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Numerical Solution for the Helmholtz Equation with Mixed Boundary Condition

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Abstract

We consider the numerical solution for the Helmholtz equation in $\mathbb{R}^2$ with mixed boundary conditions. The solvability of this mixed boundary value problem is established by the boundary integral equation method. Based on the Green formula, we express the solution in terms of the boundary data. The key to the numerical realization of this method is the computation of weakly singular integrals. Numerical performances show the validity and feasibility of our method. The numerical schemes proposed in this paper have been applied in the realization of probe method for inverse scattering problems.

Keywords: Helmholtz equation; Green formula; potential theory; boundary integral equation; numerics.
Mathematics subject classification: 35J05, 31A10, 65N99

1. Introduction

Boundary value problems for the Helmholtz equation, except for their intrinsic importance, play an important role in obstacle scattering problems, which have been studied widely in recent years. Generally, we must analyze their solvability and give some numerical schemes to solve these problems. Noticing that the direct problems should be solved iteratively in some problems such as inverse scattering, the efficiency and amount of computations should be considered carefully from the numerical point of views. In this paper, we apply the boundary integral method (BIM) to solve the boundary value problem (BVP) for the Helmholtz equation with mixed boundary conditions. Such a problem arises in the probe method for inverse scattering problem by multiple obstacles. Compared with the classical BIM method, the new ingredient in this paper is to take the boundary Dirichlet/Neumann data of the solution as the density function based on the Green formula, rather than to introduce the general density function. Once we determine the Cauchy data

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of the solution on the boundary, the solution to the BVP of the Helmholtz equation can be obtained from the integral expression explicitly. Moreover, the domain for the Helmholtz equation in this paper is non-connected where the mixed boundary conditions are specified for the equation.

Let $D \subset \mathbb{R}^2$ be a bounded domain with three parts D_j $(j = 1,2,3)$, namely, $D = \bigcup_{j=1}^{3} D_j$. We assume that each D_j is a simply connected bounded domain with C^2 boundary ∂D_j and $\bar{D}_i \bigcap \bar{D}_j = \emptyset$ for $i \neq j$. For given Ω with C^2 boundary satisfying $\bar{D} \subset \Omega$, we consider the following mixed boundary value problem for the Helmholtz equation

$$\begin{cases} \Delta u(x) + k^2 u(x) = 0 & x \in \Omega \backslash \bar{D}, \\ u(x) = f(x) & x \in \partial\Omega, \\ u(x) = g_1(x) & x \in \partial D_1, \\ \frac{\partial u}{\partial \nu}(x) = g_2(x) & x \in \partial D_2, \\ \frac{\partial u}{\partial \nu}(x) + i\lambda u(x) = g_3(x) & x \in \partial D_3, \end{cases} \tag{1.1}$$

with positive wave number $k > 0$, where ν is the unit normal vector of ∂D directed into the exterior of D and $\lambda > 0$ is the boundary impedance coefficient of D_3. This problem arises in the probe method for multiple obstacles applied in an inverse scattering problem, which is to reconstruct the obstacles with different types of boundary from the far-field pattern of the scattered wave.

In [4], as a key step to the test of the probe method for one obstacle, the author considered the following mixed boundary value problem

$$\begin{cases} \Delta u(x) + k^2 u(x) = 0 & x \in \Omega \backslash \bar{D}, \\ u(x) = f(x) & x \in \partial\Omega, \\ \frac{\partial u}{\partial \nu}(x) + i\lambda u(x) = g(x) & x \in \partial D, \end{cases} \tag{1.2}$$

and the corresponding Dirichlet-to-Neumann map

$$\Lambda_{\partial D} : f(x) \longmapsto \partial_\nu u(x)|_{\partial\Omega}, \tag{1.3}$$

defined by (1.2), which can be viewed as a special case of $D = D_3$ and $D_1 = D_2 = \emptyset$ in (1.1). By applying boundary integral equation method, the solvability of problem (1.2) and an efficient method for solving the Neumann data $\partial_\nu u(x)$ in $\partial\Omega$ are set up. However, in order to test the validity of the probe method for multiple obstacles with different types of boundary, we should treat the problem (1.1) and Dirichlet-to-Neumann map (1.3) defined by (1.1).

Firstly, the solvability of (1.1) should be clarified for suitable boundary data $f, g_i(i = 1,2,3)$ for which the Dirichlet-to-Neumann map (1.3) is well-defined. Secondly, an efficient numerical method to solve (1.1) is necessary. In fact, we can determine the boundary data $\partial_\nu u|_{\partial\Omega}, \partial_\nu u|_{\partial D_1}, u|_{\partial D_2}$ and $u|_{\partial D_3}$ from a coupled boundary integral equations, therefore $u(x)$ in $\Omega\backslash\bar{D}$ can be calculated by the Green formula. To solve it numerically, the discrete form of integral equations must be given. In this process, the key is how to deal with singular integrals.