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Upper Bounds for Korn's Constants in General Domains

Cristinel Mardare[1,*] and Thai Ha Nguyen[2]

[1] *Laboratoire Jacques-Louis Lions, Sorbonne Université, Paris, France.*
[2] *Department of Mathematics, City University of Hong Kong, Kowloon, Hong Kong SAR, P.R. China.*

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Abstract. We estimate the constants appearing in Korn inequalities in terms of the norm of a linear and continuous inverse to the divergence operator defined on a same domain, and of a few scalar parameters modeling the shape of the domain.

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1 Introduction

It is well-known that, given any domain $\Omega \subset \mathbb{R}^d, d \geq 2$, and any non-empty relatively open subset Γ_0 of the boundary of Ω, there exist constants $C_1 = C_1(\Omega)$, $C_2 = C_2(\Omega,\Gamma_0), C_3 = C_3(\Omega)$ and $C_4 = C_4(\Omega)$ such that

$$\inf_{\boldsymbol{r}\in \mathrm{Rig}(\Omega)} \|\boldsymbol{u}-\boldsymbol{r}\|_{H^1(\Omega)} \leq C_1 \|\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}\|_{L^2(\Omega)}, \qquad \forall \boldsymbol{u} \in H^1(\Omega;\mathbb{R}^d), \tag{1.1}$$

$$\|\boldsymbol{u}\|_{H^1(\Omega)} \leq C_2 \|\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}\|_{L^2(\Omega)}, \qquad \forall \boldsymbol{u} \in H^1_{\Gamma_0}(\Omega;\mathbb{R}^d), \tag{1.2}$$

$$\|\boldsymbol{u}\|_{H^1(\Omega)} \leq C_3 \|\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}\|_{L^2(\Omega)} + C_4 \|\boldsymbol{u}\|_{L^2(\Omega)}, \quad \forall \boldsymbol{u} \in H^1(\Omega;\mathbb{R}^d). \tag{1.3}$$

*Corresponding author. *Email addresses:* `tthnguyen3-c@my.cityu.edu.hk` (T. H. Nguyen), `cristinel.mardare@sorbonne-universite.fr` (C. Mardare)

Inequalities (1.2) and (1.3) constitute respectively the first and second Korn inequalities, according to most textbooks, especially in the theory of elasticity. Various proofs have been given to these inequalities, see, e.g. Duvaut and Lions [6], Fichera [7], Friedrichs [8], Gobert [9], Hlaváček [10], Hlaváček and Nečas [11], Miyoshi [16], Mosolov and Myasnikov [17], Nitsche [18], Temam [19].

The dependence of these constants on the domain Ω, and on Γ_0 for the second constant, is however not well known, save for an upper bound of order of $(r/R)^d$ for domains Ω contained in a ball with radius R and star-shaped with respect to a ball of radius r, and for some sharper upper bounds for particular domains Ω, see, e.g. Horgan [12, 13], Kondratev and Oleinik [14], Ciarlet *et al.* [5]. These sharper bounds are often needed in solid and fluid mechanics in domains depending on a small parameter, where the magnitude of Korn's constant with respect to this parameter is essential in justifying dimensionally reduced models by convergence theorems when the parameter goes to zero or to infinity.

The objective of this paper is to give new proofs to the three Korn inequalities mentioned above, based on a new approach that has the advantage of yielding constants that depend explicitly on several parameters associated with the domain Ω. More specifically, this new approach yields constants $C_1,\dots,C_4$ that depend explicitly on (an upper bound $K(\Omega)$ of) the norm of a linear and continuous inverse to the divergence operator in the domain Ω (see Lemma 3.1), beside the constants appearing in Poincaré's, Poincaré-Wirtinger's, and trace, inequalities in Sobolev spaces.

The paper is organised as follows. Section 2 specifies the notation and definitions used in all ensuing sections. Section 3 estimates the constant C_1 appearing in Korn's inequality (1.1). The key result is the first inequality of Theorem 3.1, which provides a first estimate of the constant C_1 (see Corollary 3.1). Then the results of Theorem 3.1 and Corollary 3.1 are generalized, and improved, in Theorem 3.2. Section 4 estimates the constant C_2 appearing in Korn's inequality (1.2). The main result is Theorem 4.1, which is proved by combining Theorem 3.1 with Poincaré's inequality, trace inequality, and an inequality about the eigenvalues of symmetric matrices (Lemma 4.1). Section 5 estimates the constants C_3 and C_4 appearing in Korn's inequality (1.3). The main result is Theorem 5.3, which generalizes the previous Theorem 5.2 (at the expense of considerably more difficult and at places technical proof), itself a simpler generalization of Theorem 5.1. The proofs of all three theorems rely on the inequalities established in Theorem 3.2, combined with a method based on Fubini's theorem to estimate the norm of the anti-symmetric matrix appearing in these inequalities. Finally, Section 6 summarises all the estimates obtained in this paper for the constants C_1, C_2, C_3 and C_4, with all the details necessary to apply them in future works without having to read the entire paper.

2 Main definitions and notation

The Euclidean norm in $\mathbb{R}^d$, the Frobenius norm in $\mathbb{R}^{m\times n}$, the d-dimensional Lebesgue measure, and the $(d-1)$-dimensional Haussdorff measure, all are denoted by $|\cdot|$. In particular, if Ω is a domain in $\mathbb{R}^d$ and Γ_0 is a relatively open subset of its boundary, then

$$|\Omega| = \int_\Omega \mathrm{d}x, \quad |\Gamma_0| = \int_{\Gamma_0} \mathrm{d}\sigma.$$

A domain in $\mathbb{R}^d$ is by definition a bounded and connected open subset $\Omega \subset \mathbb{R}^d$ with a Lipschitz-continuous boundary $\Gamma := \partial\Omega$ in the sense of Adams [1], the set Ω being then locally on only one side of Γ. A generic point in $\mathbb{R}^d$ is denoted $x = (x_1, x_2, \dots, x_d)$. Partial derivative operators, in the classical or weak sense, with respect to the coordinates x_j are denoted by $\partial_j := \partial/\partial x_j$.

