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ON THE CONVERGENCE OF WEISSMAN-TAYLOR ELEMENT FOR REISSNER-MINDLIN PLATE

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Abstract. In this paper, we study the Weissman-Taylor rectangular element for the Reissner-Mindlin plate [12] model and provide a convergence analysis for the transverse displacement and the rotation. We show that the element is stable and locking free, thereby improve the results of [8] and [9].

Key Words. Reissner-Mindlin plate, Locking-free, Weissman-Taylor element

1. Introduction

The Reissner-Mindlin plate model is widely used by engineers. A direct finite element approximation often yields poor results due to the shear locking, namely, the numerical solution is significantly smaller than the exact one. The development of general procedures to overcome this drawback is an active research area. Many methods have been proposed so far. However, a rigorous convergence and stability proof is missing for most of these methods, even if numerical tests show that they work properly. This is the case for the rectangular element proposed by Weissman and Taylor[12]. The element was analyzed in [8] and [9]. Nevertheless, whether the element is locking free is unclear in the previous analysis.

In this paper, we show that the transverse displacement and the rotation are convergent uniformly with respect to the plate thickness for the rectangular Weissman-Taylor element. Therefore the element is locking-free. For simplicity, we consider only a square mesh. However, the analysis is valid for a rectangular mesh as well.

The paper is organized as follows. The Reissner-Mindlin plate model is reviewed in Section 2; the Weissman-Taylor element is introduced in Section 3; the error analysis is presented in Section 4; and finally, a conclusion is given in Section 5.

Throughout the paper, C denotes a genetic constant, which is not necessarily the same at different places. However, C is independent of the mesh size h and the plate thickness t. We shall use standard notations of the Sobolev space.

2. Reissner-Mindlin Plate Model

Let Ω be a rectangle representing the mid-surface of the plate. Assume that the plate is clamped along the boundary $\partial\Omega$. Let ω and $\boldsymbol{\phi}$ denote the transverse displacement and the rotation, respectively, which are determined by the following

Problem 2.1. *Find* $(\boldsymbol{\phi}, \omega) \in \boldsymbol{H}_0^1(\Omega) \times H_0^1(\Omega)$*, such that*

$$(1) \qquad a(\boldsymbol{\phi}, \boldsymbol{\psi}) + \lambda t^{-2}(\nabla\omega - \boldsymbol{\phi}, \nabla v - \boldsymbol{\psi}) = (g, v), \quad \forall(\boldsymbol{\psi}, v) \in \boldsymbol{H}_0^1(\Omega) \times H_0^1(\Omega).$$

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Here g is the scaled transverse loading, t is the plate thickness, $\lambda = E\kappa/(2+2\nu)$ is the shear modulus, E is the Young's modulus, ν is the Poisson ratio and κ is the shear correction factor. The bilinear form a is defined by $a(\boldsymbol{\eta}, \boldsymbol{\psi}) = (\mathcal{C}\mathcal{E}\boldsymbol{\eta}, \mathcal{E}\boldsymbol{\psi})$, here $\mathcal{C}\tau$ is defined for any 2×2 symmetric matrix τ as

$$\mathcal{C}\tau := \frac{E}{12(1-\nu^2)}\left[(1-\nu)\tau + \nu\,\mathrm{tr}(\tau)\boldsymbol{I}\right].$$

Introducing the shear strain

$$\boldsymbol{\gamma} := \lambda t^{-2}(\nabla\omega - \boldsymbol{\phi})$$

as an independent variable, we get the following mixed problem

Problem 2.2. *Find* $(\boldsymbol{\phi}, \omega, \boldsymbol{\gamma}) \in \boldsymbol{H}_0^1(\Omega) \times H_0^1(\Omega) \times \boldsymbol{L}^2(\Omega)$, *such that*

$$a(\boldsymbol{\phi}, \boldsymbol{\psi}) + (\boldsymbol{\gamma}, \nabla v - \boldsymbol{\psi}) = (g, v), \qquad \forall(\boldsymbol{\psi}, v) \in \boldsymbol{H}_0^1(\Omega) \times H_0^1(\Omega), \tag{2}$$

$$\lambda^{-1}t^2(\boldsymbol{\gamma}, \boldsymbol{s}) - (\nabla\omega - \boldsymbol{\phi}, \boldsymbol{s}) = 0, \qquad \forall \boldsymbol{s} \in \boldsymbol{L}^2(\Omega). \tag{3}$$

The existence and uniqueness of the solution of Problem 2.2 and the following regularity result can be found in [4, 9].

