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ON THE CONVERGENCE OF A C^0 FINITE ELEMENT METHOD FOR THIN PLATE BENDING*1)

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§1. Introduction

Using the finite element method to solve Kirchhoff's equation of thin plate bending usually requires that the finite elements have C^1 continuity. This requirement causes a rapid increase in the degrees of freedom of the elements and in the cost of the solution. Mixed, hybrid and more general nonconforming methods are thus introduced to attack the difficulty. Those methods, though free from the requirement and some what convenient in calculation have some disadvantages, and so new methods are still being searched for by engineers and scientists.

In a recent paper [1], Ortiz and Morris proposed a new method of using the C^0 finite elements. At each step of the method, a variational problem, which contains the derivatives of the unknown function of only the first order, needs to be solved. But unlike the usual mixed methods[2, 3], the method does not solve the deflection w and moments M_{ij} or w and Δw simultaneously. The potential energy of the bending plate is expressed as a functional of the solutions $u_i = \partial w/\partial x_i$, $u = 1, 2$, in the method and then the energy functional is minimized under the constraint $\frac{\partial u_1}{\partial x_2} = \frac{\partial u_2}{\partial x_1}$. This constrained extreme problem is solved by using the penalty method. It was reported in [1] that excellent accuracy was attained in the numerical test.

The purpose of this paper is to analyse the convergence properties of Ortiz and Morris's method (OM method) and to estimate the error bounds of the finite element approximation obtained by using the OM method. Also, we will study how to choose the penalty parameter such that it matches the partition parameter to get the optimal accuracy.

For the convenience of our description, we outline the OM method in the following. Assume, for the sake of simplicity, the plate is clamped at the boundary. For other boundary conditions, we refer the readers to the original paper [1]. Suppose the plate occupies a convex domain $\Omega \subset R^2$ with piecewise smooth boundary $\partial\Omega$. We are going

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to minimize the potential energy

$$\tilde{P}(w) = \frac{1}{2}\int_\Omega D\Big[(1-v)\sum_{i,j=1}^{2}\Big(\frac{\partial^2 w}{\partial x_i \partial x_j}\Big)^2 + \nu\Big(\frac{\partial^2 w}{\partial x_1^2} + \frac{\partial^2 w}{\partial x_2^2}\Big)^2\Big]dx - \int_\Omega qw dx. \tag{1.1}$$

On space $H_0^2(\Omega)$, where $w(x) = x(x_1, x_2)$ is the deflection, $q = q(x)$ is the load and $dx = dx_1 dx_2$. Set

$$u_i = \frac{\partial w}{\partial x_i}, \quad i = 1,2; \quad u = (u_1, u_2).$$

Suppose $\varphi(x)$ is the solution of the following boundary value problem:

$$\begin{cases} -\Delta\varphi = q, & x \in \Omega, \\ \varphi = 0, & x \in \partial\Omega. \end{cases} \tag{1.2}$$

Let $n = (n_1, n_2)$ be the outward normal direction of $\partial\Omega$. Using Green's formula, we see, since $w = 0$ on $\partial\Omega$,

$$\begin{aligned} \int_\Omega qw dx &= \int_\Omega -\Delta\varphi w dx = \int_\Omega \Big(\frac{\partial\varphi}{\partial x_1}\frac{\partial w}{\partial x_1} + \frac{\partial\varphi}{\partial x_2}\frac{\partial w}{\partial x_2}\Big)dx - \int_{\partial\Omega} w\frac{\partial\varphi}{\partial n}ds \\ &= \int_\Omega \Big(u_1\frac{\partial\varphi}{\partial x_1} + u_2\frac{\partial\varphi}{\partial x_2}\Big)dx = \int_\Omega u\cdot\nabla\varphi dx. \end{aligned} \tag{1.3}$$

Substitute it into (1.1), and the potential energy becomes

$$P(u) = \frac{1}{2}\int_\Omega D\Big[(1-\nu)\sum_{i,j=1}^{2}\Big(\frac{\partial u_i}{\partial x_j}\Big)^2 + \nu\Big(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\Big)^2\Big]dx - \int_\Omega u\cdot\nabla\varphi dx. \tag{1.4}$$

The necessary and sufficient condition for the existence of a function w such that $\frac{\partial w}{\partial x_i} = u_i$, for a given function $u \in V = H_0^1(\Omega) \times H_0^1(\Omega)$, is that u has null rotation:

$$Bu = \operatorname{rot} u = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} = 0. \tag{1.5}$$

To minimize functional $P(u)$ on space V under constraint (1.5), we use the penalty method. Set $\lambda > 0$ large enough, and then minimize

$$\begin{aligned} P_\lambda(u) &= P(u) + \frac{\lambda}{2}\int D(Bu)^2 dx \\ &= \frac{1}{2}\int_\Omega D\Big[(1-\nu)\sum_{i,j=1}^{2}\Big(\frac{\partial u_i}{\partial x_j}\Big)^2 + \nu\Big(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\Big)^2 + \lambda\Big(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\Big)^2\Big]dx \\ &\quad - \int_\Omega u\cdot\nabla\varphi dx \end{aligned} \tag{1.6}$$

on space V. Suppose $u_\lambda = (u_{\lambda,1}, u_{\lambda,2}) \in V$ is the minimum point of $P_\lambda(u)$. Substitute u_λ for u on the right-hand side of the following equation, and solve the problem