Boldface letters denote vectors, matrices, vector fields and matrix fields to distinguish them from scalars and real-valued functions.

The space of all real square matrices of order n, $n = 1,2,\dots$, are denoted $\mathbb{M}^n$. Then

$$\mathbb{M}^n = \mathbb{S}^n \otimes \mathbb{A}^n,$$

where

$$\mathbb{S}^n := \{\boldsymbol{S} \in \mathbb{M}^n; \boldsymbol{S}^T = \boldsymbol{S}\}, \quad \mathbb{A}^n := \{\boldsymbol{A} \in \mathbb{M}^n; \boldsymbol{A}^T = -\boldsymbol{A}\}$$

respectively denote the space of all real symmetric matrices of order n and the space of all real anti-symmetric matrices of order n.

Given any sufficiently smooth field $\boldsymbol{u} = (u_i) : \Omega \subset \mathbb{R}^d \to \mathbb{R}^d$, its gradient is the matrix field

$$\boldsymbol{\nabla u} = (\partial_j u_i) : \Omega \to \mathbb{M}^d,$$

where the index of the derivative (j is this case) is the column index. Then

$$\boldsymbol{\nabla u} = \boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u} + \boldsymbol{\nabla}_{\mathrm{a}}\boldsymbol{u} \quad \text{in } \Omega,$$

where

$$\begin{aligned}
\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u} &= \big(e_{ij}(\boldsymbol{u})\big) := \frac{1}{2}\big(\boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^T\big), \\
\boldsymbol{\nabla}_{\mathrm{a}}\boldsymbol{u} &= \big(a_{ij}(\boldsymbol{u})\big) := \frac{1}{2}\big(\boldsymbol{\nabla u} - (\boldsymbol{\nabla u})^T\big)
\end{aligned}$$

respectively denote the symmetric and anti-symmetric parts of $\nabla \boldsymbol{u}$. Note that the components of these matrix fields satisfy

$$\partial_j u_i = e_{ij}(\boldsymbol{u}) + a_{ij}(\boldsymbol{u}),$$

where

$$e_{ij}(\boldsymbol{u})=\frac{1}{2}(\partial_j u_i+\partial_i u_j),\quad a_{ij}(\boldsymbol{u})=\frac{1}{2}(\partial_j u_i-\partial_i u_j).$$

Given any domain $\Omega\subset\mathbb{R}^d$, any $1\le p\le+\infty$, and any integer $n\ge 1$, the notation $L^p(\Omega;\mathbb{R}^n)$ denotes the spaces of all (equivalence classes modulo the equality a.e. (almost everywhere)) vector fields $\boldsymbol{u}=(u_i):\Omega\to\mathbb{R}^n$ with components u_i is the usual Lebesgue space $L^p(\Omega)$. The space $L^p(\Omega;\mathbb{R}^n)$ is equipped with the norm denoted and defined by

$$\|\boldsymbol{u}\|_{L^p(\Omega)}:=\|\,|\boldsymbol{u}|\,\|_{L^p(\Omega)},$$

where $|\boldsymbol{u}|:=(\sum_{i=1}^n|u_i|^p)^{1/p}$. Since Ω is a domain, so in particular bounded by the above definition, the subspace

$$L^2_0(\Omega):=\left\{v\in L^2(\Omega);\int_\Omega v(x)\,\mathrm{d}x=0\right\}\subset L^2(\Omega),$$

and the average

$$\fint_\Omega f:=\frac{1}{|\Omega|}\int_\Omega f(x)\,\mathrm{d}x\in\mathbb{R}$$

of functions $f\in L^2(\Omega)$ are well defined.

Given any domain $\Omega\subset\mathbb{R}^d$, the notation $H^1(\Omega)$ denotes the usual Sobolev space of functions $f\in L^2(\Omega)$ that possess (weak) first order partial derivatives in $L^2(\Omega)$. The notation $H^1(\Omega;\mathbb{R}^n)$ denotes the spaces of all (equivalence classes modulo the equality a.e.) vector fields $\boldsymbol{u}=(u_i):\Omega\to\mathbb{R}^n$ with components $u_i\in H^1(\Omega)$, equipped with the norm denoted and defined by

$$\|\boldsymbol{u}\|_{H^1(\Omega)}:=\left(\|\boldsymbol{u}\|^2_{L^2(\Omega)}+\|\boldsymbol{\nabla}\boldsymbol{u}\|^2_{L^2(\Omega)}\right)^{1/2}.$$

Given any relatively open subset $\Gamma_0\subset\Gamma$ of the boundary $\Gamma:=\partial\Omega$,

$$H^1_{\Gamma_0}(\Omega):=\left\{v\in H^1(\Omega);\ v|_{\Gamma_0}=0\right\}$$

denotes the kernel of the trace operator $v\in H^1(\Omega)\to v|_{\Gamma_0}\in L^2(\Gamma_0)$. If $\Gamma_0=\Gamma$, we use the shorter notation

$$H^1_0(\Omega):=H^1_\Gamma(\Omega),$$

and we denote by $H^{-1}(\Omega)$ the dual space of $H^1_0(\Omega)$ equipped with the norm (remember that Ω is bounded, so Poincaré inequality holds in $H^1_0(\Omega)$)

$$\|v\|_{H^1_0(\Omega)}:=\|\boldsymbol{\nabla}v\|_{L^2(\Omega)}$$

for all $v \in H_0^1(\Omega)$. Thus,

$$\|f\|_{H^{-1}(\Omega)} := \sup_{v \in H_0^1(\Omega), \|v\|_{H_0^1(\Omega)} \le 1} |\ll f, v \gg|,$$

