Journal of Computational Mathematics, Vol.25, No.6, 2007, 719–729.

A FOURTH ORDER DERIVATIVE-FREE OPERATOR MARCHING METHOD FOR HELMHOLTZ EQUATION IN WAVEGUIDES*

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Abstract

A fourth-order operator marching method for the Helmholtz equation in a waveguide is developed in this paper. It is derived from a new fourth-order exponential integrator for linear evolution equations. The method improves the second-order accuracy associated with the widely used step-wise coupled mode method where the waveguide is approximated by segments that are uniform in the propagation direction. The Helmholtz equation is solved using a one-way reformulation based on the Dirichlet-to-Neumann map. An alternative version closely related to the coupled mode method is also given. Numerical results clearly indicate that the method is more accurate than the coupled mode method while the required computing effort is nearly the same.

Mathematics subject classification: 65N99, 78A50.
Key words: Helmholtz equation, Waveguides, Dirichlet-to-Neumann map, Operator marching.

1. Introduction

For acoustic [1], microwave and optical waveguides [2], it is often necessary to solve the Helmholtz equation

$$u_{xx} + u_{zz} + \kappa^2(x, z)u = 0 \tag{1.1}$$

in a domain which has an extended length scale in one direction, say x. Here, x is the variable along the axis of the waveguide (*i.e.*, the main propagation direction), z is the transverse variable and the wavenumber κ varies with both x and z. Typically, the original waveguide is infinite in the x direction, but we assume that the x-dependent part of the waveguide is finite. That is, κ is x-independent when $x < 0$ and $x > L$ for some $L > 0$. For a given incident wave in $x < 0$, the problem is to calculate the reflected wave for $x < 0$ and the transmitted wave for $x > L$. This is a boundary value problem and the solution can be highly oscillatory if κ is large. Finite difference [3–5] and finite element [6–9] methods have been used to solve this problem. In particular, highly accurate solutions can be obtained by the adaptive finite element method [10, 11]. However, the problem is difficult to solve by a direct discretization of the Helmholtz equation when $L \gg 1$. The finite difference and finite element methods give rise to large linear systems that are difficult to solve, because the coefficient matrix is complex, non-Hermitian and indefinite.

Typically, we are interested in waveguides that change slowly in the propagation direction. That is, the variation of κ with x is small over the scale of a typical wavelength (*i.e.*, $1/\kappa$). In this case, approximate one-way models [1] which have a first-order derivative in x are widely

* Received May 26, 2006; final revised October 16, 2006; accepted January 9, 2007.

used. These one-way models can be efficiently solved by marching forward in the x direction. However, for problems where the reflected waves are important and when the overall change of the waveguide (over a large propagation distance) is significant, it is still necessary to solve the Helmholtz equation. Standard discretization schemes of the Helmholtz equation require that a step size in the x-direction be smaller than a fraction of the typical wavelength. For slowly varying waveguides, it is possible to develop more efficient numerical methods [12] where the step size in x is only restricted by the variation of the waveguide in the x direction. In the step-wise coupled mode method [1, 2, 13], the waveguide is approximated by a sequence of x-invariant segments. For the segment from x_{j-1} to $x_j = x_{j-1} + h$, the wavenumber $\kappa(x, z)$ is approximated by $\kappa(x_{j-1/2}, z)$, where $x_{j-1/2} = x_j - h/2$, and the solution is expanded in the eigenfunctions of the operator $\partial_z^2 + \kappa^2(x_{j-1/2}, z)$. When κ has a weak dependence on x, the segment length h can be larger than a typical wavelength. Notice that this is a second-order method, so that the numerical solution should have an $\mathcal{O}(h^2)$ error. But the coefficient in the error term vanishes when the waveguide becomes x-independent.

The second-order methods developed in [14] is equivalent to the step-wise coupled mode method, but it uses a one-way reformulation of the Helmholtz equation in terms of the Dirichlet-to-Neumann map. The advantage of such a one-way operator marching scheme is that the required computer memory is independent of the total distance in the x direction, *i.e.* L. On the other hand, the required computer memory of the coupled mode method depends linearly on L. The two methods require nearly the same computing effort in terms of the floating point operations. The fourth-order method developed in [14] reduces the $\mathcal{O}(h^2)$ error of the coupled mode method to $\mathcal{O}(h^4)$ and it preserves the property that h can be larger than a typical wavelength when κ varies with x slowly. However, this method needs to evaluate the derivatives of κ. This can be very inconvenient, for example, when κ itself is calculated from a coordinate transform when the original waveguide has a more complicated geometry [15]. Based on a fourth-order Magnus method [16] for linear evolution equations, we derived another fourth-order operator marching method for the Helmholtz equation in [17]. The method does not require the derivative of κ, but it cannot be applied to the more general Helmholtz equation as in [15] due to the existence of a commutator in the fourth-order Magnus method [16].

In this paper, we develop a new derivative-free fourth-order operator marching method that can be applied to the more general case. It is based on a new fourth-order exponential integrator for linear evolution equations. This exponential integrator may be useful in other applications and its fourth-order of accuracy is proved in this paper. While there are many high order numerical methods for linear evolution equations, only a few methods can be used to derive efficient operator marching schemes for the Helmholtz equation. The new fourth-order operator marching method is given using a one-way reformulation based on the Dirichlet-to-Neumann map, but we also present a version of this method which is similar to the standard step-wise coupled mode method [1, 2, 13]. Although the fourth-order methods in [14, 17] also have variants similar to the widely used coupled mode methods, this connection has not been revealed before. Numerical examples are used to illustrate the fourth-order accuracy of the proposed method.

2. One-way Re-formulations

The waveguide is assumed to be x-invariant for $x < 0$ and $x > L$; thus, we let

$$\kappa = \kappa_0(z) \quad \text{for} \quad x < 0, \quad \kappa = \kappa_\infty(z) \quad \text{for} \quad x > L.$$