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An Augmented Approach for the Pressure Boundary Condition in a Stokes Flow

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Abstract. An augmented method is proposed for solving stationary incompressible Stokes equations with a Dirichlet boundary condition along parts of the boundary. In this approach, the normal derivative of the pressure along the parts of the boundary is introduced as an additional variable and it is solved by the GMRES iterative method. The dimension of the augmented variable in discretization is the number of grid points along the boundary which is $\mathcal{O}(N)$. Each GMRES iteration (or one matrix-vector multiplication) requires three fast Poisson solvers for the pressure and the velocity. In our numerical experiments, only a few iterations are needed. We have also combined the augmented approach for Stokes equations involving interfaces, discontinuities, and singularities.

Key words: Incompressible Stokes equations; pressure boundary condition; augmented method; interface problem; immersed interface method; fast Poisson solver; GMRES method.

1 Introduction

In this paper, we consider the following stationary incompressible Stokes equations:

$$\nabla p = \nabla \cdot \mu \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T \right) + \mathbf{g}(\mathbf{x}), \qquad (x,y) \in [a,\ b] \times [c,\ d], \tag{1.1}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{1.2}$$

$$\mathbf{u}(x,c) = \mathbf{u}_1(x), \qquad \mathbf{u}(x,d) = \mathbf{u}_2(x), \tag{1.3}$$

$$\mathbf{u}(a,y) = \mathbf{u}(b,y), \qquad p(a,y) = p(b,y), \tag{1.4}$$

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where $\mathbf{u} = (u,v)^T$ is the velocity, p is the pressure, μ is the viscosity, $\mathbf{x} = (x,y)$ is the Cartesian coordinate variable, $\mathbf{u}_1(x)$ and $\mathbf{u}_2(x)$ are two given vector functions, and $\mathbf{g} = (g_1, g_2)$ is an external force. In other words, we have a Dirichlet boundary condition for the velocity at the two sides $y = c$ and $y = d$, and a periodic boundary condition for all variables at $x = a$ and $x = b$, see Fig. 1 for an illustration. Such Stokes equations have many applications. In this paper, we assume that the viscosity is constant or piecewise constant.

Note that if the viscosity is constant, then by applying the divergence operator to the momentum equation, we get

$$\Delta p = \nabla \cdot \mathbf{g}, \tag{1.5}$$

due to the incompressibility condition. A fast Poisson solver, *e.g.*, Fishpack [1] can be applied to solve the pressure if the boundary condition of the pressure is Dirichlet, Neumann, or periodic. After the pressure is solved, the velocity can be solved from

$$\Delta u = \frac{p_x - g_1}{\mu}, \qquad \Delta v = \frac{p_y - g_2}{\mu}, \tag{1.6}$$

by applying the same fast Poisson solver twice. This approach is called *the three Poisson equations approach.* For example, if all variables (p and $\mathbf{u}$) are periodic, then the stationary Stokes equations can be solved by calling a fast Poisson solver three times.

If a Dirichlet boundary condition is prescribed for the velocity, then it is well known that the pressure along the boundary is not a free variable and it is uniquely determined (up to a constant) from the governing equation (1.1)-(1.4). In other words, we can not specify the boundary condition for the pressure arbitrarily. Otherwise the problem is over-determined. This brings some difficulties to numerical schemes using uniform Cartesian grids. Note that from the momentum equation, we do have a boundary condition (1.7) for the pressure, but it is coupled with the velocity. For time-dependent Stokes or Navier-Stokes equations, some approaches in dealing with pressure boundary conditions are discussed in [5, 8, 15]. However, for *stationary* Stokes equations, one cannot use the three fast Poisson solver approach directly because the pressure boundary condition is coupled with the velocity as follows

$$\left.\frac{\partial p}{\partial n}\right|_{y=c,y=d} = \Big(\mu\,(\Delta \mathbf{u}) \cdot \mathbf{n} + \mathbf{g} \cdot \mathbf{n}\Big)\Big|_{y=c,y=d}. \tag{1.7}$$

In this paper, we propose a novel method by introducing an augmented variable that is only defined along the boundary where a Dirichlet boundary condition of the velocity is prescribed. The idea of the new method is to set $\partial p/\partial n$ as part of the unknowns, which we call the augmented variable, along the boundary, which should satisfy the momentum equation projected to the same boundary. Given a guess of the augmented variable, we can solve the Stokes equations easily by calling a fast Poisson solver three times. Since the augmented variable is defined along the boundary, we can get a small system of equations