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Legendre Rational Spectral Method for Nonlinear Klein-Gordon Equation†

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Abstract. A Legendre rational spectral method is proposed for the nonlinear Klein-Gordon equation on the whole line. Its stability and convergence are proved. Numerical results coincides well with the theoretical analysis and demonstrate the efficiency of this approach.

Key words: Legendre rational spectral method; nonlinear Klein-Gordon equation on the whole line.

AMS subject classifications: 65N35, 41A20, 41A25.

1 Introduction

The usual spectral methods are only available for differential equations on bounded domains. But it is also important to consider spectral methods for unbounded domains. For this purpose, we may use Hermite and Laguerre approximations on infinite intervals, see [2, 3, 6, 9]. We can also reformulate some problems on infinite intervals to singular problems on finite intervals, and then use specific Jacobi approximations to solve them numerically, see [4, 5]. Another effective method is based on various rational approximations, see [1, 7, 8].

This paper deals with the Legendre rational spectral method for the nonlinear Klein-Gordon equation on the whole line, which plays an important role in quantum mechanics. In the next section, we first recall some basic results on the Legendre rational approximation, and then propose the Legendre rational spectral scheme for the nonlinear Klein-Gordon equation. We also state the main results on its stability and convergence, and present some numerical results demonstrating the spectral accuracy in space of this method. In section 3, we prove the stability and convergence of the proposed scheme. Although we only consider the nonlinear Klein-Gordon equation in this paper, the main idea and techniques are also applicable to other nonlinear problems in unbounded domains.

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2 Rational spectral method for Klein-Gordon equation

2.1 Legendre rational orthogonal approximation

Let $\Lambda = \{x \mid -\infty < x < \infty\}$ and let $\chi(x)$ be certain weight function. For $1 \leq p \leq \infty$, we define the weighted space $L^p_\chi(\Lambda)$ and its norm $||v||_{L^p_\chi}$ as usual. In particular, we denote by $(u,v)_\chi$ and $||v||_\chi$ the inner product and the norm of the space $L^2_\chi(\Lambda)$. For any integer $m \geq 0$, we define the weighted Sobolev space $H^m_\chi(\Lambda)$ in the usual way. Its inner product, semi-norm and norm are denoted by $(u,v)_{m,\chi}$, $|v|_{m,\chi}$ and $||v||_{m,\chi}$, respectively. For any $r > 0$, we define the space $H^r_\chi(\Lambda)$ and its norm $||v||_{r,\chi}$ by space interpolation. If $\chi(x) \equiv 1$, then we denote $H^r_\chi(\Lambda)$, $|v|_{r,\chi}, ||v||_{r,\chi}, ||v||_\chi$ and $(u,v)_\chi$ by $H^r(\Lambda), |v|_r, ||v||_r, ||v||$ and (u,v), respectively. In addition, $||v||_\infty = ||v||_{L^\infty(\Lambda)}$.

Let $L_l(x)$ be the Legendre polynomial of degree l. The Legendre rational function of degree l is defined by (see [8])

$$R_l(x) = L_l\left(\frac{x}{\sqrt{x^2+1}}\right).$$

Let $\omega(x) = (x^2+1)^{-\frac{3}{2}}$. Then

$$\int_\Lambda R_l(x) R_m(x) \omega(x) dx = (l+\frac{1}{2})^{-1}\delta_{l,m}.$$

Thus, for any function $v \in L^2_\omega(\Lambda)$,

$$v(x) = \sum_{l=0}^{\infty} \hat{v}_l R_l(x), \qquad \hat{v}_l = \left(l+\frac{1}{2}\right)\int_\Lambda v(x) R_l(x) \omega(x) dx.$$

Now, let N be any positive integer, and

$$\mathcal{R}_N = \mathrm{span}\{R_0, R_1, \cdots, R_N\}.$$

The $L^2_\omega(\Lambda)$-orthogonal projection $P_N : L^2_\omega(\Lambda) \to \mathcal{R}_N$ is defined by

$$(P_N v - v, \phi)_\omega = 0, \qquad \forall \phi \in \mathcal{R}_N.$$

The $H^1_\omega(\Lambda)$-orthogonal projection $P^1_N : H^1_\omega(\Lambda) \to \mathcal{R}_N$ is defined by

$$(P^1_N v - v, \phi)_{1,\omega} = 0, \qquad \forall \phi \in \mathcal{R}_N.$$

For the description of approximation results, we introduce the spaces $H^r_{\omega,Z}(\Lambda),\ Z = A, B, C$. For any integer $r \geq 0$,

$$H^r_{\omega,Z}(\Lambda) = \{v \mid v \text{ is a measurable on } \Lambda \text{ and } ||v||_{r,\omega,Z} < \infty\},$$

equipped with the norms

$$||v||_{r,\omega,Z} = \left(\sum_{k=0}^{r} ||(x^2+1)^{\frac{r+k+\gamma_Z}{2}} \partial^k_x v||^2_\omega\right)^{\frac{1}{2}},$$

where $\gamma_A = 0, \gamma_B = 1$ and $\gamma_C = -1$. For any $r > 0$, we define these spaces and their norms by space interpolation.

In the sequel, c denotes a generic positive constant independent of any function and N. We have the following results (see [8]).