where $\ll f, v \gg := f(v)$, so that

$$|\ll f, v \gg| \le \|f\|_{H^{-1}(\Omega)} \|\boldsymbol{\nabla} v\|_{L^2(\Omega)}$$

for all $f \in H^{-1}(\Omega)$ and all $v \in H_0^1(\Omega)$. Finally, we recall that a vector field $\boldsymbol{r} \in H^1(\Omega;\mathbb{R}^d)$ such that $\boldsymbol{\nabla}_s \boldsymbol{r} = \boldsymbol{0}$ in $L^2(\Omega;\mathbb{S}^d)$ is called an infinitesimal rigid vector field of Ω, and that the set of all infinitesimal rigid vector fields, which is denoted and defined by

$$\mathrm{Rig}(\Omega) := \left\{ \boldsymbol{r} \in H^1(\Omega;\mathbb{R}^d);\ \boldsymbol{\nabla}_s \boldsymbol{r} = \boldsymbol{0} \text{ in } L^2(\Omega;\mathbb{S}^d) \right\},$$

satisfies

$$\begin{aligned}\mathrm{Rig}(\Omega) = \big\{ \boldsymbol{r} : \Omega \to \mathbb{R}^d;\ &\text{there exist } \boldsymbol{a} \in \mathbb{R}^d \text{ and } \boldsymbol{B} \in \mathbb{A}^d \\ &\text{such that } \boldsymbol{r}(x) = \boldsymbol{a} + \boldsymbol{B}x \text{ for all } x \in \mathbb{R}^d \big\}.\end{aligned}$$

Note that the space $\mathrm{Rig}(\Omega)$ is finite-dimensional, so the infimum in Korn's inequality (1.1) is attained.

3 Korn's inequalities for fields in $H^1(\Omega;\mathbb{R}^n)/\mathrm{Rig}(\Omega)$

The definition of a domain $\Omega \subseteq \mathbb{R}^d$ is given in the previous section.

The main objective of this section is to establish the first Korn inequality mentioned in the introduction, together with an estimate of the constant C_1 in terms of the constant K appearing in Lemma 3.1 below about the divergence equation. Since the space $\mathrm{Rig}(\Omega)$ is finite dimensional, this inequality can be recast as follows: Given any domain $\Omega \subset \mathbb{R}^d$, there exists a constant $C_1 = C_1(\Omega)$ with the following property: For every $\boldsymbol{u} \in H^1(\Omega;\mathbb{R}^d)$, there exists an infinitesimal rigid vector field $\boldsymbol{r}(\boldsymbol{u}) \in \mathrm{Rig}(\Omega)$ such that

$$\|\boldsymbol{u} - \boldsymbol{r}(\boldsymbol{u})\|_{H^1(\Omega)} \le C_1 \|\boldsymbol{\nabla}_s \boldsymbol{u}\|_{L^2(\Omega)}. \tag{3.1}$$

We will establish explicit estimates for both the constant C_1 and the rigid vector field $\boldsymbol{r}(\boldsymbol{u})$ in the above inequality.

The starting point of these estimates is the following well-known result about the surjectivity of the divergence operator. The constant $K=K(\Omega)$ appearing in this result will play a fundamental role in this paper, as all Korn constants in this paper depend on it.

Lemma 3.1 (Divergence Equation). *Given any domain $\Omega\subset\mathbb{R}^d$, there exists a constant $K=K(\Omega)$ with the following property: For every $f\in L^2_0(\Omega)$, there exists a vector field $\boldsymbol{v}=\boldsymbol{v}(f)\in H^1_0(\Omega;\mathbb{R}^d)$ such that*

$$\operatorname{div}\boldsymbol{v}=f \quad in \ \ \Omega, \quad \|\boldsymbol{\nabla v}\|_{L^2(\Omega)}\le K\|f\|_{L^2(\Omega)}.$$

Proof. See, e.g. Bogovskii [2], Borchers and Sohr [3], or Ciarlet [4]. $\square$

The next theorem show how Korn's inequalities can be deduced from Lemma 3.1 with a constant of the same order.

Theorem 3.1. *Given any domain Ω in $\mathbb{R}^d$, let $K=K(\Omega)$ denote the constant appearing in Lemma 3.1. Then, for all vector fields $\boldsymbol{u}\in H^1(\Omega;\mathbb{R}^d)$,*

$$\begin{aligned}\left\|\boldsymbol{\nabla}_a\boldsymbol{u}-\fint_\Omega\boldsymbol{\nabla}_a\boldsymbol{u}(x)\,\mathrm{d}x\right\|_{L^2(\Omega)}&\le 2K\sqrt{d}\|\boldsymbol{\nabla}_s\boldsymbol{u}\|_{L^2(\Omega)},\\ \left\|\boldsymbol{\nabla}\boldsymbol{u}-\fint_\Omega\boldsymbol{\nabla}_a\boldsymbol{u}(x)\,\mathrm{d}x\right\|_{L^2(\Omega)}&\le (1+2K\sqrt{d})\|\boldsymbol{\nabla}_s\boldsymbol{u}\|_{L^2(\Omega)}.\end{aligned}$$

Proof. Let $\boldsymbol{u}\in H^1(\Omega,\mathbb{R}^d)$ and $\boldsymbol{B}:=\fint_\Omega\boldsymbol{\nabla}_{\mathrm{a}}\boldsymbol{u}(x)\,\mathrm{d}x$. Since $\boldsymbol{\nabla u}=\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}+\boldsymbol{\nabla}_{\mathrm{a}}\boldsymbol{u}$, it suffices to prove that

$$\|\boldsymbol{\nabla}_{\mathrm{a}}\boldsymbol{u}-\boldsymbol{B}\|_{L^2(\Omega)}\le 2K\sqrt{d}\|\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}\|_{L^2(\Omega)}.$$

