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An Existence Result for a Mathematical Model of Koiter's Type

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Abstract. In this paper we first introduce a new shell model that can be applied for all kinds of geometries of the middle surface of the shell. Then we show that our model is close to Koiter's nonlinear shell model in a specific sense. Finally, we establish the existence of a minimizer for this new model.

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1 Introduction

A nonlinearly elastic shell with constant thickness is a three-dimensional elastic body whose reference configuration consists of all points that lie within a small given distance from a given surface, which is called the "middle surface of the shell". The nonlinear Koiter's shell model, introduced by Koiter (see [9]) in 1966, is one of the most used two-dimensional nonlinearly elastic shell models in numerical simulations. It states that the unknown deformation $\boldsymbol{\varphi}:\omega\to\mathbb{R}^3$ of the middle surface $S=\boldsymbol{\theta}(\overline{\omega})$ of the shell subjected to applied forces should minimize a functional

$$J_K(\boldsymbol{\varphi}):=\int_\omega W_K(\boldsymbol{\varphi})\sqrt{a}dy-L_K(\boldsymbol{\varphi}), \tag{1.1}$$

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called the total energy of the deformed shell, over an appropriate set of admissible deformations. Here W_K denotes Koiter's stored energy function (which will be defined later) and L_K denotes a linear form that takes into account the applied forces. However, as far as we know, no theorem has been established in the literature proving the existence of a such minimizer.

On the other hand, several existence theorems have been established for ad hoc approximations of Koiter's shell model, that is, for models whereby Koiter's stored energy function $W_K(\boldsymbol{\varphi})$ is in (1.1) replaced by

$$\widetilde{W}_K(\boldsymbol{\varphi}) := W_K(\boldsymbol{\varphi}) + \mathcal{R}(\varepsilon,\boldsymbol{\varphi}),$$

where the additional term $\mathcal{R}(\varepsilon,\boldsymbol{\varphi})$ is negligible compared with $W_K(\boldsymbol{\varphi})$ in some meaningful sense. Bunoiu *et al.* [3] and Ciarlet and Mardare [5] proposed a well-posed two-dimensional approximation of Koiter's model for spherical and "almost spherical" shells. Giang and Mardare [8] established existence theorems for nonlinear shell models asymptotically equivalent to Koiter's model for the shells whose middle surfaces are minimal surfaces. Finally, Anicic [1,2] proposed an approximate model of Koiter's model that has a minimizer over the set of deformations whose principal radii of curvatures are bounded below by the half thickness of the shell. A different approach by Ciarlet and Mardare [6] and Mardare [10], where the authors have proposed nonlinear shell models asymptotically equivalent to the Koiter's model for all kinds of geometries, but depending on the transverse variable, so that these are three-dimensional models.

The purpose of this paper is to define a well-posed two-dimensional shell model that is approximately equivalent to that of Koiter without any restrictions on the geometry of the middle surface of the shell. Our approach is similar to that of Anicic [1,2], the difference being that in our model the space of admissible deformations is independent of the thickness of the shell. The definition of our model is based on the ideas first appearing in the papers of Giang and Mardare [8] and Anicic [1,2].

2 Notations and definitions

In all that follows, Greek indices and exponents range in the set $\{1,2\}$ while Latin indices and exponents range in the set $\{1,2,3\}$ (except when they are used for indexing sequences). The Einstein summation convention with respect to repeated indices and exponents is used.

Vector and matrix fields are denoted by boldface letters. The Euclidean norm, the inner product and the vector product of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbb{R}^3$ are respectively denoted $|\boldsymbol{u}|, \boldsymbol{u}\cdot\boldsymbol{v}$ and $\boldsymbol{u}\wedge\boldsymbol{v}$. Given any integers $m \geq 1$ and $n \geq 1$, the inner product and the Frobenius norm in $\mathbb{R}^{m\times n}$ are respectively denoted and defined by $\boldsymbol{A}:\boldsymbol{B}:=\mathrm{Tr}(\boldsymbol{A}^T\boldsymbol{B})$ and $|\boldsymbol{A}|:=(\boldsymbol{A}:\boldsymbol{A})^{1/2}$, where Tr denotes the trace operator of square matrices. The subspace of $\mathbb{R}^{n\times n}$ formed by all symmetric matrices is denoted $\mathbb{R}^{n\times n}_{\mathrm{sym}}$.

A domain in $\mathbb{R}^2$ is a bounded, connected, open subset $\omega\subset\mathbb{R}^2$ with a Lipschitz-continuous boundary $\gamma:=\partial\omega$, the set ω being locally on the same side of γ. A generic point in the set ω is denoted $y=(y_\alpha)$ and partial derivatives, in the classical or distributional sense, are denoted $\partial_\alpha:=\partial/\partial y_\alpha$ and $\partial_{\alpha\beta}:=\partial^2/(\partial y_\alpha\partial y_\beta)$.

Given any open subset ω of $\mathbb{R}^2$ and any real number $p\geq 1$, the notation $L^p(\omega;\mathbb{R}^{m\times n})$ denotes the space of matrix fields $\boldsymbol{A}=(A_{ij}):\omega\to\mathbb{R}^{m\times n}$ with components in the Lebesgue space $L^p(\omega)$. It is equipped with the norm

$$\|\boldsymbol{A}\|_p:=\left(\int_\omega|\boldsymbol{A}(y)|^p dy\right)^{1/p},\quad \forall\boldsymbol{A}\in L^p(\omega;\mathbb{R}^{m\times n}).$$

The notation $W^{1,p}(\omega;\mathbb{R}^3)$ denotes the space of vector fields $\boldsymbol{\zeta}=(\zeta_i):\omega\to\mathbb{R}^3$ with components in the Sobolev space $W^{1,p}(\omega)$. It is equipped with the norm

$$\|\boldsymbol{\zeta}\|_{1,p}:=\left(\|\boldsymbol{\zeta}\|_p^p+\|\nabla\boldsymbol{\zeta}\|_p^p\right)^{1/p},\quad \forall\boldsymbol{\zeta}\in W^{1,p}(\omega;\mathbb{R}^3),$$

where $\nabla\boldsymbol{\zeta}:=(\partial_\beta\zeta_i)$ is the matrix field with $\partial_\beta\zeta_i$ at its row i and column β.

Strong and weak convergences in any normed vector space are respectively denoted $\to$ and $\rightharpoonup$.

The middle surface of the reference configuration of a shell is defined by $S:=\boldsymbol{\theta}(\omega)$, where $\omega\subset\mathbb{R}^2$ is a domain and $\boldsymbol{\theta}\in\mathcal{C}^1(\overline{\omega};\mathbb{R}^3)$ is an immersion, i.e. the two tangent vector fields $\boldsymbol{a}_\alpha:=\partial_\alpha\boldsymbol{\theta}$ are linearly independent at every $y\in\overline{\omega}$. We assume in addition that the vector field $\boldsymbol{a}_3:\overline{\omega}\to\mathbb{R}^3$, defined by

$$\boldsymbol{a}_3(y):=\frac{\partial_1\boldsymbol{\theta}(y)\wedge\partial_2\boldsymbol{\theta}(y)}{|\partial_1\boldsymbol{\theta}(y)\wedge\partial_2\boldsymbol{\theta}(y)|},\quad \forall y\in\overline{\omega},$$

is also of class $\mathcal{C}^1$ over $\overline{\omega}$. Note that $\boldsymbol{a}_3(y)$ is a unit vector normal to the surface S at the point $\boldsymbol{\theta}(y)$. The area element on the surface S is $\sqrt{a(y)}dy$, where

$$a:=|\partial_1\boldsymbol{\theta}\wedge\partial_2\boldsymbol{\theta}|^2=\det(a_{\alpha\beta})\quad\text{in }\omega.$$

The covariant components $a_{\alpha\beta}\in\mathcal{C}^0(\overline{\omega}), b_{\alpha\beta}\in\mathcal{C}^0(\overline{\omega})$ and $c_{\alpha\beta}\in\mathcal{C}^0(\overline{\omega})$ of the first, second and third fundamental forms of $S=\boldsymbol{\theta}(\overline{\omega})$ are respectively defined by

$$a_{\alpha\beta}:=\boldsymbol{a}_\alpha\cdot\boldsymbol{a}_\beta,\quad b_{\alpha\beta}:=-\boldsymbol{a}_\beta\cdot\partial_\alpha\boldsymbol{a}_3,\quad c_{\alpha\beta}:=\partial_\alpha\boldsymbol{a}_3\cdot\partial_\beta\boldsymbol{a}_3.$$

The contravariant components of the first fundamental form are the components $a^{\alpha\beta}\in\mathcal{C}^0(\overline{\omega})$ of the inverse matrix

$$\left(a^{\alpha\beta}(y)\right):=\left(a_{\alpha\beta}(y)\right)^{-1},\quad y\in\overline{\omega}.$$

Note that both matrices $(a_{\alpha\beta}(y))$ and $(a^{\alpha\beta}(y))$ are symmetric and positive definite at every $y\in\overline{\omega}$, that the matrix $(c_{\alpha\beta}(y))\in\mathbb{R}^{2\times 2}$ is symmetric and nonnegative definite for all $y\in\overline{\omega}$, and that

$$c:=\det(c_{\alpha\beta})=|\partial_1\boldsymbol{a}_3\wedge\partial_2\boldsymbol{a}_3|^2\geq 0\quad\text{in }\overline{\omega}.$$

The mean curvature and the total curvature of the surface $S=\boldsymbol{\theta}(\overline{\omega})$ are respectively the functions denoted and defined by

$$H:=\frac{1}{2}a^{\alpha\beta}b_{\alpha\beta}=\frac{1}{2}(k_1+k_2)\in\mathcal{C}^0(\overline{\omega})$$

and

$$K:=\det\left(a^{\alpha\sigma}b_{\sigma\beta}\right)=k_1k_2\in\mathcal{C}^0(\overline{\omega}),$$

where $k_1(y)$ and $k_2(y)$ are the principal curvatures of the surface $S=\boldsymbol{\theta}(\overline{\omega})$ at the point $\boldsymbol{\theta}(y)$, defined as the eigenvalues of the 2×2 matrix $(a^{\alpha\sigma}(y)b_{\sigma\beta}(y))$.

We also define the mixed components $b^\alpha_\beta\in\mathcal{C}^0(\overline{\omega})$ of the second fundamental form by letting

$$b^\alpha_\beta=a^{\alpha\sigma}b_{\sigma\beta}\quad\text{in }\overline{\omega}.$$

A deformation of the middle surface of the shell $S=\boldsymbol{\theta}(\overline{\omega})$ is a smooth enough mapping $\boldsymbol{\varphi}:\omega\to\mathbb{R}^3$. Given an arbitrary deformation $\boldsymbol{\varphi}$, the functions

$$a_{\alpha\beta}(\boldsymbol{\varphi}):=\boldsymbol{a}_\alpha(\boldsymbol{\varphi})\cdot\boldsymbol{a}_\beta(\boldsymbol{\varphi}),\quad\text{where}\quad\boldsymbol{a}_\alpha(\boldsymbol{\varphi}):=\partial_\alpha\boldsymbol{\varphi},$$

denote the covariant components of the first fundamental form of the deformed surface $\boldsymbol{\varphi}(\omega)$, and the functions

$$G_{\alpha\beta}(\varphi):=\frac{1}{2}\left(a_{\alpha\beta}(\boldsymbol{\varphi})-a_{\alpha\beta}\right)$$

denote the covariant components of the change of metric tensor field associated with the deformation $\boldsymbol{\varphi}$ of S. The area element along the deformed surface $\boldsymbol{\varphi}(\omega)$ is $\sqrt{a(\boldsymbol{\varphi})}$, where

$$a(\boldsymbol{\varphi}):=|\partial_1\boldsymbol{\varphi}\wedge\partial_2\boldsymbol{\varphi}|^2=\det\left(a_{\alpha\beta}(\boldsymbol{\varphi})\right).$$

If the two nonlinear vectors $\boldsymbol{a}_\alpha(\boldsymbol{\varphi})$ are linearly independent, then the unit vector field

$$\boldsymbol{a}_3(\boldsymbol{\varphi}) := \frac{\partial_1\boldsymbol{\varphi}\wedge\partial_2\boldsymbol{\varphi}}{|\partial_1\boldsymbol{\varphi}\wedge\partial_2\boldsymbol{\varphi}|}$$

is well-defined and normal to the deformed surface $\boldsymbol{\varphi}(\omega)$. The functions

$$b_{\alpha\beta}(\boldsymbol{\varphi}) := -\partial_\alpha\boldsymbol{\varphi}\cdot\partial_\beta\boldsymbol{a}_3(\boldsymbol{\varphi})$$

denote the covariant components of the second fundamental form of the deformed surface $\boldsymbol{\varphi}(\omega)$, the functions

$$R_{\alpha\beta}(\boldsymbol{\varphi}) := b_{\alpha\beta}(\boldsymbol{\varphi}) - b_{\alpha\beta}$$

denote the covariant components of the change of curvature tensor field associated with the deformation $\boldsymbol{\varphi}$ of S, the functions

$$c_{\alpha\beta}(\boldsymbol{\varphi}) := \partial_\alpha\boldsymbol{a}_3(\boldsymbol{\varphi})\cdot\partial_\beta\boldsymbol{a}_3(\boldsymbol{\varphi})$$

denote the covariant components of the third fundamental form of the deformed surface $\boldsymbol{\varphi}(\omega)$, and the functions

$$P_{\alpha\beta}(\boldsymbol{\varphi}) := \frac{1}{2}\left(c_{\alpha\beta}(\boldsymbol{\varphi}) - c_{\alpha\beta}\right)$$

denote the covariant components of the change of the third fundamental form associated with the deformation $\boldsymbol{\varphi}$ of S.

The unknown deformation $\boldsymbol{\varphi}$ of the middle surface of the shell appearing in Koiter's nonlinear shell model is assumed to satisfy a boundary condition of the form

$$\boldsymbol{\varphi} = \boldsymbol{\theta}, \quad \boldsymbol{a}_3(\boldsymbol{\varphi}) = \boldsymbol{a}_3 \quad \text{on } \gamma_0,$$

where γ_0 is a non-empty relatively open subset of $\gamma := \partial\omega$. In addition, the unknown $\boldsymbol{\varphi}$ is subjected to the constraint

$$\partial_1\boldsymbol{\varphi}\wedge\partial_2\boldsymbol{\varphi} \neq 0 \quad \text{in } \omega,$$

so that the tangent plane is well-defined at each point of the deformed surface.

