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L^∞ ERROR ESTIMATES OF NONCONFORMING FINITE ELEMENTS FOR THE BIHARMONIC EQUATION[*1)]

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Abstract

The paper considers the L^∞ convergence for nonconforming finite elements, such as Morley element, Adini element and De Veubeke element, solving the boundary value problem of the biharmonic equation. The nearly optimal order L^∞ estimates are given.

§1. Introduction

The L^∞ convergence of finite element methods is an interested topic. For second order partial differential equations, there are many papers discussing this problem, but for the boundary value problem of the biharmonic equation, only Morley element was discussed in [6]. Alrough the result given in [6] does not hold, the way of the proof is meaningful. This paper will consider the nonconforming finite elements and give the nearly optimal L^∞ estimates for them.

Let Ω be a convex polygonal domain. The Dirichlet boundary value problem of the biharmonic equation is the following

$$\begin{cases} \triangle^2 u = f, & \text{in } \Omega, \\ u|_{\partial\Omega} = \dfrac{\partial u}{\partial N}\Big|_{\partial\Omega} = 0 \end{cases} \tag{1.1}$$

where $N = (N_x, N_y)$ is the unit normal of $\partial\Omega$.

For $p \in [1, \infty]$ and $m \geq 0$, let $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$ be the usual Sobolev spaces, and $\|\cdot\|_{m,p,\Omega}$ and $|\cdot|_{m,p,\Omega}$ be the Sobolev norm and semi-norm respectively. When $p = 2$, denote them by $H^m(\Omega)$, $H_0^m(\Omega)$, $\|\cdot\|_{m,\Omega}$ and $|\cdot|_{m,\Omega}$ respectively. Let $H^{-m}(\Omega)$ be the dual space of $H_0^m(\Omega)$ with norm $\|\cdot\|_{-m,\Omega}$.

It is known that $\forall f \in H^{-1}(\Omega)$, problem (1.1) has a unique solution $u \in H_0^2(\Omega) \cap H^3(\Omega)$, such that

$$\|u\|_{3,\Omega} \leq C\|f\|_{-1,\Omega}, \tag{1.2}$$

with C a positive constant.

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Define, $\forall u, v \in H^2(\Omega)$,

$$a(u,v) = \int_\Omega \left(\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y}\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial y^2} \right) dxdy. \tag{1.3}$$

Let $f \in L^2(\Omega)$. The variational form of problem (1.1) is to find $u \in H_0^2(\Omega)$, such that,

$$a(u,v) = (f,v), \qquad \forall v \in H_0^2(\Omega), \tag{1.4}$$

where $(\cdot,\cdot)$ is the L^2 product.

For $h \in (0, h_0)$ with $h_0 \in (0,1)$, let $\mathcal{T}_h$ be a subdivision of Ω by triangles or rectangles. Let h_T =diam$T$ and ρ_T the largest of the diameters of all circles contained in T. Assume that there exists a positive constant η, independent of h, such that $\eta h < \rho_T < h_T \leq h$ for all $T \in \mathcal{T}_h$. Let $V_h \subset L^2(\Omega)$ be a finite element space associated with $\mathcal{T}_h$.

Define $a_h(\cdot,\cdot)$, for $v, w \in H^2(\Omega) + V_h$, as follows;

$$a_h(v,w) = \sum_{T\in\mathcal{T}_h} \int_T \left(\frac{\partial^2 v}{\partial x^2}\frac{\partial^2 w}{\partial x^2} + 2\frac{\partial^2 v}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2}\frac{\partial^2 w}{\partial y^2} \right) dxdy. \tag{1.5}$$

The finite element approximation to problem (1.1) is to find $u_h \in V_h$, such that

$$a_h(u_h, v) = (f,v), \qquad \forall v \in V_h. \tag{1.6}$$

For $w \in L^2(\Omega)$ and $w|_T \in H^m(T)$ for all $T \in \mathcal{T}_h$, define

$$|w|_{m,h} = \left(\sum_{T\in\mathcal{T}_h} |w|_{m,T}^2 \right)^{1/2}. \tag{1.7}$$

For $w \in L^\infty(\Omega)$ and $w|_T \in W^{m,\infty}(T)$ for all $T \in \mathcal{T}_h$, define

$$|w|_{m,\infty,h} = \max_{T\in\mathcal{T}_h} |w|_{m,\infty,T}\,. \tag{1.8}$$

This paper will show that the estimate of $|u-u_h|_{1,\infty,h}$ is $\mathcal{O}(h^2|\ln h|^{5/4})$ for Morley element, Adini element and De Veubeke element.

The remainder of the paper is arranged as follows. Section 2 will give the L^∞ estimates for Morley element and its properties. Section 3 will give the proof of the L^∞ estimate for Morley element. The last two sections will consider the case of Adini element and De Veubeke element.

§2. Morley Element

From now on, let $\mathcal{T}_h$ be a subdivision of Ω by triangles and $V_h \subset L^2(\Omega)$ be a Morley finite element space associated with $\mathcal{T}_h$. Then $v \in V_h$ if and only if it has the following properties:

1) $v|_T$ is quadratic for all $T \in \mathcal{T}_h$.
2) v is continuous at the vertices and vanishes at the vertices along $\partial\Omega$.