Since all the components of the matrix field $(\boldsymbol{\nabla}_{\mathrm{a}}\boldsymbol{u}-\boldsymbol{B})$ belong to the space $L^2_0(\Omega)$, Lemma 3.1 shows that, for every $i,j=1,\dots,d$, there exists a vector field $\boldsymbol{v}_{ij}=(v^k_{ij})^d_{k=1}\in H^1_0(\Omega;\mathbb{R}^d)$ such that

$$\begin{aligned}&\operatorname{div}\boldsymbol{v}_{ij}=(\boldsymbol{\nabla}_{\mathrm{a}}\boldsymbol{u}-\boldsymbol{B})_{ij} \quad \text{in } \ \Omega,\\ &\|\boldsymbol{\nabla}_{\mathrm{a}}\boldsymbol{v}_{ij}\|_{L^2(\Omega)}\le K\|(\boldsymbol{\nabla}_{\mathrm{a}}\boldsymbol{u}-\boldsymbol{B})_{ij}\|_{L^2(\Omega)}.\end{aligned}$$

Consequently,

$$\begin{aligned}\|\boldsymbol{\nabla}_{\mathrm{a}}\boldsymbol{u}-\boldsymbol{B}\|^2_{L^2(\Omega)}&=\sum_{i,j}\int_\Omega(\boldsymbol{\nabla}_{\mathrm{a}}\boldsymbol{u}-\boldsymbol{B})_{ij}(\operatorname{div}\boldsymbol{v}_{ij})\,\mathrm{d}x\\ &=-\sum_{i,j,k}\ll\partial_k(\boldsymbol{\nabla}_{\mathrm{a}}\boldsymbol{u})_{ij},v^k_{ij}\gg\end{aligned}$$

so that, on the one hand,

$$
\begin{aligned}
\|\boldsymbol{\nabla}_{\mathrm{a}}\boldsymbol{u}-\boldsymbol{B}\|^2_{L^2(\Omega)} &\leq \sum_{i,j,k}\|\partial_k a_{ij}\|_{H^{-1}(\Omega)}\|\nabla v^k_{ij}\|_{L^2(\Omega)}\\
&\leq K\sum_{i,j}\|\nabla a_{ij}\|_{H^{-1}(\Omega)}\|(\boldsymbol{\nabla}_{\mathrm{a}}\boldsymbol{u}-\boldsymbol{B})_{ij}\|_{L^2(\Omega)}\\
&\leq K\bigg(\sum_{i,j}\|\nabla a_{ij}\|^2_{H^{-1}(\Omega)}\bigg)^{1/2}\|\boldsymbol{\nabla}_{\mathrm{a}}\boldsymbol{u}-\boldsymbol{B}\|_{L^2(\Omega)},
\end{aligned}
$$

where $a_{ij}:=(\boldsymbol{\nabla}_{\mathrm{a}}\boldsymbol{u})_{ij}$ denote the components of the antisymmetric matrix field $\boldsymbol{\nabla}_{\mathrm{a}}\boldsymbol{u}$. Note that the first inequality above is deduced by using the definition of $H^{-1}(\Omega)$ as the dual space of the space $H^1_0(\Omega)$ equipped with the norm $f\in H^1_0(\Omega)\to\|\nabla f\|_{L^2(\Omega)}$.

Let $e_{ij}:=(\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u})_{ij}$ denote the components of the symmetric matrix field $\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}$. Then, for each $i,j,k\in\{1,2,\ldots,d\}$,

$$
\|\partial_i e_{kj}\|_{H^{-1}(\Omega)}=\sup_{v\in H^1_0(\Omega),\|\nabla v\|_{L^2(\Omega)}\leq 1}\left|\int_\Omega e_{kj}\partial_i v\,\mathrm{d}x\right|\leq\|e_{kj}\|_{L^2(\Omega)}.
$$

Since

$$
\partial_k a_{ij}=\partial_i e_{kj}-\partial_j e_{ki}\quad\text{in }\ H^{-1}(\Omega)
$$

(as is deduced immediately from the definition of the functions a_{ij} and e_{ij}), we deduce that, on the other hand,

$$
\begin{aligned}
\sum_{i,j}\|\nabla a_{ij}\|^2_{H^{-1}(\Omega)} &=\sum_{i,j,k}\|\partial_k a_{ij}\|^2_{H^{-1}(\Omega)}\\
&\leq\sum_{i,j,k}\Big(\|\partial_i e_{kj}\|_{H^{-1}(\Omega)}+\|\partial_j e_{ki}\|_{H^{-1}(\Omega)}\Big)^2\\
&\leq 2\sum_{i,j,k}\Big(\|e_{kj}\|^2_{L^2(\Omega)}+\|e_{ki}\|^2_{L^2(\Omega)}\Big)\\
&=4d\|\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}\|^2_{L^2(\Omega)}.
\end{aligned}
$$

Then we infer from the above inequalities that

$$
\begin{aligned}
\|\boldsymbol{\nabla}_{\mathrm{a}}\boldsymbol{u}-\boldsymbol{B}\|_{L^2(\Omega)} &\leq K\bigg(\sum_{i,j}\|\nabla a_{ij}\|^2_{H^{-1}(\Omega)}\bigg)^{1/2}\\
&\leq 2K\sqrt{d}\|\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}\|_{L^2(\Omega)}.
\end{aligned}
$$

This completes the proof of the theorem. □

In the proof of the next Korn inequality we will need Poincaré-Wirtinger's inequality, stated below (without proof) for clarity.