The nonlinear shell model of Koiter states that the unknown deformation $\boldsymbol{\psi}$ of the middle surface $S = \boldsymbol{\theta}(\omega)$ of the shell should be a minimizer over a set of smooth enough vector fields $\boldsymbol{\varphi} : \omega \to \mathbb{R}^3$ satisfying the boundary conditions

$$\boldsymbol{\varphi} = \boldsymbol{\theta}, \quad \boldsymbol{a}_3(\boldsymbol{\varphi}) = \boldsymbol{a}_3 \quad \text{on } \gamma_0$$

of the total energy of the deformed surface $\boldsymbol{\varphi}(\omega)$, denoted and defined by

$$J_K(\boldsymbol{\varphi}) := \int_\omega W_K(\boldsymbol{\varphi}) \sqrt{a}\, dy - L_K(\boldsymbol{\varphi}),$$

where

$$W_K(\boldsymbol{\varphi}) := \frac{\varepsilon}{2} a^{\alpha\beta\sigma\tau} G_{\sigma\tau}(\boldsymbol{\varphi}) G_{\alpha\beta}(\boldsymbol{\varphi}) + \frac{\varepsilon^3}{6} a^{\alpha\beta\sigma\tau} R_{\sigma\tau}(\boldsymbol{\varphi}) R_{\alpha\beta}(\boldsymbol{\varphi}),$$

and the functions

$$a^{\alpha\beta\sigma\tau} := \frac{4\lambda\mu}{\lambda+2\mu} a^{\alpha\beta} a^{\sigma\tau} + 2\mu(a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma})$$

are the contravariant components of the two-dimensional elasticity tensor of the shell, $\lambda \geq 0$ and $\mu > 0$ are the Lamé constants of the constitutive material, and L_K is a linear functional that takes into account the applied forces. Notice that there exists a constant $c_e = c_e(\omega, \boldsymbol{\theta}, \lambda, \mu) > 0$ such that

$$\sum_{\alpha,\beta} |t_{\alpha\beta}|^2 \leq c_e a^{\alpha\beta\sigma\tau}(y) t_{\sigma\tau} t_{\alpha\beta} \tag{2.1}$$

for all $y \in \overline{\omega}$ and all symmetric matrices $(t_{\alpha\beta})$ (see, for example, [4, Theorem 3.3-2]). We also denote by

$$W_M(\boldsymbol{\varphi}) := a^{\alpha\beta\sigma\tau} G_{\sigma\tau}(\boldsymbol{\varphi}) G_{\alpha\beta}(\boldsymbol{\varphi})$$

the membrane energy appearing in Koiter's model, by

$$W_F(\boldsymbol{\varphi}) := a^{\alpha\beta\sigma\tau} R_{\sigma\tau}(\boldsymbol{\varphi}) R_{\alpha\beta}(\boldsymbol{\varphi})$$

the flexural energy, and by

$$W_T(\boldsymbol{\varphi}) := a^{\alpha\beta\sigma\tau} P_{\sigma\tau}(\boldsymbol{\varphi}) P_{\alpha\beta}(\boldsymbol{\varphi})$$

a new energy measuring the change of the third fundamental form when the middle surface of the shell undergoes a deformation $\boldsymbol{\varphi}$.

3 A new stored energy function of Koiter's type

The purpose of this section is to define a new stored energy function that is asymptotically equivalent to Koiter's for "small enough" thickness and change of metric and curvature tensors. To begin with, let

$$r(y) := \frac{1}{8} \min\left\{2, \frac{1}{|k_1(y)| + |k_2(y)|}\right\}, \quad \forall y \in \overline{\omega}, \tag{3.1}$$

where $k_1(y)$ and $k_2(y)$ are the principal curvatures defined in Section 2. Since $H,K\in\mathcal{C}^0(\overline{\omega})$, $|k_1|,|k_2|\in\mathcal{C}^0(\overline{\omega})$ and thus they are bounded from above. Therefore, $r\in\mathcal{C}^0(\overline{\omega})$ and there exists a positive number r_0 such that

$$r(y)\geq r_0, \quad \forall y\in\overline{\omega}.$$

Next, we denote

$$\boldsymbol{g}_\alpha^+ := \partial_\alpha\boldsymbol{\theta} + r\partial_\alpha\boldsymbol{a}_3$$

the "upper" vector fields and

$$\boldsymbol{g}_\alpha^- := \partial_\alpha\boldsymbol{\theta} - r\partial_\alpha\boldsymbol{a}_3$$

the "lower" vector fields associated with the middle surface of the shell and with the function r. Then we define the "upper" and the "lower" covariant tensor fields $(g_{\alpha\beta}^+)$ and $(g_{\alpha\beta}^-)$ by letting

$$g_{\alpha\beta}^+ := \boldsymbol{g}_\alpha^+\cdot\boldsymbol{g}_\beta^+ = a_{\alpha\beta} - 2rb_{\alpha\beta} + r^2c_{\alpha\beta},$$

and

$$g_{\alpha\beta}^- := \boldsymbol{g}_\alpha^-\cdot\boldsymbol{g}_\beta^- = a_{\alpha\beta} + 2rb_{\alpha\beta} + r^2c_{\alpha\beta},$$

respectively. In the same way, we define the "deformed upper vector fields" and the "deformed lower vector fields" associated with the deformation $\boldsymbol{\varphi}:\omega\to\mathbb{R}^3$ by

$$\boldsymbol{g}_\alpha^+(\boldsymbol{\varphi}) := \partial_\alpha\boldsymbol{\varphi} + r\partial_\alpha\boldsymbol{a}_3(\boldsymbol{\varphi}),$$

and

$$\boldsymbol{g}_\alpha^-(\boldsymbol{\varphi}) := \partial_\alpha\boldsymbol{\varphi} - r\partial_\alpha\boldsymbol{a}_3(\boldsymbol{\varphi}),$$

respectively. The "deformed upper" and "lower" covariant tensor fields $g_{\alpha\beta}^+(\boldsymbol{\varphi})$ and $g_{\alpha\beta}^-(\boldsymbol{\varphi})$ are also defined by letting

$$g_{\alpha\beta}^+(\boldsymbol{\varphi}) := \boldsymbol{g}_\alpha^+(\boldsymbol{\varphi})\cdot\boldsymbol{g}_\beta^+(\boldsymbol{\varphi}) = a_{\alpha\beta}(\boldsymbol{\varphi}) - 2rb_{\alpha\beta}(\boldsymbol{\varphi}) + r^2c_{\alpha\beta}(\boldsymbol{\varphi}),$$

and

$$g_{\alpha\beta}^-(\boldsymbol{\varphi}) := \boldsymbol{g}_\alpha^-(\boldsymbol{\varphi})\cdot\boldsymbol{g}_\beta^-(\boldsymbol{\varphi}) = a_{\alpha\beta}(\boldsymbol{\varphi}) + 2rb_{\alpha\beta}(\boldsymbol{\varphi}) + r^2c_{\alpha\beta}(\boldsymbol{\varphi}).$$

The functions

$$G_{\alpha\beta}^+(\boldsymbol{\varphi}) := \frac{1}{2}\big(g_{\alpha\beta}^+(\boldsymbol{\varphi}) - g_{\alpha\beta}^+\big) = G_{\alpha\beta}(\boldsymbol{\varphi}) - rR_{\alpha\beta}(\boldsymbol{\varphi}) + r^2P_{\alpha\beta}(\boldsymbol{\varphi}),$$

and

$$G^{-}_{\alpha\beta}(\boldsymbol{\varphi}) := \frac{1}{2}\big(g^{-}_{\alpha\beta}(\boldsymbol{\varphi}) - g^{-}_{\alpha\beta}\big) = G_{\alpha\beta}(\boldsymbol{\varphi}) + rR_{\alpha\beta}(\boldsymbol{\varphi}) + r^2 P_{\alpha\beta}(\boldsymbol{\varphi}),$$

denote respectively the covariant components of the change of upper and lower tensor fields of the shell.

We also define the "upper" and "lower" contravariant tensor fields

$$\big(g^{\alpha\beta}_{+}\big) := \big(g^{+}_{\alpha\beta}\big)^{-1} \quad \text{and} \quad \big(g^{\alpha\beta}_{-}\big) := \big(g^{-}_{\alpha\beta}\big)^{-1},$$

and the functions

$$\begin{aligned} g^{+} &:= \det\big(g^{+}_{\alpha\beta}\big) = (1-2rH+r^2K)^2 a, \\ g^{-} &:= \det\big(g^{-}_{\alpha\beta}\big) = (1+2rH+r^2K)^2 a, \\ g^{+}(\boldsymbol{\varphi}) &:= \det\big(g^{+}_{\alpha\beta}(\boldsymbol{\varphi})\big), \\ g^{-}(\boldsymbol{\varphi}) &:= \det\big(g^{-}_{\alpha\beta}(\boldsymbol{\varphi})\big). \end{aligned}$$

From the definition of r, on can easily prove that the matrices $(g^{+}_{\alpha\beta}), (g^{-}_{\alpha\beta}), (g^{\alpha\beta}_{+})$ and $(g^{\alpha\beta}_{-})$ are positive definite at all points $y \in \overline{\omega}$.

Next, we define

$$\begin{aligned} \widetilde{W}_M(\boldsymbol{\varphi}) &:= 2\mu\big(a^{\alpha\beta} a_{\alpha\beta}(\boldsymbol{\varphi}) - 2\big) \\ &\quad + \frac{2\lambda\mu}{\lambda+2\mu}\left(\frac{a(\boldsymbol{\varphi})}{a} - 1\right) - 4\mu\frac{\lambda+\mu}{\lambda+2\mu}\log\left(\frac{a(\boldsymbol{\varphi})}{a}\right), \\ \widetilde{W}^{+}_T(\boldsymbol{\varphi}) &:= \frac{(\lambda+\mu)g^{+}}{r^2 a}\left[\big(g^{\alpha\beta}_{+} g^{+}_{\alpha\beta}(\boldsymbol{\varphi}) - 2\big) - \log\left(\frac{g^{+}(\boldsymbol{\varphi})}{g^{+}}\right)\right], \\ \widetilde{W}^{-}_T(\boldsymbol{\varphi}) &:= \frac{(\lambda+\mu)g^{-}}{r^2 a}\left[\big(g^{\alpha\beta}_{-} g^{-}_{\alpha\beta}(\boldsymbol{\varphi}) - 2\big) - \log\left(\frac{g^{-}(\boldsymbol{\varphi})}{g^{-}}\right)\right]. \end{aligned}$$

Now we are able to define a new stored energy function meant to replace in the functional (1.1) the usual one of Koiter.

Definition 3.1 (A New Stored Energy Function)**.** *Let W_M, W_F and W_T be the functions defined in Section* 2 *and $\widetilde{W}_M, \widetilde{W}^{+}_T$ and $\widetilde{W}^{-}_T$ are as above. Then, given any constants $0 < \delta < 1$ and $C_1 > 0$ and $C_2 > 0$, define the function $\widetilde{W}_K$ by*

$$\begin{aligned} \widetilde{W}_K(\boldsymbol{\varphi}) := &\frac{\varepsilon}{2}\big[\delta W_M(\boldsymbol{\varphi}) + (1-\delta)\widetilde{W}_M(\boldsymbol{\varphi})\big] + \frac{\varepsilon^3}{6} W_F(\boldsymbol{\varphi}) \\ &+ \frac{\varepsilon^5}{10}\big[C_1 W_T(\boldsymbol{\varphi}) + C_2(4H^2 - 2K)\big(\widetilde{W}^{+}_T(\boldsymbol{\varphi}) + \widetilde{W}^{-}_T(\boldsymbol{\varphi})\big)\big] \end{aligned}$$

for all immersions $\boldsymbol{\varphi} \in W^{1,4}(\omega;\mathbb{R}^3)$ such that $\boldsymbol{a}_3(\boldsymbol{\varphi}) \in W^{1,4}(\omega;\mathbb{R}^3)$.

The next theorem shows that $\widetilde{W}_K(\boldsymbol{\varphi})$ coincides with Koiter's stored energy function $W_K(\boldsymbol{\varphi})$ at the first order with respect to small thickness and change of metric and curvature tensors.

Theorem 3.1. *For all immersions $\boldsymbol{\varphi}\in C^1(\overline{\omega};\mathbb{R}^3)$ with $\boldsymbol{a}_3(\boldsymbol{\varphi})\in C^1(\overline{\omega};\mathbb{R}^3)$ that are sufficiently close in the $C^1(\overline{\omega})$-norm to the immersion $\boldsymbol{\theta}$, the following estimate hold:*

$$\widetilde{W}_K(\boldsymbol{\varphi})=W_K(\boldsymbol{\varphi})+o\big(W_K(\boldsymbol{\varphi})\big)+\varepsilon^2\mathcal{O}\big(W_K(\boldsymbol{\varphi})\big).$$

Proof. The result is a consequence of Lemmas 3.1-3.3 established below. □

Lemma 3.1. *The following relations hold in ω for every immersion $\boldsymbol{\varphi}\in C^1(\overline{\omega};\mathbb{R}^3)$ with $\boldsymbol{a}_3(\boldsymbol{\varphi})\in C^1(\overline{\omega};\mathbb{R}^3)$ that is sufficiently close in the $C^1(\overline{\omega})$-norm to the immersion $\boldsymbol{\theta}$:*

$$\begin{aligned}P_{\alpha\beta}(\boldsymbol{\varphi})=&-b_\alpha^\sigma b_\beta^\tau G_{\sigma\tau}(\boldsymbol{\varphi})+\frac{1}{2}b_\alpha^\sigma R_{\sigma\beta}(\boldsymbol{\varphi})+\frac{1}{2}b_\beta^\sigma R_{\alpha\sigma}(\boldsymbol{\varphi})\\&+\mathcal{O}\big(W_M(\boldsymbol{\varphi})\big)+\mathcal{O}\big(W_F(\boldsymbol{\varphi})\big),\\W_T(\boldsymbol{\varphi})=&\,\mathcal{O}\big(W_M(\boldsymbol{\varphi})\big)+\mathcal{O}\big(W_F(\boldsymbol{\varphi})\big).\end{aligned}$$

Proof. This lemma is a direct consequence of [3, Lemma 2] and for this reason its proof is omitted. □

Lemma 3.2. *Given any immersion $\boldsymbol{\varphi}\in C^1(\overline{\omega};\mathbb{R}^3)$ that is sufficiently close in the $C^1(\overline{\omega})$-norm to the immersion $\boldsymbol{\theta}$, the following relation holds in ω:*

$$\widetilde{W}_M(\boldsymbol{\varphi})=W_M(\boldsymbol{\varphi})+o\big(W_M(\boldsymbol{\varphi})\big).$$

