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A CONSTRAINED OPTIMIZATION APPROACH FOR LCP [*1)]

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Abstract

In this paper, LCP is converted to an equivalent nonsmooth nonlinear equation system $H(x, y) = 0$ by using the famous NCP function–Fischer-Burmeister function. Note that some equations in $H(x, y) = 0$ are nonsmooth and nonlinear hence difficult to solve while the others are linear hence easy to solve. Then we further convert the nonlinear equation system $H(x, y) = 0$ to an optimization problem with linear equality constraints. After that we study the conditions under which the K–T points of the optimization problem are the solutions of the original LCP and propose a method to solve the optimization problem. In this algorithm, the search direction is obtained by solving a strict convex programming at each iterative point. However, our algorithm is essentially different from traditional SQP method. The global convergence of the method is proved under mild conditions. In addition, we can prove that the algorithm is convergent superlinearly under the conditions: M is P_0 matrix and the limit point is a strict complementarity solution of LCP. Preliminary numerical experiments are reported with this method.

Mathematics subject classification: 90C30, 65K05.
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1. Introduction

Consider the following linear complementarity problem (LCP)

$$\begin{gathered} y = Mx + q, \\ x \geq 0,\ y \geq 0,\ x^T y = 0, \end{gathered} \tag{1}$$

where $M \in R^{n\times n}$, $x, y \in R^n$ and $x \geq 0$ ($y \geq 0$) means that $x_i \geq 0$ ($y_i \geq 0$). In this paper, we assume that the solution set of (1) is nonempty. Let X denote the solution set of (1). For convenience, we sometimes use $w = (x, y)$ for $(x^T, y^T)^T$.

LCP has many applications in economic and engineering, see [11, 16, 23] for survey. A lot of experts studied the problem. At present, numerous algorithms were proposed for the problem, for example, interior method (see [33] and references therein), nonsmooth Newton method (see [13, 15, 19, 21, 27]) and smoothing method (see [3, 4, 6, 28] and [8] for survey).

Since the work by Mangasarian [25] it has been well known that by means of a suitable function $\phi : R^2 \to R$, the system

$$a \geq 0,\ b \geq 0,\ ab = 0 \tag{2}$$

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can be transformed into an equivalent nonlinear equation

$$\phi(a,b) = 0. \tag{3}$$

In this case, function ϕ is named as NCP-function. Then (1) can be reformulated as the following equivalent nonlinear equation system

$$\Phi(x) = \begin{pmatrix} \phi(x_1,(Mx)_1) \\ \vdots \\ \phi(x_n,(Mx)_n) \end{pmatrix}, \tag{4}$$

or

$$H(x,y) = \begin{pmatrix} \phi(x_1,y_1) \\ \vdots \\ \phi(x_n,y_n) \\ y = Mx - q \end{pmatrix}. \tag{5}$$

Many methods have been proposed to solve (4) or (5) or to minimize their natural residual

$$\Psi_1(x) = \frac{1}{2}\|\Phi(x)\|^2 \ \text{ or } \ \Psi_2(x,y) = \frac{1}{2}\|H(x,y)\|^2,$$

see [13, 18, 17, 20, 15, 14]. In this paper, we are concerned about formulation (5). Generally speaking, (5) is nonsmooth and nonlinear, hence it is not easy to solve. However, in (5), the first n components are nonsmooth and nonlinear and difficult to solve while the last n components are linear and easy to handle. Therefore, it is reasonable to handle the first part which consists of the n nonsmooth components and the second part which consists of the n linear equations separately. Based on this idea, we transform further (5) into the following equivalent minimization problem

$$\begin{array}{rl} \min_{(x,y)\in R^{2n}} & \Psi(w) = \Psi(x,y) = \frac{1}{2}\sum_{i=1}^{n}\phi(x_i,y_i)^2 \\ \text{s.t.} & y - Mx - q = 0. \end{array} \tag{6}$$

Then we propose an SQP(Sequential Quadratic Programming) type method to solve (6). However, the method is different from the traditional SQP methods. The search direction is obtained by solving the following convex programming at each iterative point

$$\begin{array}{rl} \min_{dw\in R^{2n}} & \theta(dw) = \frac{1}{2}\|Vdw + \phi(x,y)\|_2^2 + \frac{1}{2}\mu\|dw\|_2^2, \\ \text{s.t.} & (-M, I_n)dw = -y + Mx + q, \end{array} \tag{7}$$

where $dw = (dx, dy)$ and $V^T \in \partial\phi(x,y)$, which is a generalized Jacobian of $\phi(w) = \phi(x,y) = \begin{pmatrix} \phi(x_1,y_1) \\ \vdots \\ \phi(x_n,y_n) \end{pmatrix}$ at w and $\mu = \|H(w)\|^\delta$ ($\delta = (0,2]$) and $I_n \in R^{n\times n}$ is the identical matrix. The motivation of using (7) to generate search direction is from the recent results in [12, 30]. Note that (7) is a strict convex quadratical programming, it has the unique solution. Throughout the paper, we shall only use the famous Fischer-Burmeister function defined by

$$\phi(a,b) = \sqrt{a^2+b^2} - a - b, \ (a,b \in R). \tag{8}$$

It has many promised properties and attracted the attention of many researchers [17, 13, 15, 2], see [18] for a survey of its applications.