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FIRST-ORDER AND SECOND-ORDER, CHAOS-FREE, FINITE DIFFERENCE SCHEMES FOR FISHER EQUATION[*1)]

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Abstract

A new class of finite difference schemes is constructed for Fisher partial differential equation i.e. the reaction-diffusion equation with stiff source term: $\alpha u(1-u)$. These schemes have the properties that they reduce to high fidelity algorithms in the diffusion-free case namely in which the numerical solutions preserve the properties inherent in the exact solutions for arbitrary time step-size and reaction coefficient $\alpha > 0$, and all non-physical spurious solutions including bifurcations and chaos that normally appear in the standard discrete models of Fisher partial differential equation will not occur. The implicit schemes so developed obtain the numerical solutions by solving a single linear algebraic system at each step. The boundness and asymptotic behaviour of numerical solutions obtained by all these schemes are given. The approach constructing the above schemes can be extended to reaction-diffusion equations with other stiff source terms.

Key words: Reaction-diffusion equation, Fidelity algorithm.

1. Introduction

The Fisher partial differential equation (PDE.)[1,9]

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha u(1-u), \quad \alpha > 0 \tag{1.1}$$

is a mathematical model for the analysis of a number of natural phenomena: population growth with dispersal[2], flam propagation[7]and the neution population in a nuclear reaction[3], and has also been used as a test equation for the investigation of numerical integration schemes and the related issues of numerical chaos[8,4].

It is well known that the use of Euler finite difference scheme to solve the Fisher PDE. can produce bifurcations and chaos i.e. non physical spurious solutions which are contrived from the difference equation and are not a feature of the Fisher PDE. A major source of resulting

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in non physical spurious solutions is the existence of bifurcations and chaotie behaviour in numerical solutions to the corresponding ordinary differential equation (ODE.). Therefore, in numerical solution preceduces one must eliminate such non physical spurious solutions and faithfully approximates to the original initial-value problem (IVP.):

$$\begin{cases} \dfrac{du}{dt} = \alpha S(u), \alpha > 0 \\ u(0) = u^0 \end{cases} \tag{1.2}$$

where

$$S(u) = u(1-u) \tag{1.3}$$

i.e., so that the properties of particular interest that include asymptotic behaviour of solutions and the stability properties of fixed points should be all the same to IVP. (1.2)–(1.3). Consequently the corresponding numerical model is most important.

The present paper is a continuation of [11] where a new class of explicitly high fidelity algorithms for stiff IVP. (1.2) were presented, the major purpose is that the high fidelity algorithms so developed will be incorporated, in the appropriate, into finite-difference schemes for the Fisher PDE. (1.1) such that the numerical solutions arising will faithfully represent its genuine solutions, and non physical spurious solutions including bifurcations and chaos will not occur.

The paper is organized as following. In 2, the high fidelity algorithms for the diffusion-free case i.e. the IVP. (1.2)–(1.3) and their important properties are described. In §3 the full numerical schemes for the initial-boundary value problem (IBVP.) of the Fisher PDE. and the preliminaris are given. In §4 the boundness and asymptotic behaviour of numerical solutions obtained by all the schemes are discussed in detail.

2. Diffusion-Free Case

2.1. The properties of solutions for Logistie equation

In diffusion-free case the Fisher PDE.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha u(1-u)$$

become the Logistic ODE.

$$\frac{du}{dt} = \alpha u(1-u), \quad t > 0, \tag{2.1}$$

where $\alpha \gg 0$ is a stiff parameter.

For the Logistic equation the exact solution is given by

$$u(t) = \frac{u^0}{u^0 + (1-u^0)e^{-\alpha t}} = \frac{u^0 e^{\alpha t}}{(1-u^0) + u^0 e^{\alpha t}}, \tag{2.2}$$

where $u^0 > 0$ is the initial value.

Two properties are important for the Logistic equation (2.1): 1) It has two fixed points $u^* = 1$ and $u^* = 0$ in which the fixed $u^* = 1$ is stable while $u^* = 0$ is unstable; 2) The global asymptotic solution behavious is that for every positive initial value u^0 and $\alpha > 0$ the solution is monotone, and eventually tends to stable fixed point $u^* = 1$ and that $u(t) \nearrow 1$ if $0 < u^0 < 1$ and $u(t) \searrow 1$ if $u^0 > 1$.