Proof. The proof of this lemma which can be found in the proof of [3, Lemma 1], is recalled here for reader's convenience. The definition of the functions $G_{\alpha\beta}(\boldsymbol{\varphi})$ implies that

$$a_{\alpha\beta}(\boldsymbol{\varphi})=a_{\alpha\beta}+2G_{\alpha\beta}(\boldsymbol{\varphi}).$$

As a consequence, we have

$$\begin{aligned}a(\boldsymbol{\varphi}):=\det\big(a_{\alpha\beta}(\boldsymbol{\varphi})\big)&=\det\big(a_{\alpha\beta}+2G_{\alpha\beta}(\boldsymbol{\varphi})\big)\\&=a\big[1+2a^{\alpha\beta}G_{\alpha\beta}(\boldsymbol{\varphi})+4\det\big(a^{\alpha\sigma}G_{\sigma\beta}(\boldsymbol{\varphi})\big)\big].\end{aligned}\tag{3.2}$$

Next, Cayley-Hamilton theorem applied to the matrix field with components $G_\beta^\alpha(\boldsymbol{\varphi}):=a^{\alpha\sigma}G_{\sigma\beta}(\boldsymbol{\varphi})$ shows that

$$G_\sigma^\alpha(\boldsymbol{\varphi})G_\beta^\sigma(\boldsymbol{\varphi})-G_\sigma^\sigma(\boldsymbol{\varphi})G_\beta^\alpha(\boldsymbol{\varphi})+\det\big(G_\tau^\sigma(\boldsymbol{\varphi})\big)\delta_\beta^\alpha=0,$$

where δ^α_β denotes the Kronecker symbol. Therefore,

$$2\det\big(a^{\alpha\sigma}G_{\sigma\beta}(\boldsymbol{\varphi})\big)=\big(a^{\alpha\beta}G_{\alpha\beta}(\boldsymbol{\varphi})\big)^2-a^{\alpha\beta}a^{\sigma\tau}G_{\alpha\sigma}(\boldsymbol{\varphi})G_{\beta\tau}(\boldsymbol{\varphi}). \tag{3.3}$$

Then we deduce from the relations (3.2) and (3.3) that

$$\frac{a(\boldsymbol{\varphi})}{a}=1+2a^{\alpha\beta}G_{\alpha\beta}(\boldsymbol{\varphi})+2\big(a^{\alpha\beta}G_{\alpha\beta}(\boldsymbol{\varphi})\big)^2-2a^{\alpha\beta}a^{\sigma\tau}G_{\alpha\sigma}(\boldsymbol{\varphi})G_{\beta\tau}(\boldsymbol{\varphi}),$$

which in turn implies that

$$\log\left(\frac{a(\boldsymbol{\varphi})}{a}\right)=2a^{\alpha\beta}G_{\alpha\beta}(\boldsymbol{\varphi})-2a^{\alpha\sigma}a^{\beta\tau}G_{\alpha\beta}(\boldsymbol{\varphi})G_{\sigma\tau}(\boldsymbol{\varphi})+o\big(W_M(\boldsymbol{\varphi})\big).$$

The conclusion of the lemma follows by combining the above relations with the definition of the function $\widetilde{W}_M(\boldsymbol{\varphi})$. □

Lemma 3.3. *Given any immersion $\boldsymbol{\varphi}\in C^1(\overline{\omega};\mathbb{R}^3)$ with $\boldsymbol{a}_3(\boldsymbol{\varphi})\in C^1(\overline{\omega};\mathbb{R}^3)$ that is sufficiently close in the $C^1(\overline{\omega})$-norm to the immersion $\boldsymbol{\theta}$, the following relations hold in ω:*

$$\widetilde{W}_T^+(\boldsymbol{\varphi})=\mathcal{O}\big(W_M(\boldsymbol{\varphi})\big)+\mathcal{O}\big(W_F(\boldsymbol{\varphi})\big),$$
$$\widetilde{W}_T^-(\boldsymbol{\varphi})=\mathcal{O}\big(W_M(\boldsymbol{\varphi})\big)+\mathcal{O}\big(W_F(\boldsymbol{\varphi})\big).$$

Proof. The idea of the proof is similar to the one of Lemma 3.2. We will only prove the first relation, the second one being obtained in a similar manner. The definition of the functions $G^+_{\alpha\beta}(\boldsymbol{\varphi})$ implies that

$$g^+_{\alpha\beta}(\boldsymbol{\varphi})=g^+_{\alpha\beta}+2G^+_{\alpha\beta}(\boldsymbol{\varphi}).$$

Consequently,

$$\begin{aligned}g^+(\boldsymbol{\varphi}):&=\det\big(g^+_{\alpha\beta}(\boldsymbol{\varphi})\big)=\det\big(g^+_{\alpha\beta}+2G^+_{\alpha\beta}(\boldsymbol{\varphi})\big)\\&=g^+\Big[1+2g_+^{\alpha\beta}G^+_{\alpha\beta}(\boldsymbol{\varphi})+4\det\big(g_+^{\alpha\sigma}G^+_{\sigma\beta}(\boldsymbol{\varphi})\big)\Big],\end{aligned} \tag{3.4}$$

on the one hand. On the other hand, by applying Cayley-Hamilton theorem to the matrix field with components $G^{\alpha,+}_\beta(\boldsymbol{\varphi}):=a^{\alpha\sigma}+G^+_{\sigma\beta}(\boldsymbol{\varphi})$, one deduces that

$$G^{\alpha,+}_\sigma(\boldsymbol{\varphi})G^{\sigma,+}_\beta(\boldsymbol{\varphi})-G^{\sigma,+}_\sigma(\boldsymbol{\varphi})G^{\alpha,+}_\beta(\boldsymbol{\varphi})+\det\big(G^{\sigma,+}_\tau(\boldsymbol{\varphi})\big)\delta^\alpha_\beta=0,$$

where δ^α_β again denotes the Kronecker symbol, so that (by applying the trace operator to it)

$$2\det\big(g_+^{\alpha\sigma}G^+_{\sigma\beta}(\boldsymbol{\varphi})\big)=\big(g_+^{\alpha\beta}G^+_{\alpha\beta}(\boldsymbol{\varphi})\big)^2-g_+^{\alpha\beta}g_+^{\sigma\tau}G^+_{\alpha\sigma}(\boldsymbol{\varphi})G^+_{\beta\tau}(\boldsymbol{\varphi}). \tag{3.5}$$

Then we infer from the relations (3.4) and (3.5) that

$$\begin{aligned}\frac{g^+(\boldsymbol{\varphi})}{g^+} = 1 + 2g_+^{\alpha\beta} G_{\alpha\beta}^+(\boldsymbol{\varphi}) + 2\big(g_+^{\alpha\beta} G_{\alpha\beta}^+(\boldsymbol{\varphi})\big)^2 \\ - 2g_+^{\alpha\beta} g_+^{\sigma\tau} G_{\alpha\beta}^+(\boldsymbol{\varphi}) G_{\beta\tau}^+(\boldsymbol{\varphi}).\end{aligned} \tag{3.6}$$

By combining (2.1), Lemma 3.1 and the definition of $G_{\alpha\beta}^+(\boldsymbol{\varphi})$, we obtain

$$\begin{aligned}\sum_{\alpha,\beta} \big|G_{\alpha\beta}^+(\boldsymbol{\varphi})\big|^2 &= \mathcal{O}\big(a^{\alpha\beta\sigma\tau} G_{\alpha\beta}^+(\boldsymbol{\varphi}) G_{\sigma\tau}^+(\boldsymbol{\varphi})\big) \\ &= \mathcal{O}\big(W_M(\boldsymbol{\varphi}) + W_F(\boldsymbol{\varphi})\big).\end{aligned} \tag{3.7}$$

From (3.6) and (3.7), we deduce that

$$\log\left(\frac{g^+(\boldsymbol{\varphi})}{g^+}\right) = 2g_+^{\alpha\beta} G_{\alpha\beta}^+(\boldsymbol{\varphi}) + \mathcal{O}\big(W_M(\boldsymbol{\varphi}) + W_F(\boldsymbol{\varphi})\big).$$

The desired result follows by combining the above relations with the definition of the function $\widetilde{W}_T^+(\boldsymbol{\varphi})$. □

4 Existence of the minimizer

In this section, we prove that the minimization problem for the nonlinear shell model associated with the new energy function defined in Definition 3.1 has a minimizer. To this end, we need the following five lemmas.

Lemma 4.1. *Let $\lambda \geq 0$ and $\mu > 0$ be given constants and $g \in \mathcal{C}(\overline{\omega})$ and $h \in \mathcal{C}(\overline{\omega})$ be given positive functions.*

(a) Define the function $W_1 : \omega \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ by

$$W_1(y,z) := \frac{4\mu g(y)}{\lambda + 2\mu}\left[\frac{\lambda}{2}\frac{z^2}{h(y)} - (\lambda+\mu)\log\left(\frac{z^2}{h(y)}\right)\right]$$

for all $(y,z) \in \omega \times (0,+\infty)$, and by

$$W_1(y,z) := +\infty \quad \text{otherwise.}$$

Then

$$\int_\omega W_1\big(y,v(y)\big)\,dy \leq \liminf_{n\to\infty} \int_\omega W_1\big(y,v_n(y)\big)\,dy \tag{4.1}$$

whenever $v_n \rightharpoonup v$ in $L^2(\omega)$ when $n \to \infty$.

(b) *Define the function* $W_2 : \omega \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ *by*

$$W_2(y,z) := -(\lambda+\mu) g(y) \log\left(\frac{z^2}{h(y)}\right)$$

for all $(y,z) \in \omega \times (0,+\infty)$, *and by*

$$W_2(y,z) := +\infty \quad \textit{otherwise.}$$

Then

$$\int_\omega W_2\big(y, v(y)\big)\, dy \le \liminf_{n\to\infty} \int_\omega W_2\big(y, v_n(y)\big)\, dy \tag{4.2}$$

whenever $v_n \rightharpoonup v$ *in* $L^2(\omega)$ *when* $n \to \infty$.

Proof. The proof of part (b) is similar to the proof of part (a), which itself can be found in [8, Lemma 3]. We present it here for completeness. First, observe that the functions g and h are positive and $\log x \le x$ for all $x > 0$. Consequently, we have

$$W_1(y,z) \ge \frac{4\mu(\lambda+\mu)g(y)}{\lambda+2\mu}\left[\log\big(h(y)\big) - 2z\right]$$

for all $(y,z) \in \omega \times \mathbb{R}$. Then the function $F : \omega \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$, defined by

$$F(y,z) := W_1(y,z) + \frac{4\mu(\lambda+\mu)g(y)}{\lambda+2\mu}\left[2z - \log\big(h(y)\big)\right]$$

for all $(y,z) \in \omega \times \mathbb{R}$, satisfies

$$F(y,z) \ge 0, \quad \forall (y,z) \in \omega \times \mathbb{R}.$$

Now, it is easy to see that F is a Carathéodory function and the functions $z \in \mathbb{R} \to F(y,z)$ are convex for all $y \in \omega$. Then a classical theorem in the Calculus of Variations (see, e.g. Dacorogna [7, Theorem 3.23]) shows that

$$\int_\omega F\big(y, v(y)\big)\, dy \le \liminf_{n\to\infty} \int_\omega F\big(y, v_n(y)\big)\, dy$$

whenever $v_n \rightharpoonup v$ in $L^2(\omega)$ when $n \to \infty$.

Next, notice that

$$\int_\omega g(y)v(y) = \lim_{n\to\infty} \int_\omega g(y)v_n(y)\,dy,$$

since $v_n \rightharpoonup v$ in $L^2(\omega)$ and $g \in \mathcal{C}(\overline{\omega}) \subset L^2(\omega)$ (remember that ω is a bounded set by assumption). Then the conclusion follows by combining the last two relations. The proof is complete □

Lemma 4.2. *Given any constant $\lambda \geq 0, \mu > 0, \varepsilon > 0, \delta > 0$ and $C > 0$ that satisfy $C\delta \geq 20/9$, let*

$$\begin{aligned} W_3(\boldsymbol{F},\boldsymbol{N}) := &\frac{\lambda\mu}{\lambda+2\mu}\left[\delta|\boldsymbol{F}|^4 + \frac{4\varepsilon^2}{3}(\boldsymbol{F}:\boldsymbol{N})^2 + C\frac{\varepsilon^4}{5}|\boldsymbol{N}|^4\right] \\ &+\mu\left[\delta|\boldsymbol{F}^T\boldsymbol{F}|^2 + \frac{4\varepsilon^2}{3}|\boldsymbol{F}^T\boldsymbol{N}|^2 + C\frac{\varepsilon^4}{5}|\boldsymbol{N}^T\boldsymbol{N}|^2\right] \end{aligned}$$

for all $\boldsymbol{F},\boldsymbol{N} \in \mathbb{R}^{3\times 2}$. Then

$$\int_\omega W_3(\boldsymbol{F},\boldsymbol{N})\sqrt{a}\,dy \leq \liminf_{n\to\infty} \int_\omega W_3(\boldsymbol{F}_n,\boldsymbol{N}_n)\sqrt{a}\,dy$$

whenever $(\boldsymbol{F}_n,\boldsymbol{N}_n) \rightharpoonup (\boldsymbol{F},\boldsymbol{N})$ in $(L^4(\omega;\mathbb{R}^{3\times 2}))^2$ when $n\to\infty$.

