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On $\mathfrak{F}_\tau$-s-supplemented Subgroups of Finite Groups

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Abstract. Let $\mathfrak{F}$ be a non-empty formation of groups, τ a subgroup functor and H a p-subgroup of a finite group G. Let $\bar{G}=G/H_G$ and $\bar{H}=H/H_G$. We say that H is $\mathfrak{F}_\tau$-s-supplemented in G if for some subgroup $\bar{T}$ and some τ-subgroup $\bar{S}$ of $\bar{G}$ contained in $\bar{H}$, $\bar{H}\bar{T}$ is subnormal in $\bar{G}$ and $\bar{H}\cap\bar{T}\leq\bar{S}Z_{\mathfrak{F}}(\bar{G})$. In this paper, we investigate the influence of $\mathfrak{F}_\tau$-s-supplemented subgroups on the structure of finite groups. Some new characterizations about solubility of finite groups are obtained.

AMS subject classifications: 20D10, 20D15, 20D20

Key words: Subnormal subgroup, subgroup functor, soluble group.

1 Introduction

Throughout this paper, all groups considered are finite and G always denotes a group, π denotes a set of primes and p denotes a prime. Let $|G|_p$ denote the order of Sylow p-subgroups of G. All unexplained notation and terminology are standard, as in [1] and [2].

For a class of groups $\mathfrak{F}$, a chief factor L/K of G is said to be $\mathfrak{F}$-central in G if $L/K\rtimes G/C_G(L/K)\in\mathfrak{F}$. A normal subgroup N of G is called $\mathfrak{F}$-hypercentral in G if either $N=1$ or every chief factor of G below N is $\mathfrak{F}$-central in G. Let $Z_{\mathfrak{F}}(G)$ denote the $\mathfrak{F}$-hypercentre of G, that is, the product of all $\mathfrak{F}$-hypercentral normal subgroups of G. We use $\mathfrak{N}_p$ and $\mathfrak{S}$ to denote the classes of all p-nilpotent groups and soluble groups, respectively. It is well known that $\mathfrak{N}_p$ and $\mathfrak{S}$ are all S-closed saturated formations. Following Guo [3], a subgroup functor is a function τ which assigns to each group G a set of subgroups $\tau(G)$ of G satisfying that $1\in\tau(G)$ and $\theta(\tau(G))=\tau(\theta(G))$ for any isomorphism θ: $G\to G^*$. If $H\in\tau(G)$, then H is called a τ-subgroup of G. If τ is a subgroup functor, then τ is said to be

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(1) inductive if for any group G, whenever $H\in\tau(G)$ is a p-group and $N\trianglelefteq G$, then $HN/N\in\tau(G/N)$.

(2) hereditary if for group G, whenever $H\in\tau(G)$ is a p-group and $H\le E\le G$, then $H\in\tau(E)$.

(3) Φ-regular if any primitive group G, whenever $H\in\tau(G)$ is a p-group and N is a minimal normal subgroup of G, then $|G:N_G(H\cap N)|$ is a power of p.

Recall that a subgroup H of G is said to complemented in G if G has a subgroup K such that $G=HK$ and $H\cap K=1$. A subgroup H of G is said to be supplement in G if there exists a subgroup K such that $G=HK$. A subgroup H of G is said to be c-supplemented in G [4] if there exists a normal subgroup N of G such that $G=HN$ and $H\cap N\le H_G$, where H_G is the largest normal subgroup of G contained in H. For a formation $\mathfrak{F}$, a subgroup H of G is said to be $\mathfrak{F}$-supplement in G [5] if there exists a subgroup K of G such that $G=HK$ and $(H\cap K)H_G/H_G\le Z_{\mathfrak{F}}(G/H_G)$, where $Z_{\mathfrak{F}}(G/H_G)$ is the $\mathfrak{F}$-hypercenter of G/H_G. By using the above supplement subgroups, people have obtain many interesting results (see, for example, [4], [5] and [6]). As a continuation of the above researches, by using Guo-Skiba's method (see [7]), we now introduce the following notion:

Definition 1.1. Let $\mathfrak{F}$ be a non-empty formation of groups, τ a subgroup functor and H a p-subgroup of a finite group G. Let $\bar{G}=G/H_G$ and $\bar{H}=H/H_G$. We say that H is $\mathfrak{F}_\tau$-s-supplemented in G if for some subgroup $\bar{T}$ and some τ-subgroup $\bar{S}$ of $\bar{G}$ contained in $\bar{H}$, $\bar{H}\bar{T}$ is subnormal in $\bar{G}$ and $\bar{H}\cap\bar{T}\le\bar{S}Z_{\mathfrak{F}}(\bar{G})$.

It is clear that c-supplemented subgroups and $\mathfrak{F}$-supplement subgroups are all $\mathfrak{F}_\tau$-s-supplemented subgroups. But the following example shows that the converse is not true.

Example 1.1. Let $G=A\rtimes B$, where A is a cyclic group of order 5 and $B=\langle\alpha\rangle\in Aut(A)$ with $|\alpha|=4$. Put $H=\langle\alpha^2\rangle$. Since $|G:HA|=2$, HA is normal in G. It is easy to see that $H_G=Z_\infty(G)=1$. If $H_{sG}\ne 1$, then by [8, Lemma A], $O^2(G)\le N_G(H_{sG})$ and so $H_{sG}\trianglelefteq G$, which is impossible. Hence $H_{sG}=1$. Let $\tau(G)$ be the set of all S-qusinormal subgroups of G. If $S\le H$ and $S\in\tau(G)$, then $S\le H_{sG}=1$. Hence H is $\mathfrak{F}_\tau$-s-supplemented in G. But H is not $\mathfrak{F}$-supplement in G. Assume that H is $\mathfrak{F}$-supplement in G. Then G has a subgroup K such that $G=HK$ and $H\cap K=1$. It implies that H is complemented in G, and so H is complemented in B. This contradicts that B is cyclic. Therefore, H is not $\mathfrak{F}$-supplement in G. Clearly, $O_2(G)=1$, so H is not c-supplement in G.

In this paper, we investigate the influence of the $\mathfrak{F}_\tau$-s-supplemented subgroups on the structure of finite groups. Some new results of soluble groups are obtained.

2 Preliminaries

Lemma 2.1. [9, Lemma 2.5] *Let U be a subnormal subgroup of G.*