Cartesian and polar coordinates. Let the loading function f be defined by

$$(6) \quad f(r,\theta) = \sin\alpha\theta$$
$$\left(r^{\alpha}\frac{d^4\phi(r)}{dr^4} + (4\alpha+2)r^{\alpha-1}\frac{d^3\phi(r)}{dr^3} + (4\alpha^2-1)r^{\alpha-2}\frac{d^2\phi(r)}{dr^2} - (4\alpha^2-1)r^{\alpha-3}\frac{d\phi(r)}{dr} \right).$$

Here $\alpha = \frac{\pi}{\omega} < 1$, and ϕ is a smooth function such that $\phi(r) = 1$ for $0 < r < r_1$ and $\phi(r) = 0$ for $r > r_2$. See Figure 1 for the meanings of r_1 and r_2. This is a smooth function whose support is the shaded region in the figure, which is away from the reentrant corner. For such an f, the solution of (4) is given by

$$(7) \qquad \begin{aligned} v(r,\theta) &= -\left((2\alpha+1)r^{\alpha-1}\frac{d\phi(r)}{dr} + r^{\alpha}\frac{d^2\phi(r)}{dr^2} \right)\sin\alpha\theta, \\ \tilde{u}(r,\theta) &= r^{\alpha}\phi(r)\sin\alpha\theta. \end{aligned}$$