Proof. The details of the proof can be found in the proof of [8, Lemma 4]. The key ingredient is that, under the assumptions of the lemma, the following inequality holds for all $\boldsymbol{F},\boldsymbol{N},\boldsymbol{E},\boldsymbol{M} \in \mathbb{R}^{3\times 2}$:

$$\begin{aligned} &\frac{\partial^2 W_2}{\partial \boldsymbol{F}^2}(\boldsymbol{F},\boldsymbol{N})\big((\boldsymbol{E},\boldsymbol{M}),(\boldsymbol{E},\boldsymbol{M})\big) + 2\frac{\partial^2 W_2}{\partial \boldsymbol{F}\partial \boldsymbol{N}}(\boldsymbol{F},\boldsymbol{N})\big((\boldsymbol{E},\boldsymbol{M}),(\boldsymbol{E},\boldsymbol{M})\big) \\ &\quad + \frac{\partial^2 W_2}{\partial \boldsymbol{N}^2}(\boldsymbol{F},\boldsymbol{N})\big((\boldsymbol{E},\boldsymbol{M}),(\boldsymbol{E},\boldsymbol{M})\big) \geq 0. \end{aligned} \tag{4.3}$$

Once (4.3) is proved, the fact that W_2 is convex follows. The proof is complete. □

Lemma 4.3. *Given any nonnegative functions $A, C \in \mathcal{C}^0(\overline{\omega})$ and any function $B \in \mathcal{C}^0(\overline{\omega})$, let*

$$W_4(y,\boldsymbol{F},\boldsymbol{N}) := A(y)|\boldsymbol{F}|^2 + B(y)(\boldsymbol{F}:\boldsymbol{N}) + C(y)|\boldsymbol{N}|^2$$

for all $(y,\boldsymbol{F},\boldsymbol{N}) \in \omega \times \mathbb{R}^{h\times k} \times \mathbb{R}^{h\times k}$, where h and k are given positive integers. Assume that $B(y)^2 \leq 4A(y)C(y)$ for every $y \in \omega$. Then

$$\int_\omega W_4(\cdot,\boldsymbol{F},\boldsymbol{N})\sqrt{a}\,dy \leq \liminf_{n\to\infty} \int_\omega W_4(\cdot,\boldsymbol{F}_n,\boldsymbol{N}_n)\sqrt{a}\,dy$$

whenever $(\boldsymbol{F}_n,\boldsymbol{N}_n) \rightharpoonup (\boldsymbol{F},\boldsymbol{N})$ in $(L^2(\omega;(\mathbb{R}^{h\times k})))^2$ when $n\to\infty$.

Proof. The assumption $B(y)^2 \leq 4A(y)C(y)$ implies that the function W_4 is convex with respect to $(\boldsymbol{F},\boldsymbol{N})$. This implies that

$$\int_\omega W_4(\cdot,\boldsymbol{F},\boldsymbol{N})\sqrt{a}\,dy \leq \liminf_{n\to\infty} \int_\omega W_4(\cdot,\boldsymbol{F}_n,\boldsymbol{N}_n)\sqrt{a}\,dy.$$

The proof is complete. $\square$

Lemma 4.4. *Let $A,B \in \mathcal{C}^0(\overline{\omega})$ be two functions such that $A(y) \geq B(y) \geq 0$ for all $y \in \overline{\omega}$, let $\overline{\boldsymbol{N}}: \overline{\omega} \to \mathbb{R}^{3\times 2}$ be any matrix field with continuous components, and define*

$$W_5(y,\boldsymbol{N}) := A(y)|\boldsymbol{N}|^2|\overline{\boldsymbol{N}}(y)|^2 - B(y)\left|\boldsymbol{N}\overline{\boldsymbol{N}}(y)^T\right|^2$$

for all $(y,\boldsymbol{N}) \in \mathbb{R}^{3\times 2}$.

Then

$$\int_\omega W_5(\cdot,\boldsymbol{N})\sqrt{a}\,dy \leq \liminf_{n\to\infty} \int_\omega W_5(\cdot,\boldsymbol{N}_n)\sqrt{a}\,dy$$

whenever $\boldsymbol{N}_n \rightharpoonup \boldsymbol{N}$ in $L^2(\omega;\mathbb{R}^{3\times 2})$ when $n \to \infty$.

Proof. It suffices to prove that $W_5(y,\boldsymbol{N})$ is convex with respect to $\boldsymbol{N}$. By a simple calculation, we obtain

$$\frac{\partial^2 W_5}{\partial \boldsymbol{N}^2}(y,\boldsymbol{N})(\boldsymbol{M}) = 2A(y)|\boldsymbol{M}|^2|\overline{\boldsymbol{N}}(y)|^2 - 2B(y)|\boldsymbol{M}\overline{\boldsymbol{N}}(y)^T|^2$$

for all $\boldsymbol{N},\boldsymbol{M} \in \mathbb{R}^{3\times 2}$ and all $y \in \omega$. Then from the simple inequality

$$|\boldsymbol{E}\boldsymbol{F}| \leq |\boldsymbol{E}||\boldsymbol{F}|, \quad \forall \boldsymbol{E},\boldsymbol{F} \in \mathbb{R}^{3\times 2}$$

together with the assumption $A(y) \geq B(y) \geq 0$, we obtain the desired result. $\square$

Lemma 4.5. *Let s_0 be the positive root of the equation*

$$s^2 - \frac{2^8+1}{2^{10}}s - \frac{1}{64} = 0.$$

Then the following matrix fields:

$$\begin{aligned} \boldsymbol{S}_1(y) &:= \big(s_0 a_{\alpha\beta}(y) - 2r(y)b_{\alpha\beta}(y) + s_0 r^2(y)c_{\alpha\beta}(y)\big), \\ \boldsymbol{S}_2(y) &:= \big(s_0 a_{\alpha\beta}(y) + 2r(y)b_{\alpha\beta}(y) + s_0 r^2(y)c_{\alpha\beta}(y)\big) \end{aligned}$$

for all $y \in \overline{\omega}$, are symmetric and nonnegative definite at every point $y \in \overline{\omega}$.

Proof. We only give the proof for the matrix field $\boldsymbol{S}_1$, the proof for $\boldsymbol{S}_2$ is based on a similar argument. First, it is obvious that $\boldsymbol{S}_1(y)$ is symmetric. Next, by applying Cayley-Hamilton formula to the matrix field (b^α_β), it follows that

$$b^\tau_\sigma b^\sigma_\beta - 2Hb^\tau_\beta + K\delta^\tau_\beta = 0 \quad \text{in } \overline{\omega},$$

where δ^τ_β is the Kronecker symbol. From this and the relation

$$c_{\alpha\beta} = b_{\alpha\sigma} a^{\sigma\tau} b_{\tau\beta} \quad \text{in } \overline{\omega},$$

we deduce that

$$c_{\alpha\beta} = 2Hb_{\alpha\beta} - Ka_{\alpha\beta} \quad \text{in } \overline{\omega}, \tag{4.4}$$

and thus

$$\det(c_{\alpha\beta}) = K^2 a \quad \text{in } \overline{\omega}. \tag{4.5}$$

Since the matrix $\boldsymbol{A}(y) := (a_{\alpha\beta})(y)$ is symmetric and positive definite with continuous components at each $y \in \overline{\omega}$, there exists a unique symmetric and positive definite matrix $\boldsymbol{U}(y) \in \mathbb{R}^{2\times 2}_{\text{sym}}$ with continuous components such that

$$\boldsymbol{A}(y) = \boldsymbol{U}(y)^2, \tag{4.6}$$

and thus

$$\boldsymbol{A}^{-1}(y) = \big(a^{\alpha\beta}(y)\big) = \big(\boldsymbol{U}^{-1}(y)\big)^2,$$

where $\boldsymbol{U}^{-1}(y)$ is also a symmetric and positive definite matrix with components depending continuously on $y \in \overline{\omega}$.

Now we will show that

$$\operatorname{Tr}\big(\boldsymbol{U}^{-1}(y)\boldsymbol{S}_1(y)\boldsymbol{U}^{-1}(y)\big) \geq 0, \quad \forall y \in \overline{\omega}, \tag{4.7}$$

$$\det\big(\boldsymbol{U}^{-1}(y)\boldsymbol{S}_1(y)\boldsymbol{U}^{-1}(y)\big) \geq 0, \quad \forall y \in \overline{\omega}. \tag{4.8}$$

To this end, by some straightforward calculations based in particular on the definitions of H and K, the relations $|rH| \leq 1/16$ and $4H^2 - 2K \geq 0$, we deduce from (4.4) that

$$\begin{aligned}\operatorname{Tr}\big((a^{\alpha\beta})\boldsymbol{S}_1\big) &= (a^{\alpha\beta}) : \big[s_0(a_{\alpha\beta}) - 2r(b_{\alpha\beta}) + s_0 r^2 (c_{\alpha\beta})\big] \\ &= 2s_0 - 4rH + s_0 r^2(4H^2 - 2K) \\ &\geq 2s_0 - \frac{1}{4} > 0. \end{aligned} \tag{4.9}$$

Then we infer from (4.6) and (4.9) that

$$\begin{aligned}\mathrm{Tr}\big(\boldsymbol{U}^{-1}(y)\boldsymbol{S}_1(y)\boldsymbol{U}^{-1}(y)\big) &= \mathrm{Tr}\big(\boldsymbol{U}^{-1}(y)\boldsymbol{U}^{-1}(y)\boldsymbol{S}_1(y)\big)\\ &= \mathrm{Tr}\big((a^{\alpha\beta}(y))\boldsymbol{S}_1(y)\big) > 0,\end{aligned}$$

for all $y \in \overline{\omega}$. Thus, (4.7) holds.

Next, we infer from relations (4.4) and (4.5) and from the inequalities $|rH| \leq 1/16, |r^2K| \leq 1/256$ and $4H^2 - 2K \geq 0$ that

$$\begin{aligned}\det(\boldsymbol{S}_1) &= s_0^2 \det(a_{\alpha\beta}) + 4r^2 \det(b_{\alpha\beta}) + s_0^2 r^4 \det(c_{\alpha\beta})\\ &\quad - 2rs_0(a_{11}b_{22} + a_{22}b_{11} - 2a_{12}b_{12}) + s_0^2 r^2 (a_{11}c_{22} + a_{22}c_{11} - 2a_{12}c_{12})\\ &\quad - 2r^3 s_0 (b_{11}c_{22} + b_{22}c_{11} - 2b_{12}c_{12})\\ &= s_0^2 a + 4r^2 K a + s_0^2 r^4 K^2 a - 4rs_0 H a + s_0^2 r^2 (4H^2 - 2K) a - 4r^3 s_0 HK\\ &= a\big(s_0^2 + 4r^2 K + s_0^2 r^4 K^2 - 4rs_0 H - 4r^3 s_0 HK + s_0^2 r^2 (4H^2 - 2K)\big)\\ &\geq a\big(s_0^2 + 4r^2 K - 4rs_0 H - 4r^3 s_0 HK\big) \geq a\left(s_0^2 - \frac{1}{64} - \frac{s_0}{4} - \frac{s_0}{2^{10}}\right)\\ &= a\left(s_0^2 - \frac{2^8+1}{2^{10}} s_0 - \frac{1}{64}\right) = 0,\end{aligned}$$

and thus (4.8) holds. From (4.7) and (4.8), we deduce that $\boldsymbol{U}^{-1}(y)\boldsymbol{S}_1(y)\boldsymbol{U}^{-1}(y)$ is symmetric and nonnegative definite, which in turn implies that $\boldsymbol{S}_1(y)$ is symmetric and positive definite. The proof is complete. □

Now we are able to give the proof for our existence theorem, which is the main result of this paper. Note that the theorem holds with particular choices $\delta = 1/4, C_1 := 9$ and $C_2 := 90$, irrespectively of the values of the elastic coefficients λ and μ.

Theorem 4.1. *Define the functional* $\widetilde{J}_K : \boldsymbol{V}_K(\omega) \to \mathbb{R} \cup \{+\infty\}$ *by*

$$\widetilde{J}_K(\boldsymbol{\varphi}) := \int_\omega \widetilde{W}_K(\boldsymbol{\varphi})\sqrt{a}\,dy - L_K(\boldsymbol{\varphi}), \quad \forall \boldsymbol{\varphi} \in \boldsymbol{V}_K(\omega),$$

where

$$\begin{aligned}\boldsymbol{V}_K(\omega) := \Big\{ &\boldsymbol{\varphi} \in W^{1,4}(\omega;\mathbb{R}^3);\ |\partial_1\boldsymbol{\varphi} \wedge \partial_2\boldsymbol{\varphi}| > 0 \text{ a.e. in } \omega,\ \boldsymbol{a}_3(\boldsymbol{\varphi}) \in W^{1,4}(\omega;\mathbb{R}^3),\\ &\big|\big(\partial_1\boldsymbol{a}(\boldsymbol{\varphi}) + r\partial_1\boldsymbol{a}_3(\boldsymbol{\varphi})\big) \wedge \big(\partial_2\boldsymbol{a}(\boldsymbol{\varphi}) + r\partial_2\boldsymbol{a}_3(\boldsymbol{\varphi})\big)\big| > 0 \text{ a.e. in } \omega,\\ &\big|\big(\partial_1\boldsymbol{a}(\boldsymbol{\varphi}) - r\partial_1\boldsymbol{a}_3(\boldsymbol{\varphi})\big) \wedge \big(\partial_2\boldsymbol{a}(\boldsymbol{\varphi}) - r\partial_2\boldsymbol{a}_3(\boldsymbol{\varphi})\big)\big| > 0 \text{ a.e. in } \omega,\\ &\boldsymbol{\varphi}\big|_{\gamma_0} = \boldsymbol{\theta}|_{\gamma_0},\ \boldsymbol{a}_3(\boldsymbol{\varphi})\big|_{\gamma_0} = \boldsymbol{a}_3\big|_{\gamma_0}\Big\},\end{aligned}$$

$r\in\mathcal{C}(\overline{\omega})$ is the function defined in terms of $\boldsymbol{\theta}$ by (3.1), *$\widetilde{W}_K$ is the function defined in Definition* 3.1, *and $L_K: W^{1,4}(\omega;\mathbb{R}^3)\to\mathbb{R}$ is any linear and continuous function. Assume that the constants appearing in the definition of $\widetilde{W}_K$ satisfy*

$$0<\delta<\frac{\lambda+2\mu}{4(\lambda+\mu)},\quad C_1\delta\geq\frac{20}{9},$$

and

$$C_2\geq\frac{2}{(1-s_0)}\left[C_1\frac{\mu}{\lambda+2\mu}+\frac{40}{9}\frac{\mu(3\lambda+2\mu)}{\big(\lambda+2\mu-4\delta(\lambda+\mu)\big)(\lambda+2\mu)}\right],$$

where s_0 is defined in Lemma 4.5. *Then the functional $\widetilde{J}_K$ has a minimizer in $\boldsymbol{V}_K(\omega)$.*

Proof. For convenience, the proof will be divided into five steps.

