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Superconvergence of Continuous Finite Elements with Interpolated Coefficients for Initial Value Problems of Nonlinear Ordinary Differential Equation†

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Abstract. In this paper, n-degree continuous finite element method with interpolated coefficients for nonlinear initial value problem of ordinary differential equation is introduced and analyzed. An optimal superconvergence $u-u_h=\mathcal{O}(h^{n+2})$, $n\geq 2$, at $(n+1)$-order Lobatto points in each element respectively is proved. Finally the theoretical results are tested by a numerical example.

Key words: Nonlinear ordinary differential equation; continuous finite element with interpolated coefficients; Lobatto points; superconvergence.

AMS subject classifications: 65M60

1 Introduction

Consider the initial value problem of nonlinear ordinary differential equation

$$u'=f(t,u),\quad t\in I=[0,T],\ u(0)=u_0, \tag{1}$$

where $f(t,u)$ is a sufficiently smooth function.

Let J_h be a partition of I such that $J_h: 0=t_0<t_1<\cdots<t_N=T$. Set element $I_j=[t_{j-1},t_j]$, midpoint $\bar{t}_j=(t_j+t_{j-1})/2$ and half-step $h_j=(t_j-t_{j-1})/2, h=\max(h_j), j=1,\cdots,N$. Assume that J_h is quasi-uniform, i.e., there is a $C>0$ such that $h\leq Ch_j$. Define for the partition J_h the finite element space

$$S^h=\{u\in C(I):u|_{I_j}\in\mathbf{P}_n(I_j),\ j=1,\cdots,N\}$$

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where $\mathbf{P}_n(I_j)$ denotes the space of all univariable polynomials of degree $\leq n$ in I_j.

On the element I_j, an n-degree polynomial has $n+1$ parameters. The value of the left endpoint is known on the element I_j for initial value problems, so the finite element on this element has n degrees of freedom. Classical continuous finite element solution $\bar{u}_h$ of (1) can be expressed as $\bar{u}_h = \sum \varphi_\alpha(t)\bar{u}_h(t_\alpha) \in S^h$ satisfying

$$\int_{I_j} (\bar{u}_h' - f(t, \bar{u}_h))vdt = 0, \quad v \in \mathbf{P}_{n-1},\ \bar{u}_h(0) = u_0. \tag{2}$$

For the sake of simplicity, we now define n-degree continuous finite element with interpolated coefficients, $u_h \in S^h$, by

$$\int_{I_j} (u_h' - I_h f(t, u_h))vdt = 0, \quad v \in \mathbf{P}_{n-1},\ u_h(0) = u_0, \tag{3}$$

where I_h denotes the Lagrangian interpolating operator on $(n+1)$-order Lobatto points and u_h and $I_h f(t, u_h)$ satisfy

$$\begin{aligned} u_h &= \sum \varphi_\alpha(t) u_h(t_\alpha) \in S^h, \\ I_h f(t, u_h) &= \sum \varphi_\alpha(t) f(t_\alpha, u_h(t_\alpha)) \in S^h, \end{aligned}$$

where $\varphi_\alpha(t)$ are basis functions in element I_j. Note that the exact solution of (1) satisfies, for smooth function v,

$$\int_{I_j} (u' - f(t, u))vdt = 0, \tag{4}$$

and hence, subtracting (4) from (3) gives

$$\int_{I_j} (e' - f(t, u) + I_h f(t, u_h))vdt = 0, \quad v \in \mathbf{P}_{n-1},\ e(0) = 0 \tag{5}$$

where $e = u - u_h$.

For continuous finite elements, Chen [1] and Pan et al. [2] proved superconvergence for linear case $f(t, u) = au + b$ by a new element orthogonality analysis. In virtue of a simple argument Yang et al. [3] obtained superconvergence of classical finite element for nonlinear problems. The finite element method with interpolated coefficients is an economic and graceful method. This method was introduced and analyzed for semilinear parabolic problems in Zlamal et al. [4]. Later Larsson et al. [5] studied the semidiscrete linear triangular finite element u_h and obtained the following error estimate

$$\|(u_h - u)(t)\|_{L^2(\Omega)} = \mathcal{O}(h), \qquad \text{for } 0 \leq t \leq T.$$

Chen et al. [6] derived almost optimal order of convergence

$$\|(u_h - u)(t)\|_{L^2(\Omega)} = \mathcal{O}(h^2 \ln h), \qquad \text{for } 0 \leq t \leq T,$$

on piecewise uniform triangular meshes by using superconvergence techniques. Recently, Xiong et al. [7] studied superconvergence of triangular quadratic finite elements for semilinear elliptic problems. By the compendious argument we shall study superconvergence of continuous finite element with interpolated coefficients for the initial value problems of nonlinear ordinary differential equation (1). Finally, the theoretical results are tested by a numerical example.