J. Math. Study
doi: 10.4208/jms.v47n4.14.02

Vol. **47**, No. 4, pp. 379-387
December 2014

# Infinitely Many Clark Type Solutions to a $p(x)$-Laplace Equation*

Zheng Zhou[1,*] and Xin Si[2]

[1] *School of Applied Mathematical, Xiamen University of Technology, Xiamen, Fujian 361024, P.R.China.*

Received 19 August 2014; Accepted 29 November 2014

**Abstract.** In this paper, the following $p(x)$-Laplacian equation:

$$\Delta_{p(x)}u+V(x)|u|^{p(x)-2}u=Q(x)f(x,u), \quad x\in\mathbb{R}^N,$$

is studied. By applying an extension of Clark's theorem, the existence of infinitely many solutions as well as the structure of the set of critical points near the origin are obtained.

**AMS subject classifications**: 35A15, 35A05, 35J70

**Chinese Library Classifications**: O175.29, O176.3

**Key words**: Clark theorem, infinitely many solutions, p(x)-Laplace, variational methods.

## 1 Introduction

The Clark theorem [2] is an important tool in critical point theory, which is constantly and effectively applied to sublinear differential equations with symmetry. A variant of the Clark Theorem was given by Heinz in [8].

**Theorem 1.1.** *Let $X$ be a Banach space, $\Phi\in C^1(X,\mathbb{R})$. Assume that $\Phi$ satisfies the (PS) condition, is even and bounded from below, and $\Phi(0)=0$. If for any $k\in\mathbb{N}$, there exists a $k$-dimensional subsequence $X^k$ of $X$ and $\rho_k>0$ such that $\sup_{X^k\cap S_{\rho_k}}\Phi<0$, where $S_\rho=\{u\in X|\,\|u\|=\rho\}$, then $\Phi$ has a sequence of critical values $c_k<0$ satisfying $c_k\to 0$ as $k\to\infty$.*

Theorem 1.1 asserts the existence of a sequence of critical values $c_k<0$ satisfying $c_k\to 0$ as $k\to\infty$, without giving any information on the structure of the set of critical points. A

*Supported by National Natural Science Foundation of China( No.11326150) and Foundation of Fujian Educational Committee(JA11240)

*Corresponding author. *Email addresses:* zhouzhengslx@163.com(Z. Zhou), xsi@xmut.edu.cn(X. Si)

 

very interesting question arising from Theorem 1.1 is whether there are a sequence of critical points $u_k$ such that $\Phi(u_k)\to 0$ and $\|u_k\|\to 0$ as $k\to\infty$ under the assumptions of Theorem 1.1. In [11], the authors answered this question and gave the structure of the set of critical points near the original in the abstract setting of Clark's theorem. One of the results is the following.

**Theorem 1.2.** *Let $X$ be a Banach space, $\Phi\in C^1(X,\mathbb{R})$. Assume that $\Phi$ satisfies the (PS) condition, is even and bounded from below, and $\Phi(0)=0$. If for any $k\in\mathbb{N}$, there exists a $k$-dimensional subsequence $X^k$ of $X$ and $\rho_k>0$ such that $\sup_{X^k\cap S_{\rho_k}}\Phi<0$, where $S_\rho=\{u\in X|\,\|u\|=\rho\}$, then at least one of the following conclusions holds.*

*(i) There exist a sequence of critical points $u_k$ satisfying $\Phi(u_k)<0$ for all $k$ and $\|u_k\|\to 0$ as $k\to\infty$.*

*(ii) There exists $r>0$ such that for any $0<a<r$, there exists a critical point $u$ such that $\|u\|=a$ and $\Phi(u)=0$.*

In [11], the authors got some variants of Clark Theorem which were applied to indefinite problems such as problems on periodic solutions of first order Hamiltonian systems. And the Theorem 1.2 is applied to a $p$-Laplace equation on $\mathbb{R}^N$. i.e.,

$$\begin{cases} -\Delta_p u+V(x)|u|^{p-2}u=Q(x)f(x,u), & x\in\mathbb{R}^N,\\ u\in W^{1,p}(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $p>1$. Assuming

(a1) there exists $\delta>0$, $1\le\gamma<p$, $C>0$ such that $f\in C(\mathbb{R}^N\times[-\delta,\delta],\mathbb{R})$, $f$ is odd in $u$, $|f(x,u)|\le C|u|^{\gamma-1}$, and $\lim_{u\to 0}F(x,u)/|u|^p=+\infty$ uniformly in some ball $B_r(x_0)\subset\mathbb{R}^N$;

(a2) $V,Q\in C(\mathbb{R}^N,\mathbb{R}^1)$, $V(x)\ge\alpha_0$ and $0<Q(x)\le\beta_0$ for some $\alpha_0>0$, $\beta_0>0$, and $M\triangleq Q^{\frac{p}{p-\gamma}}V^{\frac{-\gamma}{p-\gamma}}\in L^1(\mathbb{R}^N)$.

With conditions (a1) and (a2), equation (1.1) has infinitely many solutions $u_k$ such that $\|u_k\|_{L^\infty}\to 0$ as $k\to\infty$.

In this paper, the following $p(x)$-Laplacian equation $(p(x)>1)$ is considered:

$$\begin{cases} -\Delta_{p(x)}u+V(x)|u|^{p(x)-2}u=Q(x)f(x,u), & x\in\mathbb{R}^N,\\ u\in W^{1,p(x)}(\mathbb{R}^N), \end{cases} \tag{1.2}$$

where $\Delta_{p(x)}u=\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u), p(x)\in C(\mathbb{R}^N)$.

When $p(x)\equiv const.$, problem (1.2) is the equation (1.1), which was studied in [11]. The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is a natural generalization of the classical Sobolev space $W^{1,p}(\Omega)$. The variable exponent $p(x)$-Laplacian equations arise from nonlinear elastic mechanics (see [15]) and electrorheological fluids(see [1,12]). And the