Lemma 2.3. *Let* $(\boldsymbol{\phi}, \omega, \boldsymbol{\gamma}) \in \boldsymbol{H}_0^1(\Omega) \times H_0^1(\Omega) \times \boldsymbol{L}^2(\Omega)$ *be the solution of Problem 2.2, then the following regularity estimates hold*

$$\|\boldsymbol{\phi}\|_2 + \|\boldsymbol{\gamma}\|_0 \le C\|g\|_{-1}, \tag{4}$$

$$\|\omega\|_2 \le C(\|g\|_{-1} + t^2\|g\|_0), \qquad t\|\boldsymbol{\phi}\|_3 \le C\|g\|_0, \tag{5}$$

$$\|\boldsymbol{\gamma}\|_{\boldsymbol{H}(\mathrm{div})} \le C\|g\|_0, \qquad t\|\boldsymbol{\gamma}\|_1 \le C(\|g\|_{-1} + t\|g\|_0). \tag{6}$$

3. Finite element approximation

Let $\mathcal{T}_h$ be a uniform square partition of the domain Ω with the mesh size h, which is the refinement of a coarser partition $\mathcal{T}_{2h}$ with the mesh size $2h$. Let F_K be the affine mapping from the reference square $\hat{K} = [-1, 1]^2$ onto the element K, which is defined by

$$F_K(\xi, \eta) = (x_k + h\xi, y_k + h\eta),$$

where (x_k, y_k) is the center of K. Denote $\hat{v}(\xi, \eta) = v(x_k + h\xi, y_k + h\eta)$.

Define

$$\begin{aligned}
W_h &= \{v \in H_0^1(\Omega) \mid \hat{v}_{|\hat{K}} \in Q_1(\hat{K})\ \forall K \in \mathcal{T}_h\},\\
B_{Nc} &= \{v \in L^2(\Omega) \mid \hat{v}_{|\hat{K}} \in (1-\xi^2, 1-\eta^2)\ \forall K \in \mathcal{T}_h\},\\
\widehat{\boldsymbol{\Gamma}}_h &= \{\boldsymbol{\gamma} \in \boldsymbol{L}^2(\Omega) \mid \boldsymbol{\gamma}_{|K} \in P_1(K)^2\ \forall K \in \mathcal{T}_h\},\\
\boldsymbol{\Gamma}_h^R &= \{\boldsymbol{\chi} \in \boldsymbol{L}^2(\Omega) \mid \hat{\boldsymbol{\chi}}_{|\hat{K}} \in Q_{0,1} \times Q_{1,0}\ \forall K \in \mathcal{T}_h\},\\
\boldsymbol{\Gamma}_h &= \{\boldsymbol{\chi} \in \boldsymbol{H}_0(\mathrm{rot}, \Omega) \mid \hat{\boldsymbol{\chi}}_{|\hat{K}} \in Q_{0,1} \times Q_{1,0}\ \forall K \in \mathcal{T}_h\},
\end{aligned}$$

where $(1-\xi^2, 1-\eta^2)$ is the non-conforming bubble space generated by $1-\xi^2$ and $1-\eta^2$, $Q_{0,1} = (1, \eta), Q_{1,0} = (1, \xi)$.

Set

$$W_h^* = W_h \oplus B_{Nc}, \boldsymbol{V}_h^* = [W_h]^2 \oplus B_{Nc}^2, \boldsymbol{V}_h = [W_h]^2.$$