NUMERICAL MATHEMATICS
A Journal of Chinese Universities (English Series)

Vol. **15** No. 2, pp. 97-106
May 2006

Inverse Eigenvalue Problem in Structural Dynamics Design†

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Received August 28, 2003; Accepted (in received version) July 12, 2004

Abstract. A kind of inverse eigenvalue problem in structural dynamics design is considered. The problem is formulated as an optimization problem. The properties of this problem are analyzed, and the existence of the optimum solution is proved. The directional derivative of the objective function is obtained and a necessary condition for a point to be a local minimum point is given. Then a numerical algorithm for solving the problem is presented and a plane-truss problem is discussed to show the applications of the theories and the algorithm.

Key words: Eigenvalue; inverse problem; numerical method; structural design; dynamics.

AMS subject classifications: 15A18, 65F15

1 Introduction

Structural dynamics design is to design a structure subject to the dynamic characteristics requirement, i.e., determine physical and geometrical parameters such that the structure has the given frequencies and (or) mode shapes. This problem often arises in engineering connected with vibration. Recently, Joseph [1], Li et al. [2,3] converted the structural dynamics design to the following inverse eigenvalue problem.

GIEP Let $x = (x_1, \cdots, x_m)^T$, and let $A(x)$ and $B(x)$ be real $n \times n$ symmetric matrix-valued functions analytic in R^m, and $B(x)$ be positive definite whenever $c \in \Omega$ which is a closed subset of R^m. Given $p (p \leq n)$ real eigenvalues $\lambda_1^* \geq \cdots \geq \lambda_p^*$, find $x \in \Omega$ such that the generalized eigenvalue problem $A(x)y(x) = \lambda(x)B(x)y(x)$ has the prescribed eigenvalues $\lambda_1^*, \cdots, \lambda_p^*$.

Assume that $m = n = p$, and that the given eigenvalues $\lambda_1^*, \cdots, \lambda_n^*$ are distinct. In [1-4], GIEP is converted to the following nonlinear problem:

$$\lambda_i(x) = \lambda_i^*, \qquad i = 1, \cdots, n, \tag{1}$$

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†This research is partially supported by the National Natural Science Foundation of China (No. 10271055).

where $\lambda_1(x) \geq \cdots \geq \lambda_n(x)$ are n eigenvalues of the generalized eigenvalue problem $A(x)y(x) = \lambda(x)B(x)y(x)$, which is solved by using the Newton's method. In particular, [4] considered the improvement of GIEP and its numerical algorithm in the presence of multiple prescribed eigenvalues.

In structural design, the number m of design variables $x_1, \cdots, x_m$, the number p of prescribed eigenvalues λ_i, and the order n of matrices $A(x), B(x)$ are not equal. In this case the number p of equations in (1) is not equal to the number m of unknowns, thus GIEP is ill-posed. In this paper, we propose the proper formulation of inverse eigenvalue problems in structural dynamics design, convert the problem to a constrained optimization problem, analyze the properties of the optimization problem, prove that the optimum solution exists, give a necessary condition for a point to be a local minimum point, and present a numerical algorithm to solve this problem.

To facilitate the discussion, we shall adopt the following notations. By $R^{m\times n}$ is meant the set of all $m \times n$ real matrices, $R^n = R^{n\times 1}$. $SR^{n\times n}$ is the set of all real symmetric matrices. $SR_+^{n\times n}$ is the set of all real symmetric positive definite matrices. I_m is the $m \times m$ identity matrix. φ_n is the set of all permutations of $1, \cdots, n$. $\|\cdot\|$ denotes the Euclidean vector norm. $\lambda(A)$ stands for the set of all eigenvalues of the matrix A. $f'(x; v)$ denotes the directional derivative of function $f(x)$ at $x \in R^m$ along the direction $v \in R^m$.

2 The formulation of problem and its properties

Let Ω be a closed domain in R^m, $A(x) \in SR^{n\times n}$ and $B(x) \in SR_+^{n\times n}$ be an analytic matrix-valued functions on Ω. Given p real eigenvalues $\lambda_1^* \geq \cdots \geq \lambda_p^*$, define $f : \Omega \to R$

$$f(x) = \min_{\pi\in\varphi_n} \sum_{i=1}^{p} [\lambda_{\pi(i)}(x) - \lambda_i^*]^2, \tag{2}$$

where $\lambda_1(x) \geq \cdots \geq \lambda_n(x)$ are n eigenvalues of $A(x)y(x) = \lambda(x)B(x)y(x)$. $\lambda_1(x), \cdots, \lambda_n(x)$ are usually called the eigenvalues of symmetric matrix pencil $\{A(x), B(x)\}$.

The parameter x_i often varies in a special range. Let $l = (l_1, \cdots, l_m)^T$ and $u = (u_1, \cdots, u_m)^T \in R^m$. The notion $l < u (l \leq u)$ means that $l_i < u_i (l_i \leq u_i)$ for $i = 1, \cdots, m$. Moreover, the set Ω with vectors x satisfying $l \leq x \leq u$ is given by $\Omega = \{x \in R^m | l \leq x \leq u\}$. The inverse eigenvalue problem in structural dynamics design can be described as the following optimization problem.

MGIEP Given $l, u \in R^m$, $l < u$, $\Omega = \{x \in R^m | l \leq x \leq u\}$, and $p (p \leq n)$ real eigenvalues $\lambda_1^* \geq \cdots \geq \lambda_p^*$. $A(x) \in SR^{n\times n}$ and $B(x) \in SR_+^{n\times n}$ are analytic matrix-valued functions on Ω. Find $x^* \in \Omega$ such that $f(x^*) = \min_{x\in\Omega} f(x)$, where $f(x)$ is defined in (2).

The following lemma can be proved by mathematical induction.

Lemma 2.1. *Let D be a domain in R^m, $B(x) \in SR_+^{n\times n}$ be an analytic matrix-valued function on D. Then there exists a unique lower triangular matrix-valued function $L(x)$ with positive diagonal elements, such that $L(x)$ is analytic on D and satisfies*

$$B(x) = L(x)L(x)^T. \tag{3}$$

The above result implies that the matrix $B(x)$ has an analytic Cholesky decomposition on D. By Lemma 2.1, we have the following result.

Lemma 2.2. *Suppose that $A(x)$ and $B(x)$ be as in MGIEP, the analytic Cholesky decomposition of $B(x)$ on Ω be* (3), *and*

$$C(x) = L(x)^{-1}A(x)L(x)^{-T}, \tag{4}$$