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NUMERICAL STUDIES OF 2D FREE SURFACE WAVES WITH FIXED BOTTOM*1)

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Abstract

The motion of surface waves under the effect of bottom is a very interesting and challenging phenomenon in the nature. we use boundary integral method to compute and analyze this problem. In the linear analysis, the linearized equations have bounded error increase under some compatible conditions. This contributes to the cancellation of instable Kelvin-Helmholtz terms. Under the effect of bottom, the existence of equations is hard to determine, but given some limitations it proves true. These limitations are that the swing of interfaces should be small enough, and the distance between surface and bottom should be large enough. In order to maintain the stability of computation, some compatible relationship must be satisfied like that of [5]. In the numerical examples, the simulation of standing waves and breaking waves are calculated. And in the case of shallow bottom, we found that the behavior of waves are rather singular.

Key words: Fixed bottom, 2D surface wave, Boundary integral method, Linear analysis, Energy analysis.

1. Introduction

It is well known that the solution to the Dirichlet and Neumann problems for Laplace's equation may be expressed in terms of boundary integrals of source or dipole distributions. In this method, the boundary is always labelled as Lagrange markers. Numerical methods with Lagrange markers were attempted for vortex sheets long ago by Rosenhead. Such Methods for more general fluid interfaces were first proposed by Birkhoff [6]. The first successful boundary intergral method (BIM) was developed by Longuet-Higgins and Cokelet [21], who calculated plunging breakers. BIM for the exact, time-dependent equations have been developed and used in many other works, including Vinje, Brevig [33], Baker, Meiron, Orszag [2], Pullin [26], New, Mclver, Peregrine [24], Dold [10], Schwartz, Fenton [29]. Yeung [35] reviewed these early works. Methods of boundary integral type have been used even for the ill-posed cases of fluid interface motion, including vortex sheets and Reyleigh-Taylor instability (Moore [22], Krasny [20], Kerr [19], Tryggvason [32], Shelley [30]), a regularization or filtering of high wave numbers is necessary for numerical stability.

Flows such as those generated by surface waves over bottom topography or due to a solid body in motion underneath an interface require Neumann boundary conditions at the solid boundaries in addition to the free-surface conditions. Since the fluid can not penetrate a solid boundary, the normal fluid velocity at the body must equal the normal body velocity. The bottom boundary and interface are assumed to be 2π-periodic in the horizontal direction. Using the complex variables, we parametrize the free surface and solid boundary by $z_F(\alpha, t)$ and $z_B(\alpha, t)$ respectively. The bottom is assumed stationary. We take α as the Lagrange coordinate; i.e., dz_F/dt is the velocity of the lower fluid at the surface. The dipole moment and

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source moment can be determined from the potential and boundaries by resolving two linear integral equations. It is important that the integral equations are two Fredholm equations of the second kind, they can be solved by simple iterative procedures. Verchota [34] and Kenig [17][18] proved the existence of solution in the bounded domain with Lipscitz boundary. Beale, Hou and Lowengrub [4] proved that the Fredholm equation has solution in $H^s, s \geq 0$ in half plane with smooth boundary. But it is hard to extend this result to system. We use an iterative sequence to construct a solution and at the same time look for the sufficient conditions to guarantee the existence of this solution. We find it is suffice to force the distance between surface and bottom large enough and the perturbation of surface is small enough. And the later condition is the same to the idea that the free surface is sufficiently close to equilibrium, which is well-known as the condition of well-posedness of free surface according to W.Craig [8] and H. Yosihara [36]. Although they are not satisfactory conditions, it is compatible with the computation and we expect the proof of existence of more relaxed conditions, or even the removement of such limitations.

The stability of numerical methods is closely related to the question studied in section 2 of the well-posedness of arbitrary linearizations, since the numerical error can be expected to satisfy the linear equations to first approximations. Beale, Hou and Lowengrub [5] presented a convergence proof of a boundary integral for water waves with or without surface tension. Following a framework developed in [4] for linearized motion perturbed about an arbitrary smooth solution at the continuous level, they found that very delicate balances among terms with singular integrals and derivatives must be preserved at the discrete level in order to maintain numerical stability. They also realized that suitable numerical filtering is necessary at certain places to prevent the discretization from introducing new instabilities in the high modes. This filtering depends on the choices for approximating spatial derivatives and quadrature rules for singular integrals. Besides filtering, Hou and Zhang [16] discovered a new stablizing method which compensates the unstable terms, the new method can be expanded to 3-D water waves. In order to illustrate the necessity of filtering, we develop a group of numerical experiments to show the differences that filtering brings with bottom. While the comparison under the case without bottom was shown in [5]. When the bottom is considered, it doesn't bring any singularity to the velocity, which make the numerical analysis and computation comparatively easy to work on, provided realizing the solvablity of the linearized Fredholm equations (see (29), (30)).

The advantage of using alternating trapezoidal quadrature is that the approximation is spectrally accurate. Sidi and Israeli [31] analyzed the spectral accuracy of a midpoint rule approximation for a periodic singular integrate. They realized that the alternating quadrature rule applied to singular integrals gives spectral accuracy. Shelley [30] used this scheme in the context of studying the cortex sheet singularity by vortex methods. By using the spectral accuracy of the alternating trapezoidal rule, Hou, Lowengrub and Krasny [14] simplifed the proof of the convergence of the point vortex method for vortex sheets.

The rest of the paper is organized as follows: The following in Section 1 is devoted to describe the boundary integral reformulation introduced by Beale, Hou and Lowengrub [5] and their ideas to remove numerical instabilities. In Section 2, we present our linearize analysis in continuous level. The numerical analysis is given in Section 3. Finally, in Section 5, some numerical examples are included to demonstrate the robustness of the method. Numerical simulation of shallow and deep water proceed to a time where it approximates the singularity. The method remains stable even in the full nonlinear regime of motion.

1.1. Analytical Formulas

We consider the 2D incompressible, inviscid and irrotational fluid bounded by upper free boundary and lower fixed bottom. Based on the potential theory and partial differential equations, we can regard the interface as dipole layer, and bottom as source layer with potential flow, thus there exist potential function and stream function.

Assuming there is a source with strength m at $z(e)$, then at any place z except $z(e)$ the