INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 3, Number 4, Pages 413–436

A POSTERIORI ERROR ANALYSIS FOR FEM OF THERMISTOR PROBLEMS

WALTER ALLEGRETTO AND NINGNING YAN

Abstract. In this paper, we present what we believe is the first a posteriori finite element error analysis for the system of equations that governs micromachined microsensors. Our main result is the establishment of an efficient and reliable a posteriori error estimator Θ.

Key Words. Thermistor problems, adaptive finite element methods, a posteriori error analysis.

1. Introduction

The system of equations that govern thermistor behavior has a long history. Recently, it has been the subject of intensive investigations, see e.g. [7] for a survey of the subject. We are interested in this paper in the version of the system that has been recently proposed as a model for micromachined microsensors, [4], [5], [6], [7]. The equations incorporate terms that account for heat losses to the surrounding gas and radiation effects. Some of these are expressed as nonlocal terms, and to avoid physically contradictory effects at high gas pressures, the system of equations is expressed as an obstacle problem, [3]. In this article, the existence of solutions and their long time behaviors was considered. The error analysis of numerical approximations, based on finite volume methods, were considered in [2]. In this paper, we present what we believe to be the first a posteriori error analysis for the finite element approximation of the obstacle system introduced in [3]. In particular we obtain an efficient and reliable a posteriori error estimator Θ. Our analysis involves, in part, the adaptation of results earlier obtained for elliptic equations. To the best of our knowledge, these results are new even for the classical thermistor systems, as described in [7].

2. Finite Element Approximation of Thermistor problems

Let $\Omega \subset R^2$ be a polygonal domain, $J = (0,T)$, $J_t = (0,t)$. Let $(\cdot,\cdot)$ be the inner product on Ω. In this paper we adopt the standard notation $W^{m,q}(\Omega)$ for Sobolev spaces on Ω with norm $\|\cdot\|_{m,q,\Omega}$ and seminorm $|\cdot|_{m,q,\Omega}$. Denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$ with norm $\|\cdot\|_{m,\Omega}$ and seminorm $|\cdot|_{m,\Omega}$. Set

$$H_0^1(\Omega) \equiv \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}.$$

We denote by $L^s(0,T;W^{m,p}(\Omega))$ the Banach space of all L^s integrable functions from $(0,T)$ into $W^{m,p}(\Omega)$ with norm $\|v\|_{L^s(0,T;W^{m,p}(\Omega))} =$

Received by the editors June 11, 2004 and, in revised form, January 22, 2005.
2000 *Mathematics Subject Classification.* 49J20, 65N30.
This work is supported by NSERC (canada) and the Special Funds for Major State Basic Research Projects (No. G2000067102), National Natural Science Foundation of China (No. 60474027).

$(\int_0^T \|v\|^s_{W^{m,p}(\Omega)}dt)^{\frac{1}{s}}$ for $s \in [1,\infty)$ and the standard modification for $s=\infty$. Similarly, one define the spaces $H^1(0,T;W^{m,p}(\Omega))$ and $C^l(0,T;W^{m,p}(\Omega))$. The details can be found in [16]. In addition c or C denotes a general positive constant independent of h. Let $\|v\|_{-1,W}$ represent the negative norm of v defined by

$$\|v\|_{-1,\Omega} = \sup_{w\in H_0^1(\Omega), w\neq 0} \frac{(v,w)}{\|w\|_{1,\Omega}},$$

$$\|v\|_{-1,\Omega\times J_t} = \sup_{w\in H^1(0,t;H_0^1(\Omega)), w\neq 0} \frac{\int_0^t (v,w)dt}{\|w\|_{1,\Omega\times J_t}}.$$

Let $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$, $\partial\Omega_D \cap \partial\Omega_N = \emptyset$. Set

$$H_D^1(\Omega) \equiv \{v \in H^1(\Omega) : v|_{\partial\Omega_D} = 0\}, \qquad H^1_{\phi_0}(\Omega) \equiv \{v \in H^1(\Omega) : v|_{\partial\Omega_D} = \phi_0\}.$$

Consider the model problem of thermistor problems: find $(u,\phi) \in K \times H^1_{\phi_0}(\Omega)$ with $u_t \in H^{-1}(\Omega)$ for each t such that

$$(u_t, u-v) + (k(u)\nabla u, \nabla(u-v)) + \eta(\int_\Omega G(x,y)u(y)dy, u-v)$$

$$+\alpha(u^4, u-v) \leq (\sigma(u)|\nabla\phi|^2, u-v) \quad \forall v \in K, \tag{2.1}$$

$$(\sigma(u)\nabla\phi, \nabla w) = 0 \quad \forall w \in H_D^1(\Omega), \tag{2.2}$$

$$u(x,0) = u_0(x) \geq 0, \tag{2.3}$$

where $(\cdot,\cdot)$ denotes the standard $L^2(\Omega)$ inner product, and

$$K = \{v \in H_0^1(\Omega) : \ v \geq 0\}.$$

Throughout this paper, we assume that $u_0 \in H_0^1(\Omega)$, $\phi_0 \in C^\infty(\partial\Omega_D)$, η, α are constants, $0 \leq G(x,y) < \infty$. In the physically significant case, ϕ_0 is piecewise constant function (in space) on the components of $\partial\Omega_D$ which represent the contacts. Moreover, it is assumed that $0 < c \leq \sigma(s), k(s) \leq C < \infty$, and there exists a constant $C_0 > 0$ such that

$$|\sigma(s) - \sigma(s')| + |k(s) - k(s')| \leq C_0|s - s'|, \quad s, s' \in R.$$

Using (2.2) and Green's formula, (2.1)-(2.3) can be rewritten to be

$$(u_t, u-v) + (k(u)\nabla u, \nabla(u-v)) + \eta(\int_\Omega G(x,y)u(y)dy, u-v)$$

$$+\alpha(u^4, u-v) \leq (\sigma(u)\phi\nabla\phi, \nabla(v-u)) \quad \forall v \in K, \tag{2.4}$$

$$(\sigma(u)\nabla\phi, \nabla w) = 0 \quad \forall w \in H_D^1(\Omega), \tag{2.5}$$

$$u(x,0) = u_0(x). \tag{2.6}$$

Let us consider the finite element approximation of problem (2.4)-(2.6). Let T_u^h be a regular partition of Ω. Let h_{τ_u} be the size of the element τ_u in T_u^h, $h = \max_{\tau_u \in T_u^h}\{h_{\tau_u}\}$. Set the finite element space S_u^h to be the standard conforming piecewise linear finite element space on T_u^h. Let $V_u^h = S_u^h \cap H_0^1(\Omega)$, $K^h = \{v \in V_u^h : v \geq 0\}$. Then, it is easy to see that $K^h \subset K$. Similarly, let T_ϕ^h be another regular partition of Ω. Let h_{τ_ϕ} be the size of the element τ_ϕ in T_ϕ^h. Set the finite element