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SUPERCONVERGENCE OF DISCONTINUOUS GALERKIN METHOD FOR NONSTATIONARY HYPERBOLIC EQUATION*1)

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Abstract

For the first order nonstationary hyperbolic equation taking the piecewise linear discontinuous Galerkin solver, we prove that under the uniform rectangular partition, such a discontinuous solver, after postprossesing, can have two and half approximative order which is half order higher than the optimal estimate by Lesaint and Raviart under the rectangular partition.

Key words: Discontinuous Galerkin method, Hyperbolic equation, Nonstationary, Superconvergence.

1. Introduction

Consider the first order hyperbolic equation in two-dimensional nonstationary case,

$$\begin{aligned} u_t + u_x + u_y + u &= f, \qquad && in\ \Omega \\ u(x,y,t) &= 0, && on\ \Gamma_- \\ u(x,y,0) &= u_0(x,y) && \end{aligned} \tag{1}$$

where the domain $\Omega = [0,1] \times [0,1]$ for the sake of simplicity , and the boundary $\Gamma_- = \{(0,y) \times (x,0)\}$ and $\Gamma_+ = \{(1,y) \times (x,1)\}$. We assume throughout that the solution $u, u_t \in H^4$.

Earlier in 1973, Strang[1] has indicated that the continuous Galerkin method(CGM) with piecewise polynomials has two shortcomings for solving the above equation: first, that they resulted in an implicit scheme, rather than an explicit scheme; and second, that the convergence rate would be reduced by an order compared to that of the ordinary polynomial approximation.

However, if the discontinuous Galerkin method(DGM) was employed, the situation would be improved in two ways: on one hand, the scheme becomes explicit, and on the other, under the rectangular partition, there is proved to result without loss in the order of convergence rate; that is, it gains the same order as the ordinary polynomial approximation (Lesiant- Raviart's 1974 [2]). Even under the general triangulation of domains with no particular geometry, only half an order was lost in the convergence rate(Johnson et al. 1986 [3]).

Recently, we [4] have found that for piecewise bilinear elements, under uniform rectangular partition, there is a half order increase in the convergence rate as against Lesiant-Raviart's optimal estimate. This again confirms a conclusion previously reached by us [5] that a careful selection of the partition would effect the convergence rate in a considerable degree, whether CGM or DGM be employed.

The above results of Lesiant-Raviart[2], Johnson[3], and us[4] were done for stationary problem only, but this paper will consider the nonstationary case.

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2. Preliminaries

Decompose Ω into the elements

$$e = [x_e - h_e, x_e + h_e] \times [y_e - k_e, y_e + k_e]$$

where (x_e, y_e) is the center of e, $2h_e$ and $2k_e$ are the length and height, respectively, of e.

Assume that W_h is a finite element space of discontinuous piecewise bilinear functions. The bilinear form correspondering to stationary equation(1) has been extended from C^1 to W_h in [4] as follows: $\forall v \in W_h$,

$$\begin{aligned} B(w,v) \quad = \quad & \textstyle\sum_e\{\int_e w(v - v_x - v_y) + \int_{y_e-k_e}^{y_e+k_e}[w(x_e + h_e - 0, y) \\ & v(x_e + h_e - 0, y) - w(x_e - h_e - 0, y)v(x_e - h_e + 0, y)]dy \\ & + \textstyle\int_{x_e-h_e}^{x_e+h_e}[w(x, y_e + k_e - 0)v(x, y_e + k_e - 0) \\ & -w(x, y_e - k_e - 0)v(x, y_e - k_e + 0)]dx\}, \end{aligned} \tag{2}$$

which returns to $B(w,v) = \int_\Omega (w_x + w_y + w)v$ when $w \in C^1$. Notice that the above line integrals can be rewritten into:

$$\begin{aligned} & \textstyle\sum_e \int_{y_e-k_e}^{y_e+k_e} w(x_e + h_e - 0, y)v(x_e + h_e - 0, y)dy \\ = \quad & \textstyle\sum_e \int_{y_e-k_e}^{y_e+k_e} w(x_e - h_e - 0, y)v(x_e - h_e - 0, y)dy \qquad . \\ & + \textstyle\int_0^1 w(1-0, y)v(1-0, y)dy - \int_0^1 w(-0, y)v(-0, y)dy. \end{aligned}$$

Hence we also have

$$\begin{aligned} B(w,v) \quad = \quad & \textstyle\sum_e\{\int_e w(v - v_x - v_y) + \int_{y_e-k_e}^{y_e+k_e} w(x_e - h_e - 0, y) \\ & [v(x_e - h_e - 0, y) - v(x_e - h_e + 0, y)]dy \\ & + \textstyle\int_{x_e-h_e}^{x_e+h_e} w(x, y_e - k_e - 0)[v(x, y_e - k_e - 0) - v(x, y_e - k_e + 0)]dx \\ & + \textstyle\int_{\Gamma_+} wvds - \int_{\Gamma_-} wvds\}. \end{aligned} \tag{3}$$

Notice that (3) includes a jump, of v, across the boundaries between adjacent elements. B is