Step 1. We prove that the functional $\widetilde{J}_K$ is well-defined and coercive. Let $\boldsymbol{\varphi}\in\boldsymbol{V}_K(\omega)$. Notice that for an arbitrary symmetric and positive definite matrix $\boldsymbol{M}\in\mathbb{R}^{2\times 2}_{\rm sym}$, we have

$$\frac{1}{2}\mathrm{Tr}(\boldsymbol{M})\geq\big(\det(\boldsymbol{M})\big)^{1/2}.$$

Then we infer from (4.6) that (recall that $\boldsymbol{U}$ denotes the square root of the matrix field $\boldsymbol{A}=(a_{\alpha\beta})$)

$$\begin{aligned}\frac{1}{2}a^{\alpha\beta}a_{\alpha\beta}(\boldsymbol{\varphi})&=\frac{1}{2}\mathrm{Tr}\big(\boldsymbol{A}^{-1}\big(a_{\alpha\beta}(\boldsymbol{\varphi})\big)\big)=\frac{1}{2}\mathrm{Tr}\big(\boldsymbol{U}^{-1}\boldsymbol{U}^{-1}\big(a_{\alpha\beta}(\boldsymbol{\varphi})\big)\big)\\&=\frac{1}{2}\mathrm{Tr}\big(\boldsymbol{U}^{-1}\big(a_{\alpha\beta}(\boldsymbol{\varphi})\big)\boldsymbol{U}^{-1}\big)\geq\sqrt{\det\big(\boldsymbol{U}^{-1}\big(a_{\alpha\beta}(\boldsymbol{\varphi})\big)\boldsymbol{U}^{-1}\big)}\\&=\sqrt{\det\big(\boldsymbol{U}^{-1}\boldsymbol{U}^{-1}\big(a_{\alpha\beta}(\boldsymbol{\varphi})\big)\big)}=\sqrt{\det\big(a^{\alpha\sigma}a_{\sigma\beta}(\boldsymbol{\varphi})\big)}\\&=\sqrt{\frac{a(\boldsymbol{\varphi})}{a}}.\end{aligned}$$

Then from the definition of the function $\widetilde{W}_M(\boldsymbol{\varphi})$, we obtain

$$\begin{aligned}\widetilde{W}_M(\boldsymbol{\varphi})&\geq 4\mu(x-1)+\frac{2\lambda\mu}{\lambda+2\mu}(x^2-1)-8\mu\frac{\lambda+\mu}{\lambda+2\mu}\log x\\&\geq 4\mu\left(x-1-\frac{\lambda}{2(\lambda+2\mu)}-\frac{2(\lambda+\mu)}{\lambda+2\mu}\log x\right),\end{aligned}$$

where $x:=\sqrt{a(\boldsymbol{\varphi})/a}$. Then an elementary computation of the infimum of the function in the right-hand side shows that

$$\widetilde{W}_M(\boldsymbol{\varphi})\geq C_3,$$

where C_3 is a constant depending only on λ and μ. By a similar argument, there exists a constant C_4 depending only on $\boldsymbol{\theta}, r_0, \lambda$ and μ such that

$$\widetilde{W}_T^+(\boldsymbol{\varphi}) \geq C_4, \quad \widetilde{W}_T^-(\boldsymbol{\varphi}) \geq C_4.$$

Besides, the uniform positive-definiteness of the two-dimensional elasticity tensor (see the inequality (2.1)) implies that there exist positive constants $C_5 = C_5(\lambda,\mu,\omega), C_6 = C_6(\lambda,\mu,\omega)$ and $C_7 = C_7(\lambda,\mu,\omega)$ such that, for all $\boldsymbol{\varphi} \in \boldsymbol{V}_K(\omega)$,

$$\begin{aligned}
W_M(\boldsymbol{\varphi}) &= \frac{1}{4} a^{\alpha\beta\sigma\tau} \left(a_{\alpha\beta}(\boldsymbol{\varphi}) - a_{\alpha\beta}\right) \left(a_{\sigma\tau}(\boldsymbol{\varphi}) - a_{\sigma\tau}\right) \\
&\geq C_5 \sum_{\alpha,\beta} |a_{\alpha\beta}(\boldsymbol{\varphi}) - a_{\alpha\beta}|^2 \\
&\geq C_6 \sum_{\alpha} |a_{\alpha\alpha}(\boldsymbol{\varphi})|^2 - C_7, \\
W_F(\boldsymbol{\varphi}) &= a^{\alpha\beta\sigma\tau} \left(b_{\alpha\beta}(\boldsymbol{\varphi}) - b_{\alpha\beta}\right) \left(b_{\sigma\tau}(\boldsymbol{\varphi}) - b_{\sigma\tau}\right) \\
&\geq C_5 \sum_{\alpha,\beta} |b_{\alpha\beta}(\boldsymbol{\varphi}) - b_{\alpha\beta}|^2 \geq 0, \\
W_T(\boldsymbol{\varphi}) &= \frac{1}{4} a^{\alpha\beta\sigma\tau} \left(c_{\alpha\beta}(\boldsymbol{\varphi}) - c_{\alpha\beta}\right) \left(c_{\sigma\tau}(\boldsymbol{\varphi}) - c_{\sigma\tau}\right) \\
&\geq C_6 \sum_{\alpha,\beta} |c_{\alpha\beta}(\boldsymbol{\varphi})|^2 - C_7.
\end{aligned}$$

Combining the above inequalities with Poincaré's inequality in the domain ω (which is bounded with Lipschitz continuous boundary by assumption, cf. Section 2) and using that $\boldsymbol{\varphi}$ and $\boldsymbol{a}_3(\boldsymbol{\varphi})$ both belong to $W^{1,4}(\omega;\mathbb{R}^3)$, one deduces that the functional $\widetilde{J}_K : \boldsymbol{V}_K(\omega) \to \mathbb{R} \cup \{+\infty\}$ is well-defined as an extended real number in $\mathbb{R} \cup \{+\infty\}$, that $\widetilde{J}_K(\boldsymbol{\varphi})$ is bounded from below, and that $\widetilde{J}_K$ is coercive in the following sense: If a sequence $(\boldsymbol{\varphi}_n)_{n=1}^\infty \subseteq \boldsymbol{V}_K(\omega)$ satisfies

$$\sup_n \widetilde{J}_K(\boldsymbol{\varphi}_n) < \infty,$$

then the sequences $(\boldsymbol{\varphi}_n)$ and $\boldsymbol{a}_3(\boldsymbol{\varphi}_n)$, $n \geq 1$, are both bounded in $W^{1,4}(\omega;\mathbb{R}^3)$.

Step 2. Let $(\boldsymbol{\varphi}_n)_{n=1}^\infty \subseteq \boldsymbol{V}_K(\omega)$ denote an infimizing sequence of the functional $\widetilde{J}_K$ over the set $\boldsymbol{V}_K(\omega)$. Since $\boldsymbol{V}_K(\omega)$ contains at least one element, namely $\boldsymbol{\theta}$, we have

$$\inf_{\boldsymbol{\varphi} \in \boldsymbol{V}_K(\omega)} \widetilde{J}_K(\boldsymbol{\varphi}) \leq \widetilde{J}_K(\boldsymbol{\theta}) < \infty.$$

Then the coerciveness of $\widetilde{J}_K$ implies that the sequences $(\boldsymbol{\varphi}_n)$ and $(\boldsymbol{a}_3(\boldsymbol{\varphi}_n)), n \geq 1$, are both bounded in $W^{1,4}(\omega;\mathbb{R}^3)$. This space being reflexive, there exists a subsequence, still denoted $(\boldsymbol{\varphi}_n)_{n=1}^{\infty} \subseteq \boldsymbol{V}_K(\omega)$ for conciseness, such that

$$\begin{aligned} &\boldsymbol{\varphi}_n && \rightharpoonup \boldsymbol{\psi} \quad \text{in } W^{1,4}(\omega;\mathbb{R}^3),\\ &\partial_1\boldsymbol{\varphi}_n \wedge \partial_2\boldsymbol{\varphi}_n && \rightharpoonup \boldsymbol{\xi} \quad \text{in } L^2(\omega;\mathbb{R}^3),\\ &\boldsymbol{a}_3(\boldsymbol{\varphi}_n) && \rightharpoonup \boldsymbol{\zeta} \quad \text{in } W^{1,4}(\omega;\mathbb{R}^3), \end{aligned} \tag{4.10}$$

where

$$\begin{aligned} &\boldsymbol{\psi} \in W^{1,4}(\omega;\mathbb{R}^3) \quad \text{with} \quad \boldsymbol{\psi}|_{\gamma_0} = \boldsymbol{\theta}|_{\gamma_0},\\ &\boldsymbol{\xi} \in L^2(\omega;\mathbb{R}^3),\\ &\boldsymbol{\zeta} \in W^{1,4}(\omega;\mathbb{R}^3) \quad \text{with} \quad \boldsymbol{\zeta}|_{\gamma_0} = \boldsymbol{a}_3|_{\gamma_0}. \end{aligned} \tag{4.11}$$

Moreover, since

$$\begin{aligned} \partial_1\boldsymbol{\varphi}_n \wedge \partial_2\boldsymbol{\varphi}_n &= \frac{1}{2}[\partial_1(\boldsymbol{\varphi}_n \wedge \partial_2\boldsymbol{\varphi}_n) + \partial_2(\partial_1\boldsymbol{\varphi}_n \wedge \boldsymbol{\varphi}_n)]\\ &\rightharpoonup \frac{1}{2}[\partial_1(\boldsymbol{\psi} \wedge \partial_2\boldsymbol{\psi}) + \partial_2(\partial_1\boldsymbol{\psi} \wedge \boldsymbol{\psi})] = \partial_1\boldsymbol{\psi} \wedge \partial_2\boldsymbol{\psi} \quad \text{in } D'(\omega,\mathbb{R}^3), \end{aligned}$$

we have

$$\boldsymbol{\xi} = \partial_1\boldsymbol{\psi} \wedge \partial_2\boldsymbol{\psi},$$

which implies

$$\partial_1\boldsymbol{\varphi}_n \wedge \partial_2\boldsymbol{\varphi}_n \rightharpoonup \partial_1\boldsymbol{\psi} \wedge \partial_2\boldsymbol{\psi} \quad \text{in } L^2(\omega;\mathbb{R}^3). \tag{4.12}$$

By a similar argument, we deduce that

$$\partial_1\boldsymbol{a}_3(\boldsymbol{\varphi}_n) \wedge \partial_2\boldsymbol{a}_3(\boldsymbol{\varphi}_n) \rightharpoonup \partial_1\boldsymbol{\zeta} \wedge \partial_2\boldsymbol{\zeta} \quad \text{in } L^2(\omega;\mathbb{R}^3), \tag{4.13}$$

$$\partial_1\big(\boldsymbol{\varphi}_n + \boldsymbol{a}_3(\boldsymbol{\varphi}_n)\big) \wedge \partial_2\big(\boldsymbol{\varphi}_n + \boldsymbol{a}_3(\boldsymbol{\varphi}_n)\big) \rightharpoonup \partial_1(\boldsymbol{\psi} + \boldsymbol{\zeta}) \wedge \partial_2(\boldsymbol{\psi} + \boldsymbol{\zeta}) \quad \text{in } L^2(\omega;\mathbb{R}^3), \tag{4.14}$$

Notice that $r \in \mathcal{C}^0(\overline{\omega})$. By combining (4.12)-(4.14), one deduce that

$$\begin{aligned} \big(\partial_1\boldsymbol{\varphi}_n + r\partial_1\boldsymbol{a}_3(\boldsymbol{\varphi}_n)\big) \wedge \big(\partial_2\boldsymbol{\varphi}_n + r\partial_2\boldsymbol{a}_3(\boldsymbol{\varphi}_n)\big) &\rightharpoonup (\partial_1\boldsymbol{\psi} + r\partial_1\boldsymbol{\zeta}) \wedge (\partial_2\boldsymbol{\psi} + r\partial_2\boldsymbol{\zeta}),\\ \big(\partial_1\boldsymbol{\varphi}_n - r\partial_1\boldsymbol{a}_3(\boldsymbol{\varphi}_n)\big) \wedge \big(\partial_2\boldsymbol{\varphi}_n - r\partial_2\boldsymbol{a}_3(\boldsymbol{\varphi}_n)\big) &\rightharpoonup (\partial_1\boldsymbol{\psi} - r\partial_1\boldsymbol{\zeta}) \wedge (\partial_2\boldsymbol{\psi} - r\partial_2\boldsymbol{\zeta}) \end{aligned}$$

in $L^2(\omega;\mathbb{R}^3)$.

Step 3. Next, we decompose $\widetilde{W}_K(\boldsymbol{\varphi})$ into a sum of six particular functions, as follows. From the definition of $W_M(\boldsymbol{\varphi})$ and $\widetilde{W}_M(\boldsymbol{\varphi})$, we have

$$\begin{aligned}
&\delta W_M(\boldsymbol{\varphi})+(1-\delta)\widetilde{W}_M(\boldsymbol{\varphi})\\
=&\delta\left[\frac{\lambda\mu}{\lambda+2\mu}\left(a^{\alpha\beta}a_{\alpha\beta}(\boldsymbol{\varphi})\right)^2+\mu a^{\alpha\sigma}a^{\beta\tau}a_{\alpha\beta}(\boldsymbol{\varphi})a_{\sigma\tau}(\boldsymbol{\varphi})\right]\\
&+a^{\alpha\beta}a_{\alpha\beta}(\boldsymbol{\varphi})\left[(1-\delta)2\mu-\delta\left(\frac{4\lambda\mu}{\lambda+2\mu}+2\mu\right)\right]\\
&+(1-\delta)\frac{4\mu}{\lambda+2\mu}\left[\frac{\lambda}{2}\frac{a(\boldsymbol{\varphi})}{a}-(\lambda+\mu)\log\left(\frac{a(\boldsymbol{\varphi})}{a}\right)\right]\\
&+\delta\left(\frac{4\lambda\mu}{\lambda+2\mu}+2\mu\right)-(1-\delta)\left(4\mu+\frac{2\lambda\mu}{\lambda+2\mu}\right).
\end{aligned}$$

Note that the coefficient of $a^{\alpha\beta}a_{\alpha\beta}(\boldsymbol{\varphi})$ is positive thanks to the assumption that $\delta<(\lambda+2\mu)/(4(\lambda+\mu))$. Also notice that

$$a^{\alpha\beta}a_{\alpha\beta}=2,\quad a^{\alpha\sigma}a_{\alpha\beta}a^{\beta\tau}=a^{\sigma\tau}.$$

Consequently,

$$\widetilde{W}_K(\boldsymbol{\varphi})=\mathcal{A}_1(\boldsymbol{\varphi})+\mathcal{A}_2(\boldsymbol{\varphi})+\mathcal{A}_3(\boldsymbol{\varphi})+\mathcal{A}_4(\boldsymbol{\varphi})+\mathcal{A}_5(\boldsymbol{\varphi})+\mathcal{A}_6(\boldsymbol{\varphi})+\mathcal{A}_7, \tag{4.15}$$

where the functions in the right-hand side are defined as follows.