Lemma 3.2 (Poincaré-Wirtinger). *Given any domain $\Omega \subset \mathbb{R}^d$, there exists a constant $W(\Omega)$ such that, for all $\boldsymbol{v} \in H^1(\Omega;\mathbb{R}^d)$,*

$$\left\| \boldsymbol{v} - \fint_\Omega \boldsymbol{v}(x)\,\mathrm{d}x \right\|_{L^2(\Omega)} \leq W(\Omega) \|\boldsymbol{\nabla} \boldsymbol{v}\|_{L^2(\Omega)}.$$

We are now in a position to estimate the constant C_1 appearing in Korn's inequality (1.1) stated in the introduction.

Corollary 3.1. *Given any domain Ω in $\mathbb{R}^d$, let $K = K(\Omega)$ and $W = W(\Omega)$ respectively denote the constants appearing in Lemmas* 3.1 *and* 3.2*. With every vector field $\boldsymbol{u} \in H^1(\Omega;\mathbb{R}^d)$, we associate the infinitesimal rigid vector field $\boldsymbol{r}(\boldsymbol{u}) \in \mathrm{Rig}(\Omega)$ defined by, for all $x \in \Omega$,*

$$\boldsymbol{r}(\boldsymbol{u})(x) := \fint_\Omega \boldsymbol{u}(x)\,\mathrm{d}x + \left(\fint_\Omega \boldsymbol{\nabla}_a \boldsymbol{u}(x)\,\mathrm{d}x \right) \left(x - \fint_\Omega x\,\mathrm{d}x \right).$$

Then, for all vector fields $\boldsymbol{u} \in H^1(\Omega;\mathbb{R}^d)$,

$$\begin{aligned} &\|\boldsymbol{u} - \boldsymbol{r}(\boldsymbol{u})\|_{L^2(\Omega)} \leq W(1 + 2K\sqrt{d}) \|\boldsymbol{\nabla}_s \boldsymbol{u}\|_{L^2(\Omega)}, \\ &\inf_{\boldsymbol{r} \in \mathrm{Rig}(\Omega)} \|\boldsymbol{u} - \boldsymbol{r}\|_{H^1(\Omega)} \leq \|\boldsymbol{u} - \boldsymbol{r}(\boldsymbol{u})\|_{H^1(\Omega)} \leq (1 + W)(1 + 2K\sqrt{d}) \|\boldsymbol{\nabla}_s \boldsymbol{u}\|_{L^2(\Omega)}. \end{aligned}$$

Proof. Let

$$\boldsymbol{B} := \fint_\Omega \boldsymbol{\nabla}_{\mathrm{a}} \boldsymbol{u}(x)\,\mathrm{d}x, \quad \boldsymbol{a} := \fint_\Omega \big(\boldsymbol{u}(x) - \boldsymbol{B}x\big)\,\mathrm{d}x.$$

Since the matrix $\boldsymbol{B}$ is anti-symmetric, the vector field $\boldsymbol{r}(\boldsymbol{u}) : \Omega \to \mathbb{R}^d$,

$$\big(\boldsymbol{r}(\boldsymbol{u})\big)(x) := \boldsymbol{a} + \boldsymbol{B}x, \quad \forall x \in \Omega$$

belongs to $\mathrm{Rig}(\Omega)$. Hence,

$$\inf_{\boldsymbol{r} \in \mathrm{Rig}(\Omega)} \|\boldsymbol{u} - \boldsymbol{r}\|_{H^1(\Omega)} \leq \|\boldsymbol{u} - \boldsymbol{r}(\boldsymbol{u})\|_{H^1(\Omega)}.$$

Since $\int_\Omega (\boldsymbol{u} - \boldsymbol{r}(\boldsymbol{u}))(x)\,\mathrm{d}x = \boldsymbol{0}$, Poincaré-Wirtinger inequality (Lemma 3.2) further implies that

$$\|\boldsymbol{u} - \boldsymbol{r}(\boldsymbol{u})\|_{L^2(\Omega)} \leq W \big\| \boldsymbol{\nabla}\big(\boldsymbol{u} - \boldsymbol{r}(\boldsymbol{u})\big) \big\|_{L^2(\Omega)} = W \|\boldsymbol{\nabla} \boldsymbol{u} - \boldsymbol{B}\|_{L^2(\Omega)}.$$

Then both inequalities of the theorem follow from the inequality

$$\|\boldsymbol{\nabla u}-\boldsymbol{B}\|_{L^2(\Omega)}\leq(1+2K\sqrt{d})\|\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}\|_{L^2(\Omega)}$$

established in Theorem 3.1. □

We conclude this section by showing that the above Korn inequalities also hold with other choices of infinitesimal rigid vector fields $\boldsymbol{r}(\boldsymbol{u})$ in their left-hand side. This will be useful in Section 5, to obtain better (meaning smaller) constants C_3 and C_4 in Korn's inequality (1.3) stated in the introduction.

To this end, we need two lemmas. The first one is about the continuity of the trace operator for fields in $H^1(\Omega)$, recalled below for reader's convenience.

Lemma 3.3 (Trace Operator)**.** *Given any domain $\Omega\subset\mathbb{R}^d$ and any non-empty relatively open subset Γ_0 of the boundary of Ω, there exists a constant $T(\Omega,\Gamma_0)$ such that, for all $\boldsymbol{u}\in H^1(\Omega;\mathbb{R}^d)$,*

$$\|\boldsymbol{u}\|_{L^2(\Gamma_0)}\leq T(\Omega,\Gamma_0)\|\boldsymbol{u}\|_{H^1(\Omega)}.$$

Note the abuse of notation in the left-hand side of the above inequality, where $\boldsymbol{u}$ denotes the image of $\boldsymbol{u}$ by the trace operator from $H^1(\Omega;\mathbb{R}^d)$ into $L^2(\Gamma_0;\mathbb{R}^d)$.

