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A NOTE ON THE CONSTRUCTION OF FUNCTION SPACES FOR DISTRIBUTED-MICROSTRUCTURE MODELS WITH SPATIALLY VARYING CELL GEOMETRY

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Abstract. We construct Lebesgue and Sobolev spaces of functions defined on a continuous distribution of domains $\{Y_x \subset \mathbb{R}^m : x \in \Omega\}$. The resulting spaces can be viewed as a generalisation of the Bochner spaces $L_p(\Omega; W_q^l(Y))$ for the case that Y depends on $x \in \Omega$. Furthermore, we introduce a Lebesgue space of functions defined on the boundaries $\{\partial Y_x : x \in \Omega\}$. The latter construction relies on a uniform Lipschitz parametrisation of the above collection of boundaries, interpreted as a higher-dimensional manifold. The results are applied to prove existence, uniqueness and upper and lower bounds for a distributed-microstructure model of reactive transport in a heterogeneous porous medium.

Key Words. Lebesgue spaces, Sobolev spaces, distributed-microstructure model, direct integral, reaction–diffusion, homogenisation.

1. Introduction

Transport in porous media is governed by at least two highly different spatial scales: the *pore scale* and the *macroscopic scale*, the latter of which is usually of interest in applications. In cases where two or more transport processes happen simultaneously on highly different time scales, it has been shown by periodic homogenisation that *distributed-microstructure models* (or *two-scale models*) are appropriate [3, 2]. Such models consist of averaged equations describing the fast transport processes and of local microscopic cell problems accounting for the slow transport. The most studied example is flow in fissured media [1, 25].

¿From a mathematical point of view, these models are interesting due to the non-standard coupling of the equations and the unusual choice of solution spaces. In [25], the authors show that the variational formulation of a distributed-microstructure model with a cell geometry that varies at different points of the medium naturally leads to function spaces of the form $L_2(\Omega; H^1(Y_x))$ where Y_x is another domain depending on $x \in \Omega$. The construction of such spaces and particularly of their trace spaces is quite intricate and it is the major aim of this paper.

We briefly recall the model from [25] and how a variational formulation is derived. If $\Omega \subset \mathbb{R}^n$ is the macroscopic flow region, then at each $x \in \Omega$ the local geometry is described by a solid matrix block $Y_x \subset Y \subset \mathbb{R}^n$ surrounded by the pore $Y \setminus \overline{Y}_x$. The domain Y_x can depend on the macroscopic space coordinate $x \in \Omega$ in order to account for a heterogeneous medium. For $x \in \Omega$, $y \in Y_x$ and $t \geq 0$, let $u(x,t)$ be the fluid density in the pore space and $U(x,y,t)$ that in the matrix blocks. The

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model equations consist of the (averaged) mass balance of fluid within the pores[1]

(1a)
$$\frac{\partial}{\partial t}(a(x)u) - \mathrm{div}_x(A(x)\nabla_x u) = \frac{1}{|Y|}\int_{\partial Y_x} k(\gamma_x U(x,y,t) - u(x,t))\cdot\nu \, d\sigma_y, \quad x \in \Omega,\ t > 0,$$

where $\gamma_x U(t,x,y)$ denotes the trace of U at $y \in \partial Y_x$, and a family of local mass balances in the matrix blocks parameterised by $x \in \Omega$,

$$\frac{\partial}{\partial t}(b(x)U) - \mathrm{div}_y(B(x)\nabla_y U) = 0, \qquad x \in \Omega,\ y \in Y_x,\ t > 0. \tag{1b}$$

The exchange condition reads

$$-B(x)\nabla_y U \cdot \nu_x = k(\gamma_x U(x,y,t) - u(x,t)), \qquad x \in \Omega,\ y \in \partial Y_x,\ t > 0. \tag{1c}$$

Following [25], a variational formulation of (1) is given as follows. Let $V := L_2(\Omega; H^1(Y_x))$ be an anisotropic Sobolev space (see Def. 4). We look for a pair of functions $u \in L_2(0,T;H^1(\Omega))$ and $U \in L_2(0,T;V)$ satisfying (1a) in the usual weak sense and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\int_{Y_x} bU\Psi \, dy\, dx + \int_\Omega\int_{Y_x} B\nabla_y U \cdot \nabla_y \Psi \, dy\, dx + \int_\Omega\int_{\partial Y_x} k(\gamma_x U - u)\,\gamma_x\Psi \, d\sigma_y \, dx = 0 \qquad \forall \Phi \in V.$$

In [25], the authors prove that the system (1) is wellposed in the above sense. However, a systematic discussion of the properties spaces of the form $L_2(\Omega; H^1(Y_x))$ is missing. Moreover, the cell boundaries Γ_x need to have some regularity with respect to $x \in \Omega$ in order to justify terms of the form

$$\int_\Omega\int_{\partial Y_x} \gamma_x U \, \gamma_x \Psi \, d\sigma_y \, dx.$$

It is the aim of this paper to fill this gap by constructing general spaces $L_p(\Omega; W_q^l(Y_x))$ and $L_p(\Omega; L_q(\partial Y_x))$ and proving some elementary properties of them like separability and reflexivity. While for the former space, it is sufficient that the higher-dimensional set $Q := \cup_{x\in\Omega}(\{x\} \times Y_x)$ is Lebesgue measurable, it turns out that for the latter space of functions defined on a family of cell *boundaries*, the situation is more intricate. We construct a uniform parametrisation of the cell boundaries ∂Y_x under quite general conditions on the geometry. With this framework at hand, objects like the *distributed trace operator* $\gamma U(x,y) := \gamma_x U(x,\cdot)(y)$ are easily constructed. Afterwards, the results are applied to a semilinear two-scale reaction–diffusion system, which has also been discussed in [17] under stronger restrictions on the cell geometry. Modifying techniques from [14, 9], we prove boundedness, existence and uniqueness of weak solutions.

We mention some related work for constant microstructure: The analysis of a similar two-scale reaction–diffusion system has been shown in [10]. Homogenisation results for a general diffusion–convection–reaction–adsorption system can be found in [12, 13]. For numerical approaches to two-scale models, see [21, 1, 18]. A huge list of further references is also given in [11]. We emphasise that in the present paper and in all of the above cited work, a change of the microstructure *w.r.t. time* is not considered. For homogenisation and two-scale models with evolving microstructure, we refer to [22, 16].

This paper is organised as follows. In section 2, we discuss function spaces on cell *domains*. Function spaces on the cell *boundaries* are treated in section 3. In

[1]The model (1) corresponds to the *regularised-microstructure* case in [25].