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On L^2-Stability Analysis of Time-Domain Acoustic Scattering Problems with Exact Nonreflecting Boundary Conditions

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Abstract. This paper is devoted to stability analysis of the acoustic wave equation exterior to a bounded scatterer, where the unbounded computational domain is truncated by the exact time-domain circular/spherical nonreflecting boundary condition (NRBC). Different from the usual energy method, we adopt an argument that leads to L^2-*a priori* estimates with minimum regularity requirement for the initial data and source term. This needs some delicate analysis of the involved NRBC. These results play an essential role in the error analysis of the interior solvers (e.g., finite-element /spectral-element/spectral methods) for the reduced scattering problems. We also apply the technique to analyze a time-domain waveguide problem.

AMS subject classifications: 65R10, 65N35, 65E05, 65M70

Key words: Wave equation, time-domain scattering problems, exact nonreflecting boundary conditions, stability analysis, a priori estimates.

1 Introduction

In this paper, we consider the time-domain acoustic scattering problem:

$$\partial_t^2 U = c^2 \Delta U + F, \quad \text{in } \Omega_\infty := \mathbb{R}^d \setminus \bar{D},\ t>0,\ d=2,3; \tag{1.1}$$

$$U = U_0, \quad \partial_t U = U_1, \quad \text{in } \Omega_\infty,\ t=0; \tag{1.2}$$

$$U=0, \quad \text{on } \Gamma_D,\ t>0; \quad \partial_t U + c\partial_{\boldsymbol{n}} U = o(|\boldsymbol{x}|^{(1-d)/2}),\ |\boldsymbol{x}| \to \infty,\ t>0, \tag{1.3}$$

where D is a bounded obstacle (scatterer) with Lipschitz boundary Γ_D, $c>0$ is the wave speed, and $\boldsymbol{n} = \boldsymbol{x}/|\boldsymbol{x}|$. Assume that the data F, U_0 and U_1 are compactly supported in a 2D disk or a 3D ball B of radius b, which contains the obstacle D.

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The acoustic wave propagates in the free space exterior to D, so the first important issue is to reduce the unbounded domain to a bounded domain. One efficient way is to set up an artificial boundary and impose a transparent/non-reflecting boundary condition (NRBC) thereon (see e.g., [8]). It is advantageous to use the exact NRBC, as it can be placed as close as possible to the scatterer, and the reduced problem, so as the discretized problem, can be best mimic to the continuous problem. Though such a NRBC is global in time and space in nature, fast and accurate numerical and/or semi-numerical means were developed for its evaluation and/or seamless integration with some solver in the reduced domain (see e.g., [3,14,15]).

This paper is largely concerned with the analysis of the reduced scattering problem by the exact circular/spherical NRBC. We remark that in [5,15], the usual energy method (i.e., testing the equation with $\partial_t U$) was used to obtain H^1-type estimates under strong regularity assumptions for the initial data and source term. Moreover, this approach did not lead to optimal L^2-estimates. In this paper, we resort to an argument in [4,7], which, together with a delicate analysis of the involved NRBC, leads to $L^\infty(L^2)$- and $L^2(L^2)$-*a priori* estimates for the reduced problem with a minimum regularity requirement for the initial data and source term. With this at our disposal, we can also analyze a waveguide problem considered in [18].

The paper is organized as follows. We present the reduced problem and carry out the *a priori* estimates in the forthcoming section. In the last section, we apply the argument to analyze a waveguide problem.

2 $L^\infty(L^2)$- and $L^2(L^2)$-a priori estimates

2.1 The reduced problem

We first reduce the scattering problem (1.1)-(1.3) to a bounded domain via the exact circular/spherical NRBC (see e.g., [3,8,15]), leading to

$$\partial_t^2 U = c^2 \Delta U + F, \qquad \text{in } \Omega := B\setminus\bar{D},\ t>0,\ d=2,3; \tag{2.1}$$

$$U=U_0, \quad \partial_t U = U_1, \qquad \text{in } \Omega,\ t=0; \qquad U=0, \quad \text{on } \Gamma_D,\ t>0; \tag{2.2}$$

$$\partial_r U - \mathcal{T}_d(U)=0, \qquad \text{at } r=b,\ t>0, \tag{2.3}$$

where the time-domain DtN boundary condition at the artificial boundary $\Gamma_b := \partial B$, is given, in polar/spherical coordinates, by

$$\mathcal{T}_d(U)=\begin{cases} \Big(-\dfrac{1}{c}\dfrac{\partial U}{\partial t}-\dfrac{U}{2r}\Big)\Big|_{r=b}+\displaystyle\sum_{|n|=0}^{\infty}\sigma_n(t)*\widehat{U}_n(b,t)e^{\mathrm{i}n\phi}, & d=2,\\ \Big(-\dfrac{1}{c}\dfrac{\partial U}{\partial t}-\dfrac{U}{r}\Big)\Big|_{r=b}+\displaystyle\sum_{n=0}^{\infty}\sum_{|m|=0}^{n}\sigma_{n+1/2}(t)*\widehat{U}_{nm}(b,t)Y_n^m(\theta,\phi), & d=3. \end{cases} \tag{2.4}$$