The second one states the following variants of Poincaré-Wirtinger inequality, whose proofs are given for completeness.

Lemma 3.4 (Poincaré-Wirtinger)**.** *Let Ω be a domain in $\mathbb{R}^d$ and let $W(\Omega)$ be the constant defined in Lemma* 3.2*.*

(a) Given any non-empty open subset D of Ω, there exists a constant $W(\Omega,D)$ such that

$$W(\Omega,D)\leq 2\sqrt{\frac{|\Omega|}{|D|}}W(\Omega),$$

and, for all $\boldsymbol{v}\in H^1(\Omega;\mathbb{R}^d)$,

$$\left\|\boldsymbol{v}-\fint_D \boldsymbol{v}(x)\,\mathrm{d}x\right\|_{L^2(\Omega)}\leq W(\Omega,D)\|\boldsymbol{\nabla v}\|_{L^2(\Omega)}.$$

(b) Given any non-empty relatively open subset Γ_1 of a Lipschitz hypersurface Γ in $\overline{\Omega}$ (e.g. $\Gamma=\partial\Omega$), there exists a constant $W(\Omega,\Gamma_1)$ such that

$$W(\Omega,\Gamma_1)\leq W(\Omega)+\big(1+W(\Omega)\big)T(\Omega,\Gamma_1)\sqrt{\frac{|\Omega|}{|\Gamma_1|}},$$

where $T(\Omega,\Gamma_1)$ *denotes the constant defined in Lemma* 3.3*, and, for all* $\boldsymbol{v}\in H^1(\Omega;\mathbb{R}^d)$*,*

$$\left\|\boldsymbol{v}-\fint_{\Gamma_1}\boldsymbol{v}(x)\,\mathrm{d}x\right\|_{L^2(\Omega)}\le W(\Omega,\Gamma_1)\|\boldsymbol{\nabla v}\|_{L^2(\Omega)}.$$

Proof. Let $\boldsymbol{v}\in H^1(\Omega;\mathbb{R}^d)$. Then, using in particular Cauchy-Schwarz inequality and Lemma 3.2, we have

$$\begin{aligned}
\left\|\boldsymbol{v}-\fint_D\boldsymbol{v}\right\|_{L^2(\Omega)}&\le\left\|\boldsymbol{v}-\fint_\Omega\boldsymbol{v}\right\|_{L^2(\Omega)}+|\Omega|^{1/2}\left|\fint_D\left(\boldsymbol{v}-\fint_\Omega\boldsymbol{v}\right)\right|\\
&\le\left\|\boldsymbol{v}-\fint_\Omega\boldsymbol{v}\right\|_{L^2(\Omega)}+\sqrt{\frac{|\Omega|}{|D|}}\left\|\boldsymbol{v}-\fint_\Omega\boldsymbol{v}\right\|_{L^2(D)}\\
&\le\left(1+\sqrt{\frac{|\Omega|}{|D|}}\right)\left\|\boldsymbol{v}-\fint_\Omega\boldsymbol{v}\right\|_{L^2(\Omega)}\\
&\le 2\sqrt{\frac{|\Omega|}{|D|}}W(\Omega)\|\boldsymbol{\nabla v}\|_{L^2(\Omega)}.
\end{aligned}$$

To prove the second inequality of the lemma, we first notice that

$$\begin{aligned}
\left\|\boldsymbol{v}-\fint_\Omega\boldsymbol{v}\right\|_{L^2(\Gamma_1)}&\le T(\Omega,\Gamma_1)\left\|\boldsymbol{v}-\fint_\Omega\boldsymbol{v}\right\|_{H^1(\Omega)}\\
&\le T(\Omega,\Gamma_1)\left(\left\|\boldsymbol{v}-\fint_\Omega\boldsymbol{v}\right\|_{L^2(\Omega)}+\|\nabla f\|_{L^2(\Omega)}\right).
\end{aligned}$$

Using in particular Lemma 3.3, we next deduce that

$$\begin{aligned}
\left\|\boldsymbol{v}-\fint_{\Gamma_1}\boldsymbol{v}\right\|_{L^2(\Omega)}&\le\left\|\boldsymbol{v}-\fint_\Omega\boldsymbol{v}\right\|_{L^2(\Omega)}+|\Omega|^{1/2}\left|\fint_{\Gamma_1}\left(\boldsymbol{v}-\fint_\Omega\boldsymbol{v}\right)\right|\\
&\le\left\|\boldsymbol{v}-\fint_\Omega\boldsymbol{v}\right\|_{L^2(\Omega)}+\sqrt{\frac{|\Omega|}{|\Gamma_1|}}\left\|\boldsymbol{v}-\fint_\Omega\boldsymbol{v}\right\|_{L^2(\Gamma_1)}\\
&\le\left\|\boldsymbol{v}-\fint_\Omega\boldsymbol{v}\right\|_{L^2(\Omega)}+\sqrt{\frac{|\Omega|}{|\Gamma_1|}}T(\Omega,\Gamma_1)\left(\left\|\boldsymbol{v}-\fint_\Omega\boldsymbol{v}\right\|_{L^2(\Omega)}+\|\boldsymbol{\nabla v}\|_{L^2(\Omega)}\right).
\end{aligned}$$

Then, by using the usual Poincaré-Wirtinger inequality (Lemma 4.2), we finally deduce that

$$\left\|\boldsymbol{v}-\fint_{\Gamma_1}\boldsymbol{v}\right\|_{L^2(\Omega)}\le\left[\left(1+\sqrt{\frac{|\Omega|}{|\Gamma_1|}}T(\Omega,\Gamma_1)\right)W(\Omega)+\sqrt{\frac{|\Omega|}{|\Gamma_1|}}T(\Omega,\Gamma_1)\right]\|\boldsymbol{\nabla v}\|_{L^2(\Omega)},$$

which is precisely the second inequality of the lemma. $\square$

We are now in a position to prove the following theorem, which generalizes both Theorem 3.1 and Corollary 3.1.