The function $\mathcal{A}_1(\boldsymbol{\varphi}):\omega\to\mathbb{R}\cup\{+\infty\}$ is defined by

$$\begin{aligned}
\mathcal{A}_1(\boldsymbol{\varphi}):=&\frac{\varepsilon}{2}(1-\delta)\frac{4\mu}{\lambda+2\mu}\left[\frac{\lambda}{2}\frac{a(\boldsymbol{\varphi})}{a}-(\lambda+\mu)\log\left(\frac{a(\boldsymbol{\varphi})}{a}\right)\right]\\
&-\frac{\varepsilon^5}{10}C_2(4H^2-2K)\frac{(\lambda+\mu)g^+}{r^2a}\log\left(\frac{g^+(\boldsymbol{\varphi})}{g^+}\right)\\
&-\frac{\varepsilon^5}{10}C_2(4H^2-2K)\frac{(\lambda+\mu)g^-}{r^2a}\log\left(\frac{g^-(\boldsymbol{\varphi})}{g^-}\right)
\end{aligned}$$

at the points of ω, where $g^+(\boldsymbol{\varphi})>0$ and $g^-(\boldsymbol{\varphi})>0$, and by $\mathcal{A}_1(\boldsymbol{\varphi}):=+\infty$ at the points of ω where $g^+(\boldsymbol{\varphi})=0$ or $g^-(\boldsymbol{\varphi})=0$.

The function $\mathcal{A}_2(\boldsymbol{\varphi}):\omega\to\mathbb{R}$ is defined by

$$\mathcal{A}_2(\boldsymbol{\varphi}):=\frac{\lambda\mu}{\lambda+2\mu}\left[\frac{\varepsilon}{2}\delta\left(a^{\alpha\beta}a_{\alpha\beta}(\boldsymbol{\varphi})\right)^2+\frac{2\varepsilon^3}{3}\left(a^{\alpha\beta}b_{\alpha\beta}(\boldsymbol{\varphi})\right)^2+\frac{\varepsilon^5}{10}C_1\left(a^{\alpha\beta}c_{\alpha\beta}(\boldsymbol{\varphi})\right)^2\right]$$

$$+\mu\left[\frac{\varepsilon}{2}\delta a^{\alpha\sigma}a^{\beta\tau}a_{\alpha\beta}(\boldsymbol{\varphi})a_{\sigma\tau}(\boldsymbol{\varphi})+\frac{2\varepsilon^3}{3}a^{\alpha\sigma}a^{\beta\tau}b_{\alpha\beta}(\boldsymbol{\varphi})b_{\sigma\tau}(\boldsymbol{\varphi})\right.$$
$$\left.+\frac{\varepsilon^5}{10}C_1a^{\alpha\sigma}a^{\beta\tau}c_{\alpha\beta}(\boldsymbol{\varphi})c_{\sigma\tau}(\boldsymbol{\varphi})\right].$$

The function $\mathcal{A}_3(\boldsymbol{\varphi}):\omega\to\mathbb{R}$ is defined by

$$\mathcal{A}_3(\boldsymbol{\varphi}):=\frac{\varepsilon}{2}\frac{\lambda\mu}{\lambda+2\mu}\left[l\frac{\lambda+2\mu}{4\mu(\lambda+\mu)}a^{\alpha\beta}a_{\alpha\beta}a^{\sigma\tau}a_{\sigma\tau}(\boldsymbol{\varphi})-\frac{8\varepsilon^2}{3}a^{\alpha\beta}b_{\alpha\beta}a^{\sigma\tau}b_{\sigma\tau}(\boldsymbol{\varphi})\right.$$
$$\left.+\frac{\varepsilon^4}{5}\left(\frac{320}{9l}\frac{\mu(\lambda+\mu)}{\lambda+2\mu}\right)a^{\alpha\beta}c_{\alpha\beta}a^{\sigma\tau}c_{\sigma\tau}(\boldsymbol{\varphi})\right],$$

where

$$l:=2\mu\left(1-4\delta\frac{\lambda+\mu}{\lambda+2\mu}\right).$$

The function $\mathcal{A}_4(\boldsymbol{\varphi}):\omega\to\mathbb{R}$ is defined by

$$\mathcal{A}_4(\boldsymbol{\varphi}):=\frac{\varepsilon}{2}\mu\left[l\frac{\lambda+2\mu}{2\mu(\lambda+\mu)}a^{\alpha\sigma}a_{\alpha\beta}a^{\beta\tau}a_{\sigma\tau}(\boldsymbol{\varphi})-\frac{8\varepsilon^2}{3}a^{\alpha\sigma}b_{\alpha\beta}a^{\beta\tau}b_{\sigma\tau}(\boldsymbol{\varphi})\right.$$
$$\left.+\frac{\varepsilon^4}{5}\left(\frac{160}{9l}\frac{\mu(\lambda+\mu)}{\lambda+2\mu}\right)a^{\alpha\sigma}c_{\alpha\beta}a^{\beta\tau}c_{\sigma\tau}(\boldsymbol{\varphi})\right].$$

The function $\mathcal{A}_5(\boldsymbol{\varphi}):\omega\to\mathbb{R}$ is defined by

$$\mathcal{A}_5(\boldsymbol{\varphi}):=\frac{\varepsilon^5}{10}\left[2(1-s_0)(\lambda+\mu)C_2(4H^2-2K)a^{\sigma\tau}c_{\sigma\tau}(\boldsymbol{\varphi})\right.$$
$$-\left[\frac{\lambda\mu}{\lambda+2\mu}\left(\frac{320}{9l}\frac{\mu(\lambda+\mu)}{\lambda+2\mu}\right)a^{\alpha\beta}c_{\alpha\beta}a^{\sigma\tau}c_{\sigma\tau}(\boldsymbol{\varphi})\right.$$
$$\left.+\mu\left(\frac{160}{9l}\frac{\mu(\lambda+\mu)}{\lambda+2\mu}\right)a^{\alpha\sigma}c_{\alpha\beta}a^{\beta\tau}c_{\sigma\tau}(\boldsymbol{\varphi})\right]$$
$$\left.-C_1\left[\frac{2\lambda\mu}{\lambda+2\mu}a^{\alpha\beta}c_{\alpha\beta}a^{\sigma\tau}c_{\sigma\tau}(\boldsymbol{\varphi})+2\mu a^{\alpha\sigma}c_{\alpha\beta}a^{\beta\tau}c_{\sigma\tau}(\boldsymbol{\varphi})\right]\right].$$

The function $\mathcal{A}_6(\boldsymbol{\varphi}):\omega\to\mathbb{R}$ is defined by

$$\mathcal{A}_6(\boldsymbol{\varphi}):=\frac{\varepsilon^5}{10}(\lambda+\mu)C_2(4H^2-2K)$$
$$\times\left[\frac{g^+}{r^2a}g_+^{\alpha\beta}g^+_{\alpha\beta}(\boldsymbol{\varphi})+\frac{g^-}{r^2a}g_-^{\alpha\beta}g^-_{\alpha\beta}(\boldsymbol{\varphi})-2(1-s_0)a^{\sigma\tau}c_{\sigma\tau}(\boldsymbol{\varphi})\right].$$

The function $\mathcal{A}_7:\omega\to\mathbb{R}$ is defined by

$$\mathcal{A}_7:=\frac{\varepsilon}{2}\left[\delta\left(\frac{4\lambda\mu}{\lambda+2\mu}+2\mu\right)-(1-\delta)\left(\frac{2\lambda\mu}{\lambda+2\mu}+4\mu\right)\right]+\frac{\varepsilon^3}{3}a^{\alpha\beta\sigma\tau}b_{\alpha\beta}b_{\sigma\tau}$$
$$+\frac{\varepsilon^5}{10}\left[\frac{1}{4}C_1a^{\alpha\beta\sigma\tau}c_{\alpha\beta}c_{\sigma\tau}-(\lambda+\mu)C_2(4H^2-2K)\left(\frac{2g^+}{r^2a}+\frac{2g^-}{r^2a}\right)\right].$$

Step 4. We now prove that each of the following integrals:

$$\int_\omega \mathcal{A}_i(\boldsymbol{\varphi})\sqrt{a}dy,\quad i=1,2,\ldots,6,$$

is sequentially weakly lower semi-continuous. First, since

$$a(\boldsymbol{\varphi})=|\boldsymbol{a}_3(\boldsymbol{\varphi})\cdot(\partial_1\boldsymbol{\varphi}\wedge\partial_2\boldsymbol{\varphi})|^2 \quad \text{a.e. in } \omega,$$
$$g^+(\boldsymbol{\varphi})=\left|\boldsymbol{a}_3(\boldsymbol{\varphi})\cdot\left[\left(\partial_1\boldsymbol{\varphi}+r\partial_1\boldsymbol{a}_3(\boldsymbol{\varphi})\right)\wedge\left(\partial_2\boldsymbol{\varphi}+r\partial_2\boldsymbol{a}_3(\boldsymbol{\varphi})\right)\right]\right|^2 \quad \text{a.e. in } \omega,$$
$$g^-(\boldsymbol{\varphi})=\left|\boldsymbol{a}_3(\boldsymbol{\varphi})\cdot\left[\left(\partial_1\boldsymbol{\varphi}-r\partial_1\boldsymbol{a}_3(\boldsymbol{\varphi})\right)\wedge\left(\partial_2\boldsymbol{\varphi}-r\partial_2\boldsymbol{a}_3(\boldsymbol{\varphi})\right)\right]\right|^2 \quad \text{a.e. in } \omega,$$

the function $\mathcal{A}_1(\boldsymbol{\varphi})$ satisfies

$$\mathcal{A}_1(\boldsymbol{\varphi})\sqrt{a}=\frac{\varepsilon}{2}W_1^*\left(\cdot,\boldsymbol{a}_3(\boldsymbol{\varphi})\cdot(\partial_1\boldsymbol{\varphi}\wedge\partial_2\boldsymbol{\varphi})\right)$$
$$+\frac{\varepsilon^5}{10}W_{2,a}\left(\cdot,\boldsymbol{a}_3(\boldsymbol{\varphi})\cdot\left(\partial_1\boldsymbol{a}_3(\boldsymbol{\varphi})\wedge\partial_2\boldsymbol{a}_3(\boldsymbol{\varphi})\right)\right)$$
$$+\frac{\varepsilon^5}{10}W_{2,b}\left(\cdot,\boldsymbol{a}_3(\boldsymbol{\varphi})\cdot\left(\partial_1\boldsymbol{a}_3(\boldsymbol{\varphi})\wedge\partial_2\boldsymbol{a}_3(\boldsymbol{\varphi})\right)\right),$$

where W_1^* denotes the function W_1 defined in Lemma 4.1(a) with

$$g:=(1-\delta)\sqrt{a},\quad h:=a,$$

and $W_{2,a}$, respectively $W_{2,b}$, denotes the function W_2 in Lemma 4.1(b) with

$$g:=C_2(4H^2-2K)\frac{g^+}{r^2a}\sqrt{a},\quad h:=g^+,$$

respectively with

$$g:=C_2(4H^2-2K)\frac{g^-}{r^2a}\sqrt{a},\quad h:=g^-.$$

Since the sequences $(\boldsymbol{a}_3(\boldsymbol{\varphi}_n)\cdot(\partial_1\boldsymbol{\varphi}_n\wedge\partial_2\boldsymbol{\varphi}_n)),(t_n^+),(t_n^-)$, where

$$
\begin{aligned}
t_n^+ &:= \boldsymbol{a}_3(\boldsymbol{\varphi}_n)\cdot\big[\big(\partial_1\boldsymbol{\varphi}_n + r\partial_1\boldsymbol{a}_3(\boldsymbol{\varphi}_n)\big)\wedge\big(\partial_2\boldsymbol{\varphi}_n + r\partial_2\boldsymbol{a}_3(\boldsymbol{\varphi}_n)\big)\big],\\
t_n^- &:= \boldsymbol{a}_3(\boldsymbol{\varphi}_n)\cdot\big[\big(\partial_1\boldsymbol{\varphi}_n - r\partial_1\boldsymbol{a}_3(\boldsymbol{\varphi}_n)\big)\wedge\big(\partial_2\boldsymbol{\varphi}_n - r\partial_2\boldsymbol{a}_3(\boldsymbol{\varphi}_n)\big)\big],
\end{aligned}
$$

converge weakly in $L^2(\omega)$, respectively to the functions

$$
\begin{aligned}
&\big(\boldsymbol{\zeta}\cdot(\partial_1\boldsymbol{\psi}\wedge\partial_2\boldsymbol{\psi})\big),\\
&\boldsymbol{\zeta}\cdot[(\partial_1\boldsymbol{\psi}+r\partial_1\boldsymbol{\zeta})\wedge(\partial_2\boldsymbol{\psi}+r\partial_2\boldsymbol{\zeta})],\\
&\boldsymbol{\zeta}\cdot[(\partial_1\boldsymbol{\psi}-r\partial_1\boldsymbol{\zeta})\wedge(\partial_2\boldsymbol{\psi}-r\partial_2\boldsymbol{\zeta})]
\end{aligned}
$$

thanks to the convergences established in Step 2 above combined with the compact embedding of $W^{1,4}(\omega)$ into $L^\infty(\omega)$ (remember that ω is a bounded Lipschitz domain in $\mathbb{R}^2$), Lemma 4.1 implies that

$$
\begin{aligned}
&\int_\omega \Big[\frac{\varepsilon}{2}W_1^*\big(\cdot,\boldsymbol{\zeta}\cdot(\partial_1\boldsymbol{\psi}\wedge\partial_2\boldsymbol{\psi})\big)+\frac{\varepsilon^5}{10}W_{2,a}\big(\cdot,\boldsymbol{\zeta}\cdot[(\partial_1\boldsymbol{\psi}+r\partial_1\boldsymbol{\zeta})\wedge(\partial_2\boldsymbol{\psi}+r\partial_2\boldsymbol{\zeta})]\big)\\
&\qquad +\frac{\varepsilon^5}{10}W_{2,b}\big(\cdot,\boldsymbol{\zeta}\cdot[(\partial_1\boldsymbol{\psi}-r\partial_1\boldsymbol{\zeta})\wedge(\partial_2\boldsymbol{\psi}-r\partial_2\boldsymbol{\zeta})]\big)\Big]dy\\
&\le \frac{\varepsilon}{2}\liminf_{n\to\infty}\int_\omega W_1^*\big(\cdot,\boldsymbol{a}_3(\boldsymbol{\varphi}_n)\cdot(\partial_1\boldsymbol{\varphi}_n\wedge\partial_2\boldsymbol{\varphi}_n)\big)dy\\
&\quad +\frac{\varepsilon^5}{10}\liminf_{n\to\infty}\int_\omega W_{2,a}(\cdot,t_n^+)\,dy+\frac{\varepsilon^5}{10}\liminf_{n\to\infty}\int_\omega W_{2,b}(\cdot,t_n^-)\,dy\\
&=\liminf_{n\to\infty}\int_\omega \mathcal{A}_1(\boldsymbol{\varphi}_n)\sqrt{a}\,dy.
\end{aligned}
\tag{4.16}
$$

