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Anisotropic Superconvergence Analysis for the Wilson Nonconforming Element†

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Abstract. The regular condition (there exists a constant c independent of the element K and the mesh such that $h_K/\rho_K \leq c$, where h_K and ρ_K are diameters of K and the biggest ball contained in K, respectively) or the quasi-uniform condition is a basic assumption in the analysis of classical finite elements. In this paper, the supercloseness for consistency error and the superconvergence estimate at the central point of the element for the Wilson nonconforming element in solving second-order elliptic boundary value problem are given without the above assumption on the meshes. Furthermore the global superconvergence for the Wilson nonconforming element is obtained under the anisotropic meshes. Lastly, a numerical test is carried out which confirms our theoretical analysis.

Key words: Anisotropic; nonconforming finite element; superclose; superconvergence.

AMS subject classifications: 65N30

1 Introduction

The Wilson nonconforming element has been widely used in computational mechanics and structural engineering because of its good convergence. In many practical cases, it seems better than the bilinear conforming finite element. This phenomenon causes the great interest of many people who study finite elements. Some papers about the Wilson element have been published which deal with superconvergence. In [6], the superclose property and the global superconvergence are obtained under the regular condition. In [8], the new superconvergence at the vertexes and the midpoints of four edges of rectangular meshes is developed under the same assumption. In this paper we will give the superconvergence analysis which include the supercloseness for consistency error, the superconvergence estimate at the central point and the global superconvergence without the regular assumption or the quasi-uniform assumption. At the same time the numerical test is carried out which confirms our theoretical analysis.

It is well-known that the regular condition or quasi-uniform condition [1] is a basic condition in the analysis of finite element approximation both for conventional conforming and nonconforming

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element. However with the development of the finite element method and its applications to more fields, the above conventional mesh condition becomes a severe restriction for the finite element method. For example, the solution of some partial differential equations may have anisotropic behavior in boundary layers or corner layers. This means that the solution varies significantly only in certain directions. Under this case, an obvious idea to reflect this anisotropy is to use anisotropic meshes, i.e. a small mesh size in the direction of the rapid variation of the solution and a large mesh size in the perpendicular direction. On the other hand, anisotropic meshes can also be advantageous if the surface has strongly anisotropic curvature such as the front of an airplane wings or thin layers of different material. Recently, Zenisek and Apel published a series of papers [2-5] concentrating mainly on the interpolation error estimates for second-order problems. In [9,10], we presented a general anisotropic interpolation theorem. By this theorem, a new criterion to anisotropic interpolation is presented which improves the result of Apel. There seems to be few results yet in the literature on the superconvergence of the nonconforming elements under anisotropic meshes. So the superconvergence analysis of the nonconforming Wilson element under the anisotropic mesh is valuable both in theory and in practice.

2 Fundamental theorem and main results for Wilson element

We consider the second-order problem

$$\begin{cases} -\Delta u = f, & in \ \ \Omega, \\ \quad\; u = 0, & on \ \ \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset R^2$ is a bounded rectangular domain.

The variational problem of (1) is to find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = f(v), \forall \; v \in H_0^1(\Omega), \tag{2}$$

where

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v dxdy, \qquad f(v) = \int_\Omega f v dxdy.$$

Let J_h be a anisotropic rectangular partition of $\Omega, K \in J_h$ an arbitrary element and $\hat{K} = [-1,1] \times [-1,1]$ the reference element. Given an element $K \in J_h$, assume $O_K = (x_K, y_K)$ is its center and $a_i(x_i, y_i)$, $1 \le i \le 4$, are the four vertices of K and $2h_x, 2h_y$ are the lengths of the two edges of K in x and y direction respectively. Here, $\frac{h_x}{h_y}$ and $\frac{h}{h_y}$ may be very large and $h = \max_K(h_x, h_y)$. That is, J_h is an anisotropic mesh without the regularity assumption or the quasi-uniform assumption.

For later use we define the following finite element spaces, their interpolation operators and the discrete variational problem:

(1) For the mesh J_h, we define V_h as the Wilson element space which consists of all functions $v_h \in L^2(\Omega)$ such that v_h is a piecewise quadratic function over Ω, continuous at a_i and vanishes on $a_i \cap \partial\Omega$. More precisely, on $\hat{K}$,

$$\hat{I}\hat{v} = \sum_{i=1}^{4} \hat{N}_i(\xi,\eta)\hat{v}_i + \hat{N}_5(\xi,\eta)\int_{\hat{K}} \frac{\partial^2 \hat{v}}{\partial \xi^2} d\xi d\eta + \hat{N}_6(\xi,\eta)\int_{\hat{K}} \frac{\partial^2 \hat{v}}{\partial \eta^2} d\xi d\eta,$$

where

$$\hat{N}_1 = \frac{1}{4}(1-\xi)(1-\eta), \quad \hat{N}_2 = \frac{1}{4}(1+\xi)(1-\eta), \quad \hat{N}_3 = \frac{1}{4}(1+\xi)(1+\eta),$$