Theorem 3.2. *Given any domain Ω in $\mathbb{R}^d$, let $K=K(\Omega)$ denote the constant appearing in Lemma* 3.1. *Given any open set $\Omega'\subset\Omega$, let*

$$C_0(\Omega,\Omega'):=4K\sqrt{d}\frac{|\Omega|^{1/2}}{|\Omega'|^{1/2}}.$$

Then, for all vector fields $\boldsymbol{u}\in H^1(\Omega;\mathbb{R}^d)$, the following two inequalities hold:

$$\begin{aligned}\left\|\boldsymbol{\nabla}_a\boldsymbol{u}-\fint_{\Omega'}\boldsymbol{\nabla}_a\boldsymbol{u}\,\mathrm{d}x\right\|_{L^2(\Omega)}&\le C_0(\Omega,\Omega')\|\boldsymbol{\nabla}_s\boldsymbol{u}\|_{L^2(\Omega)},\\ \left\|\boldsymbol{\nabla}\boldsymbol{u}-\fint_{\Omega'}\boldsymbol{\nabla}_a\boldsymbol{u}\,\mathrm{d}x\right\|_{L^2(\Omega)}&\le\big(1+C_0(\Omega,\Omega')\big)\|\boldsymbol{\nabla}_s\boldsymbol{u}\|_{L^2(\Omega)}.\end{aligned}\tag{3.2}$$

Furthermore, given any non-empty set $\Omega''\subset\overline{\Omega}$ that is either on open subset of $\mathbb{R}^d$, or a relatively open subset of a Lipschitz hypersurface in $\overline{\Omega}$ (such as the boundary of Ω), define $\boldsymbol{r}(\boldsymbol{u})\in\mathrm{Rig}(\Omega)$ by letting, at each $x\in\Omega$,

$$\boldsymbol{r}(\boldsymbol{u})(x):=\left(\fint_{\Omega''}\boldsymbol{u}\right)+\left(\fint_{\Omega'}\boldsymbol{\nabla}_a\boldsymbol{u}\right)\left(x-\fint_{\Omega''}x\right).$$

Then, for all vector fields $\boldsymbol{u}\in H^1(\Omega;\mathbb{R}^d)$, the following two inequalities hold:

$$\begin{aligned}&\|\boldsymbol{u}-\boldsymbol{r}(\boldsymbol{u})\|_{L^2(\Omega)}\le W(\Omega'')\big(1+C_0(\Omega,\Omega')\big)\|\boldsymbol{\nabla}_s\boldsymbol{u}\|_{L^2(\Omega)},\\ &\inf_{\boldsymbol{r}\in\mathrm{Rig}(\Omega)}\|\boldsymbol{u}-\boldsymbol{r}\|_{H^1(\Omega)}\le\|\boldsymbol{u}-\boldsymbol{r}(\boldsymbol{u})\|_{H^1(\Omega)}\\ &\qquad\qquad\le\big(1+W(\Omega'')\big)\big(1+C_0(\Omega,\Omega')\big)\|\boldsymbol{\nabla}_s\boldsymbol{u}\|_{L^2(\Omega)}.\end{aligned}\tag{3.3}$$

Proof. We first prove the inequalities (3.2). Let $\boldsymbol{u}\in H^1(\Omega,\mathbb{R}^d)$. Since $\boldsymbol{\nabla}\boldsymbol{u}=\boldsymbol{\nabla}_\mathrm{s}\boldsymbol{u}+\boldsymbol{\nabla}_\mathrm{a}\boldsymbol{u}$, it suffices to prove that

$$\left\|\boldsymbol{\nabla}_\mathrm{a}\boldsymbol{u}-\fint_{\Omega'}\boldsymbol{\nabla}_\mathrm{a}\boldsymbol{u}\right\|_{L^2(\Omega)}\le4K\sqrt{\frac{d|\Omega|}{|\Omega'|}}\|\boldsymbol{\nabla}_\mathrm{s}\boldsymbol{u}\|_{L^2(\Omega)}.$$

To this end, notice that

$$\left|\fint_{\Omega'}\boldsymbol{\nabla}_\mathrm{a}\boldsymbol{u}-\fint_{\Omega}\boldsymbol{\nabla}_\mathrm{a}\boldsymbol{u}\right|=\left|\fint_{\Omega'}\left(\boldsymbol{\nabla}_\mathrm{a}\boldsymbol{u}-\fint_{\Omega}\boldsymbol{\nabla}_\mathrm{a}\boldsymbol{u}\right)\right|\le|\Omega'|^{-1/2}\left\|\boldsymbol{\nabla}_\mathrm{a}\boldsymbol{u}-\fint_{\Omega}\boldsymbol{\nabla}_\mathrm{a}\boldsymbol{u}\right\|_{L^2(\Omega)}.$$

Then, using the first inequality proved in Theorem 3.1 in the right-hand side above shows that

$$\begin{aligned}\left\|\boldsymbol{\nabla}_{\mathrm{a}}\boldsymbol{u}-\fint_{\Omega'}\boldsymbol{\nabla}_{\mathrm{a}}\boldsymbol{u}\right\|_{L^2(\Omega)} &\le \left\|\boldsymbol{\nabla}_{\mathrm{a}}\boldsymbol{u}-\fint_{\Omega}\boldsymbol{\nabla}_{\mathrm{a}}\boldsymbol{u}\right\|_{L^2(\Omega)}+|\Omega|^{1/2}\left|\fint_{\Omega'}\boldsymbol{\nabla}_{\mathrm{a}}\boldsymbol{u}-\fint_{\Omega}\boldsymbol{\nabla}_{\mathrm{a}}\boldsymbol{u}\right| \\ &\le \left(1+\frac{|\Omega|^{1/2}}{|\Omega'|^{1/2}}\right)\left\|\boldsymbol{\nabla}_{\mathrm{a}}\boldsymbol{u}-\fint_{\Omega}\boldsymbol{\nabla}_{\mathrm{a}}\boldsymbol{u}\right\|_{L^2(\Omega)} \\ &\le \left(2\frac{|\Omega|^{1/2}}{|\Omega'|^{1/2}}\right)2K\sqrt{d}\,\|\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}\|_{L^2(\Omega)}.\end{aligned}$$

This proves (3.2).