Second, let $\boldsymbol{F}:=(\nabla\boldsymbol{\varphi})\boldsymbol{U}^{-1}$ and $\boldsymbol{N}:=\nabla(\boldsymbol{a}_3(\boldsymbol{\varphi}))\boldsymbol{U}^{-1}$, where $\boldsymbol{U}$ is defined in (4.6). Then a series of straightforward calculations show that, for every immersion $\boldsymbol{\varphi}\in W^{1,4}(\omega;\mathbb{R}^3)$ such that $\boldsymbol{a}_3(\boldsymbol{\varphi})\in W^{1,4}(\omega;\mathbb{R}^3)$, the following relations hold a.e. in ω:

$$
\begin{aligned}
&a^{\alpha\beta}a_{\alpha\beta}(\boldsymbol{\varphi})=|\boldsymbol{F}|^2,\\
&a^{\alpha\beta}b_{\alpha\beta}(\boldsymbol{\varphi})=-\boldsymbol{F}:\boldsymbol{N},\\
&a^{\alpha\beta}c_{\alpha\beta}(\boldsymbol{\varphi})=|\boldsymbol{N}|^2,\\
&a^{\alpha\sigma}a^{\beta\tau}a_{\alpha\beta}(\boldsymbol{\varphi})a_{\sigma\tau}(\boldsymbol{\varphi})=|\boldsymbol{F}\boldsymbol{F}^T|^2,\\
&a^{\alpha\sigma}a^{\beta\tau}b_{\alpha\beta}(\boldsymbol{\varphi})b_{\sigma\tau}(\boldsymbol{\varphi})=[\boldsymbol{F}\boldsymbol{F}^T]:[\boldsymbol{N}\boldsymbol{N}^T],\\
&a^{\alpha\sigma}a^{\beta\tau}c_{\alpha\beta}(\boldsymbol{\varphi})c_{\sigma\tau}(\boldsymbol{\varphi})=|\boldsymbol{N}\boldsymbol{N}^T|^2,
\end{aligned}
\tag{4.17}
$$

where the notation ":" means the inner product between two matrices of the same type.

Using these relations in the definition of $\mathcal{A}_2(\boldsymbol{\varphi})$ implies that

$$\mathcal{A}_2(\boldsymbol{\varphi}) = \frac{\varepsilon}{2} W_3(\boldsymbol{F},\boldsymbol{N}),$$

where the function W_3 is that defined in the statement of Lemma 4.2 with the constant C replaced by the constant C_1 appearing in the statement of Theorem 4.1. Therefore, since the convergences (4.10) imply that

$$\begin{aligned} \boldsymbol{F}_n &:= (\nabla \boldsymbol{\varphi}_n)\boldsymbol{U}^{-1} && \rightharpoonup (\nabla \boldsymbol{\psi})\boldsymbol{U}^{-1} \quad \text{in } L^4(\omega,\mathbb{R}^{3\times 2}), \\ \boldsymbol{N}_n &:= \nabla(\boldsymbol{a}_3(\boldsymbol{\varphi}_n))\boldsymbol{U}^{-1} && \rightharpoonup (\nabla \boldsymbol{\zeta})\boldsymbol{U}^{-1} \quad \text{in } L^4(\omega,\mathbb{R}^{3\times 2}), \end{aligned}$$

we infer from Lemma 4.2 that

$$\begin{aligned} &\int_\omega W_3\big((\nabla \boldsymbol{\psi})\boldsymbol{U}^{-1},(\nabla \boldsymbol{\zeta})\boldsymbol{U}^{-1}\big)\sqrt{a}\,dy \\ &\leq \liminf_{n\to\infty} \int_\omega W_3(\boldsymbol{F}_n,\boldsymbol{N}_n)\sqrt{a}\,dy \\ &= \frac{2}{\varepsilon}\liminf_{n\to\infty} \int_\omega \mathcal{A}_2(\boldsymbol{\varphi}_n)\sqrt{a}\,dy. \end{aligned} \tag{4.18}$$

Third, let $\boldsymbol{F_0} := (\nabla \boldsymbol{\theta})\boldsymbol{U}^{-1}$ and $\boldsymbol{N_0} := \nabla(\boldsymbol{a}_3(\boldsymbol{\theta}))\boldsymbol{U}^{-1}$. Then, after a series of straightforward calculations, we deduce that

$$a^{\alpha\beta}a_{\alpha\beta} = |\boldsymbol{F}_0|^2, \quad a^{\alpha\beta}b_{\alpha\beta} = -\boldsymbol{F}_0 : \boldsymbol{N}_0, \quad a^{\alpha\beta}c_{\alpha\beta} = |\boldsymbol{N}_0|^2 \tag{4.19}$$

for every $y \in \overline{\omega}$. From this, (4.17) and definition of $\mathcal{A}_3(\boldsymbol{\varphi})$, we obtain

$$\mathcal{A}_3(\boldsymbol{\varphi}) = \frac{\varepsilon}{2}\frac{\lambda\mu}{\lambda+2\mu} W_{4,a}(\cdot,\boldsymbol{F},\boldsymbol{N}),$$

where the function $W_{4,a}$ is the function W_4 defined in the statement of Lemma 4.3 with

$$\begin{aligned} A(y) &= l\frac{\lambda+2\mu}{4(\lambda+\mu)}|\boldsymbol{F}_0(y)|^2, \\ B(y) &= \frac{8\varepsilon^2}{3}\big(\boldsymbol{F}_0(y) : \boldsymbol{N}_0(y)\big), \\ C(y) &= \frac{\varepsilon^4}{5}\left(\frac{320}{9l}\frac{\mu(\lambda+\mu)}{\lambda+2\mu}|\boldsymbol{N}_0(y)|^2\right). \end{aligned}$$

It is easy to see that these $A(y), B(y)$ and $C(y)$ satisfy the assumptions of Lemma 4.3. Therefore,

$$\begin{aligned}&\frac{\varepsilon}{2}\frac{\lambda\mu}{\lambda+2\mu}\int_\omega W_{4,a}\big(\cdot,(\nabla\boldsymbol{\psi})\boldsymbol{U}^{-1},(\nabla\boldsymbol{\zeta})\boldsymbol{U}^{-1}\big)\sqrt{a}dy\\ \leq&\frac{\varepsilon}{2}\frac{\lambda\mu}{\lambda+2\mu}\liminf_{n\to\infty}\int_\omega W_{4,a}(\cdot,\boldsymbol{F}_n,\boldsymbol{N}_n)\sqrt{a}dy\\ =&\liminf_{n\to\infty}\int_\omega \mathcal{A}_3(\boldsymbol{\varphi}_n)\sqrt{a}dy. \end{aligned}\tag{4.20}$$

Fourth, a series of straightforward calculations show that, for all immersion $\boldsymbol{\varphi}\in W^{1,4}(\omega;\mathbb{R}^3)$ such that $\boldsymbol{a}_3(\boldsymbol{\varphi})\in W^{1,4}(\omega;\mathbb{R}^3)$, the following relations hold a.e. in ω:

$$\begin{aligned}&a^{\alpha\sigma}a^{\beta\tau}a_{\alpha\beta}a_{\sigma\tau}(\boldsymbol{\varphi})=|\boldsymbol{F}\boldsymbol{F}_0^T|^2,\\ &a^{\alpha\sigma}a^{\beta\tau}b_{\alpha\beta}b_{\sigma\tau}(\boldsymbol{\varphi})=-(\boldsymbol{F}\boldsymbol{F}_0^T):(\boldsymbol{N}\boldsymbol{N}_0^T),\\ &a^{\alpha\sigma}a^{\beta\tau}c_{\alpha\beta}c_{\sigma\tau}(\boldsymbol{\varphi})=|\boldsymbol{N}\boldsymbol{N}_0^T|^2.\end{aligned}\tag{4.21}$$

This implies that

$$\mathcal{A}_4(\boldsymbol{\varphi})=\frac{\varepsilon}{2}\mu W_{4,b}\big(\cdot,\boldsymbol{F}\boldsymbol{F}_0^T,\boldsymbol{N}\boldsymbol{N}_0^T\big),$$

where the function $W_{4,b}$ coincides with the function W_4 defined in the statement of Lemma 4.3 with

$$A(y)=l\frac{\lambda+2\mu}{4(\lambda+\mu)},\quad B(y)=\frac{8\varepsilon^2}{3},\quad C(y)=\frac{\varepsilon^4}{5}\left(\frac{1600}{9l}\frac{\mu(\lambda+\mu)}{\lambda+2\mu}\right).$$

Again, it is easy to see that these $A(y), B(y)$ and $C(y)$ satisfy the assumptions of Lemma 4.3. Therefore, since the convergences (4.10) imply that

$$\begin{aligned}\boldsymbol{F}_n\boldsymbol{F}_0^T:=(\nabla\boldsymbol{\varphi}_n)\boldsymbol{U}^{-1}\boldsymbol{F}_0^T\quad&\rightharpoonup\ (\nabla\boldsymbol{\psi})\boldsymbol{U}^{-1}\boldsymbol{F}_0^T\quad\text{in } L^4(\omega,\mathbb{R}^{3\times3}),\\ \boldsymbol{N}_n\boldsymbol{N}_0^T:=\nabla\big(\boldsymbol{a}_3(\boldsymbol{\varphi}_n)\big)\boldsymbol{U}^{-1}\boldsymbol{N}_0^T\quad&\rightharpoonup\ (\nabla\boldsymbol{\zeta})\boldsymbol{U}^{-1}\boldsymbol{N}_0^T\quad\text{in } L^4(\omega,\mathbb{R}^{3\times3}),\end{aligned}$$

we infer from Lemma 4.3 that

$$\begin{aligned}&\frac{\varepsilon}{2}\mu\int_\omega W_{4,b}\big(\cdot,(\nabla\boldsymbol{\psi})\boldsymbol{U}^{-1}\boldsymbol{F}_0^T,(\nabla\boldsymbol{\zeta})\boldsymbol{U}^{-1}\boldsymbol{N}_0^T\big)\sqrt{a}dy\\ \leq&\frac{\varepsilon}{2}\mu\liminf_{n\to\infty}\int_\omega W_{4,b}\big(\cdot,\boldsymbol{F}_n\boldsymbol{F}_0^T,\boldsymbol{N}_n\boldsymbol{N}_0^T\big)\sqrt{a}dy\\ =&\liminf_{n\to\infty}\int_\omega\mathcal{A}_4(\boldsymbol{\varphi}_n)\sqrt{a}dy.\end{aligned}\tag{4.22}$$

Fifthly, using in particular that

$$a^{\alpha\beta}c_{\alpha\beta}=4H^2-2K,$$

we deduce from (4.17), (4.19) and (4.21) that

$$\begin{aligned}\mathcal{A}_5(\boldsymbol{\varphi}):=\frac{\varepsilon^5}{10}\Bigg[&\left(2(1-s_0)(\lambda+\mu)C_2-\frac{\lambda\mu}{\lambda+2\mu}\left(\frac{320}{9l}\frac{\mu(\lambda+\mu)}{\lambda+2\mu}\right)-C_1\frac{2\lambda\mu}{\lambda+2\mu}\right)|\boldsymbol{N}_0|^2|\boldsymbol{N}|^2\\&-\left(\mu\left(\frac{160}{9l}\frac{\mu(\lambda+\mu)}{\lambda+2\mu}\right)+2\mu C_1\right)\left|\boldsymbol{N}\boldsymbol{N}_0^T\right|^2\Bigg].\end{aligned}$$

Thus,

$$\mathcal{A}_5(\boldsymbol{\varphi})=\frac{\varepsilon^5}{10}W_{5,a}(\cdot,\boldsymbol{N}),$$

where $W_{5,a}$ is the function W_5 defined in Lemma 4.4 with $\overline{\boldsymbol{N}}$ replaced by $\boldsymbol{N}_0$,

$$\begin{aligned}&A=2(1-s_0)(\lambda+\mu)C_2-\frac{\lambda\mu}{\lambda+2\mu}\left(\frac{320}{9l}\frac{\mu(\lambda+\mu)}{\lambda+2\mu}\right)-C_1\frac{2\lambda\mu}{\lambda+2\mu},\\&B=\mu\left(\frac{160}{9l}\frac{\mu(\lambda+\mu)}{\lambda+2\mu}\right)+2\mu C_1.\end{aligned}$$

From the assumptions on C_1 and C_2, we deduce that A and B defined above satisfy $A\geq B\geq 0$ and thus by Lemma 4.4 we have

$$\begin{aligned}&\frac{\varepsilon^5}{10}\int_\omega W_{5,a}\left(\cdot,(\nabla\boldsymbol{\zeta})\boldsymbol{U}^{-1}\right)\sqrt{a}\,dy\\&\leq\frac{\varepsilon^5}{10}\liminf_{n\to\infty}\int_\omega W_{5,a}(\cdot,\boldsymbol{N}_n)\sqrt{a}\,dy\\&=\liminf_{n\to\infty}\int_\omega\mathcal{A}_5(\boldsymbol{\varphi}_n)\sqrt{a}\,dy.\end{aligned}\tag{4.23}$$

Finally, let $\boldsymbol{S}_3:\overline{\omega}\to\mathbb{R}^{2\times 2}$ be the matrix field defined by

$$\boldsymbol{S}_3(y):=\left(a_{\alpha\beta}(y)+r^2(y)c_{\alpha\beta}(y)\right),\quad\forall y\in\overline{\omega},$$

and $\boldsymbol{S}_3(\boldsymbol{\varphi}):\overline{\omega}\to\mathbb{R}^{2\times 2}$ the matrix field defined by

$$\boldsymbol{S}_3(\boldsymbol{\varphi}):=\left(a_{\alpha\beta}(\boldsymbol{\varphi})+r^2c_{\alpha\beta}(\boldsymbol{\varphi})\right).$$

