INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 4, Number 3-4, Pages 402–424

ANALYSIS OF THE $[L^2, L^2, L^2]$ LEAST-SQUARES FINITE ELEMENT METHOD FOR INCOMPRESSIBLE OSEEN-TYPE PROBLEMS

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Dedicated to Professor Max D. Gunzburger on the occasion of his 60th birthday

Abstract. In this paper we analyze several first-order systems of Oseen-type equations that are obtained from the time-dependent incompressible Navier-Stokes equations after introducing the additional vorticity and possibly total pressure variables, time-discretizing the time derivative and linearizing the non-linear terms. We apply the $[L^2, L^2, L^2]$ least-squares finite element scheme to approximate the solutions of these Oseen-type equations assuming homogeneous velocity boundary conditions. All of the associated least-squares energy functionals are defined to be the sum of squared L^2 norms of the residual equations over an appropriate product space. We first prove that the homogeneous least-squares functionals are coercive in the $H^1 \times L^2 \times L^2$ norm for the velocity, vorticity, and pressure, but only continuous in the $H^1 \times H^1 \times H^1$ norm for these variables. Although equivalence between the homogeneous least-squares functionals and one of the above two product norms is not achieved, by using these *a priori* estimates and additional finite element analysis we are nevertheless able to prove that the least-squares method produces an optimal rate of convergence in the H^1 norm for velocity and suboptimal rate of convergence in the L^2 norm for vorticity and pressure. Numerical experiments with various Reynolds numbers that support the theoretical error estimates are presented. In addition, numerical solutions to the time-dependent incompressible Navier-Stokes problem are given to demonstrate the accuracy of the semi-discrete $[L^2, L^2, L^2]$ least-squares finite element approach.

Key Words. Navier-Stokes equations, Oseen-type equations, finite element methods, least squares.

1. Problem formulation

As a first step towards the finite element solution of the time-dependent incompressible Navier-Stokes problem by using the least-squares principles, in this paper we analyze the $[L^2, L^2, L^2]$ least-squares finite element approximations to several first-order systems of Oseen-type equations all equipped with the homogeneous velocity boundary conditions. These systems are obtained from the time-dependent incompressible Navier-Stokes problem after introducing the additional vorticity and possibly total pressure variables, time-discretizing the time derivative and linearizing the non-linear terms.

Received by the editors February 9, 2006, and, in revised form, February 27, 2006.
2000 *Mathematics Subject Classification.* 65N15, 65N30, 76M10.

We start with the derivation of these first-order Oseen-type problems and introduce some background and notations. Let Ω be an open bounded and connected domain in $\mathbb{R}^N$ ($N = 2$ or 3) with Lipschitz boundary $\partial\Omega$. The time-dependent incompressible Navier-Stokes problem on the bounded domain Ω can be posed as the following initial-boundary value problem (cf. [13, 14, 15]):

Find $\mathbf{u}(\boldsymbol{x}, t) : \overline{\Omega} \times [0, T] \to \mathbb{R}^N$ *and* $p(\boldsymbol{x}, t) : \overline{\Omega} \times [0, T] \to \mathbb{R}$ *such that*

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} - \frac{1}{\lambda}\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mathbf{f} && \text{in } \Omega \times (0, T), \\ \nabla \cdot \mathbf{u} &= 0 && \text{in } \Omega \times (0, T), \\ \mathbf{u} &= \mathbf{0} && \text{on } \partial\Omega \times [0, T], \\ \mathbf{u}(\cdot, 0) &= \mathbf{u}_0(\cdot) && \text{in } \Omega, \end{aligned} \tag{1.1}$$

where the symbols Δ, ∇ and $\nabla\cdot$ stand for the Laplacian, gradient and divergence operators with respect to the spatial variable $\boldsymbol{x}$, respectively; $\mathbf{u} = (u_1, \cdots, u_N)^\top$ is the velocity vector; p is the pressure; $\lambda \geq 1$ is the Reynolds number and may be identified with the inverse viscosity constant $1/\nu$; $[0, T]$ is the time interval under consideration; $\mathbf{f} = (f_1, \cdots, f_N)^\top : \Omega \times (0, T) \to \mathbb{R}^N$ is a given vector function representing the density of body force; the initial velocity $\mathbf{u}_0 : \overline{\Omega} \to \mathbb{R}^N$ with $\mathbf{u}_0 = \mathbf{0}$ on $\partial\Omega$ is prescribed. All of them are assumed to be non-dimensionalized.

We now introduce some notations that are used throughout the article. When $N = 2$, we define the curl operator, $\nabla\times$, with respect to the spatial variable $\boldsymbol{x}$ for a smooth scalar function v by

$$\nabla \times v = \Big(\frac{\partial v}{\partial y}, -\frac{\partial v}{\partial x}\Big)^\top,$$

and for a smooth 2-component vector function $\mathbf{v} = (v_1, v_2)^\top$ by

$$\nabla \times \mathbf{v} = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}.$$

When $N = 3$, we define the curl of a smooth 3-component vector function $\mathbf{v} = (v_1, v_2, v_3)^\top$ by

$$\nabla \times \mathbf{v} = \Big(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}, \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}, \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\Big)^\top.$$

We also define the following cross products. If w is a scalar function and $\mathbf{v} = (v_1, v_2)^\top$, then

$$w \times \mathbf{v} = -\mathbf{v} \times w = (-wv_2, wv_1)^\top.$$

If $\mathbf{w} = (w_1, w_2, w_3)^\top$ and $\mathbf{v} = (v_1, v_2, v_3)^\top$, then

$$\mathbf{w} \times \mathbf{v} = (w_2v_3 - w_3v_2, w_3v_1 - w_1v_3, w_1v_2 - w_2v_1)^\top.$$

With these notations, it can be easily checked that the following identities hold: for a smooth vector function $\mathbf{u} = (u_1, \cdots, u_N)^\top$,

$$\nabla \times (\nabla \times \mathbf{u}) = -\Delta\mathbf{u} + \nabla(\nabla \cdot \mathbf{u}) \tag{1.2}$$

and

$$(\mathbf{w} \times \mathbf{v}) \cdot \mathbf{v} = 0 \tag{1.3}$$

for $\mathbf{w} = (w_1, \cdots, w_{2N-3})^\top$ and $\mathbf{v} = (v_1, \cdots, v_N)^\top$.

Introducing the additional vorticity variable $\boldsymbol{\omega}$ (cf. [2, 7, 10]),

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} \qquad \text{on } \overline{\Omega} \times [0, T],$$