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A Survey of High Order Schemes for the Shallow Water Equations

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Abstract. In this paper, we survey our recent work on designing high order positivity-preserving well-balanced finite difference and finite volume WENO (weighted essentially non-oscillatory) schemes, and discontinuous Galerkin finite element schemes for solving the shallow water equations with a non-flat bottom topography. These schemes are genuinely high order accurate in smooth regions for general solutions, are essentially non-oscillatory for general solutions with discontinuities, and at the same time they preserve exactly the water at rest or the more general moving water steady state solutions. A simple positivity-preserving limiter, valid under suitable CFL condition, has been introduced in one dimension and reformulated to two dimensions with triangular meshes, and we prove that the resulting schemes guarantee the positivity of the water depth.

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1 Overview

Free surface flows have wide applications in ocean, environmental, hydraulic engineering and atmospheric modeling, with examples including the dam break and flooding problem, tidal flows in coastal water region, nearshore wave propagation with complex bathymetry structure, Tsunami wave propagation and ocean model. Three-dimensional Navier-Stokes equations can be used to simulate such flows directly. However, in the

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case where the horizontal length scale is much greater than the vertical length scale, one can average over the depth to eliminate the vertical direction and reduce the model into two-dimensional nonlinear shallow water equations (SWEs). SWEs play a critical role in the modeling and simulation of free surface flows in rivers and coastal areas, and can predict tides, storm surge levels and coastline changes from hurricanes and ocean currents. SWEs also arise in atmospheric flows, debris flows, and certain hydraulic structures like open channels and sedimentation tanks. SWEs take the form of non-homogeneous hyperbolic conservation laws with source terms modeling the effects of bathymetry and viscous friction on the bottom. In one space dimension, SWEs are defined as follows

$$\begin{cases} h_t+(hu)_x=0, \\ (hu)_t+\left(hu^2+\frac{1}{2}gh^2\right)_x=-ghb_x, \end{cases} \tag{1.1}$$

where h denotes the water height, u is the velocity of the fluid, b represents the bottom topography and g is the gravitational constant. In this paper, we will consider the variation of the bottom as the only source term, but other terms, such as a friction term or variations of the channel width, could also be added.

Due to the large scientific and engineering applications of the SWEs, research on effective and accurate numerical methods for their solutions has attracted great attention in the past two decades. Two types of difficulties are often encountered at the simulation of the SWEs, coming from the preservation of steady state solutions and the preservation of water height positivity. The first difficulty is related to the treatment of the source terms. An essential part for the SWEs and other conservation laws with source terms is that they often admit steady-state solutions in which the flux gradients are exactly balanced by the source terms. SWEs admit the general moving water equilibrium, given by

$$m:=hu=const \qquad \text{and} \qquad E:=\frac{1}{2}u^2+g(h+b)=const, \tag{1.2}$$

where m, E are the moving water equilibrium variables. People are often interested in the still water steady-state solution, which represents a still flat water surface, and referred as "lake at rest" solution:

$$u=v=0 \qquad \text{and} \qquad h+b=const. \tag{1.3}$$

Still water steady state (1.3) is simply a special case of the moving water steady state (1.2), when the velocity reduces to zero. Traditional numerical schemes with a straightforward handling of the source term cannot balance the effect of the source term and the flux, and usually fail to capture the steady state well. They will introduce spurious oscillations near the steady state. The well-balanced schemes are specially designed to preserve exactly these steady-state solutions up to machine error with relatively coarse meshes, and therefore it is desirable to design numerical methods which have the well-balanced property. The other major difficulty often encountered in the simulations of the