Introducing the matrix

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

we note that

$$g^+(g_+^{\alpha\beta}) = J(g_{\alpha\beta}^+)J^T, \quad g^-(g_-^{\alpha\beta}) = J(g_{\alpha\beta}^-)J^T, \quad a(a^{\alpha\beta}) = J(a_{\alpha\beta})J^T.$$

Then

$$\begin{aligned} \mathcal{A}_6(\boldsymbol{\varphi}) := & \frac{\varepsilon^5}{10}(\lambda+\mu)C_2(4H^2-2K) \\ & \times \left[\frac{1}{r^2 a} \mathrm{Tr}\left(J S_1 J^T \left(g_{\alpha\beta}^+(\boldsymbol{\varphi}) \right) \right) + \frac{1}{r^2 a} \mathrm{Tr}\left(J S_2 J^T \left(g_{\alpha\beta}^-(\boldsymbol{\varphi}) \right) \right) \right. \\ & \left. + \frac{2(1-s_0)}{r^2 a} \mathrm{Tr}\left(J S_3 J^T S_3(\boldsymbol{\varphi}) \right) - 2(1-s_0) a^{\sigma\tau} c_{\sigma\tau}(\boldsymbol{\varphi}) \right], \end{aligned} \tag{4.24}$$

where S_1 and S_2 are matrix fields defined in Lemma 4.5. Since these matrix fields are symmetric and nonnegative definite, so are the matrix fields JS_1J^T and JS_2J^T. Then there exists a unique symmetric and nonnegative definite matrix fields $T_1 : \overline{\omega} \to \mathbb{R}^{2\times 2}$, $T_2 : \overline{\omega} \to \mathbb{R}^{2\times 2}$ with continuous components such that, for all $y \in \overline{\omega}$,

$$\begin{aligned} J S_1(y) J^T &= T_1(y)^2, \\ J S_2(y) J^T &= T_2(y)^2. \end{aligned}$$

By some simple calculations, we also have

$$\begin{aligned} &\mathrm{Tr}\left(S_3(y)\right) > 0, \quad \forall y \in \overline{\omega}, \\ &\det\left(S_3(y)\right) > 0, \quad \forall y \in \overline{\omega}, \end{aligned}$$

and thus $S_3(y)$ is symmetric and positive definite at all points $y \in \overline{\omega}$. This implies that $JS_3(y)J^T$ is symmetric and positive definite, and thus there exists a unique symmetric and nonnegative definite matrix field $T_3 : \overline{\omega} \to \mathbb{R}^{2\times 2}$ with continuous components such that, for all $y \in \overline{\omega}$,

$$J S_3(y) J^T = T_3(y)^2.$$

Similarly, there exists a unique symmetric and semi-positive definite matrix field with continuous components $T : \overline{\omega} \to \mathbb{R}^{2\times 2}$ such that, for all $y \in \overline{\omega}$,

$$J\left(c_{\alpha\beta}(y)\right) J^T = T(y)^2.$$

The above equalities and (4.24) yield

$$
\begin{aligned}
\mathcal{A}_6(\boldsymbol{\varphi}) := &\frac{\varepsilon^5}{10}(\lambda+\mu)C_2(4H^2-2K)\\
&\times\Big[\frac{1}{r^2a}\big|\big(\nabla\boldsymbol{\varphi}+r\nabla\boldsymbol{a}_3(\boldsymbol{\varphi})\big)\boldsymbol{T}_1\big|^2+\frac{1}{r^2a}\big|\big(\nabla\boldsymbol{\varphi}-r\nabla\boldsymbol{a}_3(\boldsymbol{\varphi})\big)\boldsymbol{T}_2\big|^2\\
&\qquad+\frac{2(1-s_0)}{r^2a}|\nabla\boldsymbol{\varphi}\boldsymbol{T}_3|^2+\frac{2(1-s_0)}{a}r^2|\nabla\boldsymbol{a}_3(\boldsymbol{\varphi})\boldsymbol{T}|^2\Big]\\
=&\frac{\varepsilon^5}{10}(\lambda+\mu)C_2\Big[W_{5,b}\big(\cdot,\big(\nabla\boldsymbol{\varphi}+r\nabla\boldsymbol{a}_3(\boldsymbol{\varphi})\big)\boldsymbol{T}_1\big)+W_{5,b}\big(\cdot,\big(\nabla\boldsymbol{\varphi}-r\nabla\boldsymbol{a}_3(\boldsymbol{\varphi})\big)\boldsymbol{T}_2\big)\\
&\qquad+W_{5,c}\big(\cdot,(\nabla\boldsymbol{\varphi})\boldsymbol{T}_3\big)+W_{5,d}\big(\cdot,\big(\nabla\boldsymbol{a}_3(\boldsymbol{\varphi})\big)\boldsymbol{T}\big)\Big],
\end{aligned}
$$

where $W_{5,b}$ is the function W_5 defined in Lemma 4.4 with

$$\overline{\boldsymbol{N}}(y)=\frac{1}{\sqrt{2}}\boldsymbol{I}_2,\quad A(y)=\frac{4H^2(y)-2K(y)}{r^2(y)a(y)},\quad B(y)=0,$$

$\boldsymbol{I}_2$ denotes the identity matrix in $\mathbb{R}^{2\times 2}$, $W_{5,c}$ is the function W_5 defined in Lemma 4.4 with

$$\overline{\boldsymbol{N}}(y)=\frac{1}{\sqrt{2}}\boldsymbol{I}_2,\quad A(y)=\frac{2(1-s_0)\big(4H^2(y)-2K(y)\big)}{r^2(y)a(y)},\quad B(y)=0,$$

and $W_{5,d}$ is the function W_5 defined in Lemma 4.4 with

$$\overline{\boldsymbol{N}}(y)=\frac{1}{\sqrt{2}}\boldsymbol{I}_2,\quad A(y)=\frac{2(1-s_0)\big(4H^2(y)-2K(y)\big)}{a(y)},\quad B(y)=0.$$

It follows from Lemma 4.4 and (4.10) that

$$
\begin{aligned}
&\frac{\varepsilon^5}{10}(\lambda+\mu)C_2\int_\omega\Big[W_{5,b}\big(\cdot,(\nabla\boldsymbol{\psi}+r\nabla\boldsymbol{\zeta})\boldsymbol{T}_1\big)+W_{5,b}\big(\cdot,(\nabla\boldsymbol{\psi}-r\nabla\boldsymbol{\zeta})\boldsymbol{T}_2\big)\\
&\qquad+W_{5,c}\big(\cdot,(\nabla\boldsymbol{\psi})\boldsymbol{T}_3\big)+W_{5,d}\big(\cdot,(\nabla\boldsymbol{\zeta})\boldsymbol{T}\big)\Big]dy\\
\le&\frac{\varepsilon^5}{10}(\lambda+\mu)C_2\int_\omega\Big[W_{5,b}\big(\cdot,\big(\nabla\boldsymbol{\varphi}_n+r\nabla\boldsymbol{a}_3(\boldsymbol{\varphi}_n)\big)\boldsymbol{T}_1\big)\\
&\qquad+W_{5,b}\big(\cdot,\big(\nabla\boldsymbol{\varphi}_n-r\nabla\boldsymbol{a}_3(\boldsymbol{\varphi}_n)\big)\boldsymbol{T}_2\big)+W_{5,c}\big(\cdot,(\nabla\boldsymbol{\varphi})\boldsymbol{T}_3\big)\\
&\qquad+W_{5,d}\big(\cdot,(\nabla\boldsymbol{a}_3(\boldsymbol{\varphi}_n)\boldsymbol{T})\Big]dy\\
=&\liminf_{n\to\infty}\int_\omega\mathcal{A}_6(\boldsymbol{\varphi}_n)\sqrt{a}\,dy.
\end{aligned}\tag{4.25}
$$

Step 5. Adding inequalities (4.16), (4.18), (4.20), (4.22), (4.23) and (4.25) shows that

$$\begin{aligned}
&\int_\omega \Big[\frac{\varepsilon}{2} W_1^*\big(\cdot,\boldsymbol{\zeta}\cdot(\partial_1\boldsymbol{\psi}\wedge\partial_2\boldsymbol{\psi})\big)+\frac{\varepsilon^5}{10}W_{2,a}\big(\cdot,\boldsymbol{\zeta}\cdot\big[(\partial_1\boldsymbol{\psi}+r\partial_1\boldsymbol{\zeta})\wedge(\partial_2\boldsymbol{\psi}+r\partial_2\boldsymbol{\zeta})\big]\big)\\
&\qquad+\frac{\varepsilon^5}{10}W_{2,b}\big(\cdot,\boldsymbol{\zeta}\cdot\big[(\partial_1\boldsymbol{\psi}-r\partial_1\boldsymbol{\zeta})\wedge(\partial_2\boldsymbol{\psi}-r\partial_2\boldsymbol{\zeta})\big]\big)\Big]dy\\
&\quad+\frac{\varepsilon}{2}\int_\omega W_3\big((\nabla\boldsymbol{\psi})\boldsymbol{U}^{-1},(\nabla\boldsymbol{\zeta})\boldsymbol{U}^{-1}\big)\sqrt{a}\,dy\\
&\quad+\frac{\varepsilon}{2}\frac{\lambda\mu}{\lambda+2\mu}\int_\omega W_{4,a}\big(\cdot,(\nabla\boldsymbol{\psi})\boldsymbol{U}^{-1},(\nabla\boldsymbol{\zeta})\boldsymbol{U}^{-1}\big)\sqrt{a}\,dy\\
&\quad+\frac{\varepsilon}{2}\mu\int_\omega W_{4,b}\big(\cdot,(\nabla\boldsymbol{\psi})\boldsymbol{U}^{-1}\boldsymbol{F}_0^T,(\nabla\boldsymbol{\zeta})\boldsymbol{U}^{-1}\boldsymbol{N}_0^T\big)\sqrt{a}\,dy\\
&\quad+\frac{\varepsilon^5}{10}\int_\omega W_{5,a}\big(\cdot,(\nabla\boldsymbol{\zeta})\boldsymbol{U}^{-1}\big)\sqrt{a}\,dy\\
&\quad+\frac{\varepsilon^5}{10}(\lambda+\mu)C_2\int_\omega\Big[W_{5,b}\big(\cdot,(\nabla\boldsymbol{\psi}+r\nabla\boldsymbol{\zeta})\boldsymbol{T}_1\big)+W_{5,b}\big(\cdot,(\nabla\boldsymbol{\psi}-r\nabla\boldsymbol{\zeta})\boldsymbol{T}_2\big)\\
&\qquad\qquad+W_{5,c}\big(\cdot,(\nabla\boldsymbol{\psi})\boldsymbol{T}_3\big)+W_{5,d}\big(\cdot,(\nabla\boldsymbol{\zeta})\boldsymbol{T}\big)\Big]dy\\
&\le\liminf_{n\to\infty}\int_\omega\Big[\sum_{i=1}^6\mathcal{A}_i(\boldsymbol{\varphi}_n)+\mathcal{A}_7\Big]\sqrt{a}\,dy\\
&=\liminf_{n\to\infty}\int_\omega\widetilde{W}_K(\boldsymbol{\varphi}_n)\sqrt{a}\,dy=\liminf_{n\to\infty}\widetilde{J}_K(\boldsymbol{\varphi}_n)+L_K(\boldsymbol{\psi}).
\end{aligned}\tag{4.26}$$

Since $\liminf_{n\to\infty}\widetilde{J}_K(\boldsymbol{\varphi}_n)<\infty$, we deduce in particular that

$$W_1^*\big(\cdot,\boldsymbol{\zeta}\cdot(\partial_1\boldsymbol{\psi}\wedge\partial_2\boldsymbol{\psi})\big)<\infty\quad\text{a.e. in }\omega,$$

so that by Lemma 4.1,

$$\boldsymbol{\zeta}\cdot(\partial_1\boldsymbol{\psi}\wedge\partial_2\boldsymbol{\psi})>0\quad\text{a.e. in }\omega.$$

A similar argument shows that

$$\boldsymbol{\zeta}\cdot\big((\partial_1\boldsymbol{\psi}+r\partial_1\boldsymbol{\zeta})\wedge(\partial_2\boldsymbol{\psi}+r\partial_2\boldsymbol{\zeta})\big)>0\quad\text{a.e. in }\omega,\tag{4.27}$$

$$\boldsymbol{\zeta}\cdot\big((\partial_1\boldsymbol{\psi}-r\partial_1\boldsymbol{\zeta})\wedge(\partial_2\boldsymbol{\psi}-r\partial_2\boldsymbol{\zeta})\big)>0\quad\text{a.e. in }\omega.\tag{4.28}$$

Since $|\boldsymbol{a}_3(\boldsymbol{\varphi}_n)|=1$ and $\boldsymbol{a}_3(\boldsymbol{\varphi}_n)\cdot\partial_\alpha\boldsymbol{\varphi}_n=0$ a.e. in ω, the convergences established in Step 3 of the proof imply that

$$|\boldsymbol{\zeta}|=1, \quad \boldsymbol{\zeta}\cdot\partial_\alpha\boldsymbol{\psi}=0 \quad \text{a.e. in } \omega.$$

The last two relations show that $\boldsymbol{\zeta}$ is a unit normal vector field to the surface $\boldsymbol{\psi}(\omega)$. Since we also proved that

$$\boldsymbol{\zeta}\cdot(\partial_1\boldsymbol{\psi}\wedge\partial_2\boldsymbol{\psi})>0 \quad \text{a.e. in } \omega,$$

we have

$$\boldsymbol{\zeta}=\boldsymbol{a}_3(\boldsymbol{\psi}) \quad \text{a.e. in } \omega. \tag{4.29}$$

Consequently, inequality (4.26) can be recast as

$$\int_\omega \left[\sum_{i=1}^{6}\mathcal{A}_i(\boldsymbol{\psi})+\mathcal{A}_7\right]\sqrt{a}\,dy \leq \liminf_{n\to\infty}\widetilde{J}_K(\boldsymbol{\varphi}_n)+L_K(\boldsymbol{\psi}),$$

or equivalently, as

$$\widetilde{J}_K(\boldsymbol{\psi})\leq\liminf_{n\to\infty}\widetilde{J}_K(\boldsymbol{\varphi}_n).$$

Since relations (4.11), (4.27)-(4.29) imply that

$$\boldsymbol{\psi}\in\boldsymbol{V}_K(\omega),$$

we have proved that the function $\boldsymbol{\psi}$ minimizes the functional $\widetilde{J}_K$ over $\boldsymbol{V}_K(\omega)$. The proof is complete. □

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