To prove inequalities (3.3), let

$$\boldsymbol{B}:=\fint_{\Omega'}\boldsymbol{\nabla}_{\mathrm{a}}\boldsymbol{u},\quad \boldsymbol{a}:=\left(\fint_{\Omega''}\boldsymbol{u}\right)-\boldsymbol{B}\left(\fint_{\Omega''}x\right),$$

and

$$\big(\boldsymbol{r}(\boldsymbol{u})\big)(x):=\boldsymbol{a}+\boldsymbol{B}x,\quad \forall x\in\Omega.$$

Since the matrix $\boldsymbol{B}$ is anti-symmetric, the vector field $\boldsymbol{r}(\boldsymbol{u}):\Omega\to\mathbb{R}^d$ belongs to $\mathrm{Rig}(\Omega)$. Hence,

$$\inf_{\boldsymbol{r}\in\mathrm{Rig}(\Omega)}\|\boldsymbol{u}-\boldsymbol{r}\|_{H^1(\Omega)}\le\|\boldsymbol{u}-\boldsymbol{r}(\boldsymbol{u})\|_{H^1(\Omega)}\le\|\boldsymbol{u}-\boldsymbol{r}(\boldsymbol{u})\|_{L^2(\Omega)}+\|\boldsymbol{\nabla}\boldsymbol{u}-\boldsymbol{B}\|_{L^2(\Omega)},$$

on the one hand.

On the other hand, since $\int_{\Omega''}(\boldsymbol{u}-\boldsymbol{r}(\boldsymbol{u}))=\boldsymbol{0}$ (thanks to the definition of the vector $\boldsymbol{a}$), we have by Lemma 3.4 that

$$\|\boldsymbol{u}-\boldsymbol{r}(\boldsymbol{u})\|_{L^2(\Omega)}\le W(\Omega,\Omega'')\big\|\boldsymbol{\nabla}\big(\boldsymbol{u}-\boldsymbol{r}(\boldsymbol{u})\big)\big\|_{L^2(\Omega)}=W\|\boldsymbol{\nabla}\boldsymbol{u}-\boldsymbol{B}\|_{L^2(\Omega)}.$$

Then the inequalities (3.3) are obtained by combining the last two inequalities with the following inequality (see Theorem 3.2):

$$\|\boldsymbol{\nabla}\boldsymbol{u}-\boldsymbol{B}\|_{L^2(\Omega)}\le\big(1+C_0(\Omega,\Omega')\big)\|\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}\|_{L^2(\Omega)}.$$

The proof is complete. □

4 Korn's inequalities for fields in $H^1_{\Gamma_0}(\Omega;\mathbb{R}^n)$

The main objective of this section is to establish the second Korn inequality mentioned in the introduction, together with an explicit estimate of its constant C_2. To this end, we need an inequality about eigenvalues of matrices, which we now prove.

Lemma 4.1. *(a) Let* $\boldsymbol{A}=(a_{ij})\in\mathbb{M}^d$ *be a matrix whose components satisfy* $a_{ij}\geq 0$ *and, for all* $i\in\{1,2,\ldots,d\}$, $\sum_{j=1}^d a_{ij}=1$ *and* $\sum_{j=1}^d a_{ji}=1$. *Then, for all vectors* $\boldsymbol{x}=(x_i)\in\mathbb{R}^d$ *and* $\boldsymbol{y}=(y_i)\in\mathbb{R}^d$ *whose components satisfy* $x_1\leq x_2\leq\cdots\leq x_d$ *and* $y_1\leq y_2\leq\cdots\leq y_d$,

$$\sum_{i=1}^d\sum_{j=1}^d a_{ij}x_iy_j\geq x_1y_d+x_2y_{d-1}+\cdots+x_dy_1.$$

(b) Let $\boldsymbol{X},\boldsymbol{Y}\in\mathbb{S}^d$ *be two symmetric matrices with eigenvalues* $-\infty<x_1\leq x_2\leq\cdots\leq x_d<\infty$ *and* $-\infty<y_1\leq y_2\leq\cdots\leq y_d<\infty$. *Then*

$$\boldsymbol{X}:\boldsymbol{Y}\geq x_1y_d+x_2y_{d-1}+\cdots+x_dy_1.$$

Proof. We first prove part (a) of the lemma. Let $(\lambda_{ij})\in\mathbb{M}^d$ denote the matrix defined by $\lambda_{ij}=1$ if $i+j=d+1$ and $\lambda_{ij}=0$ otherwise, and let $\boldsymbol{B}=(b_{ij})\in\mathbb{M}^d$ be the matrix defined by $b_{ij}:=a_{ij}-\lambda_{ij}$. Then

$$\sum_{i=1}^d\sum_{j=1}^d a_{ij}x_iy_j=\sum_{i=1}^d\sum_{j=1}^d(\lambda_{ij}+b_{ij})x_iy_j=(x_1y_d+x_2y_{d-1}+\cdots+d_dy_1)+\boldsymbol{x}^T\boldsymbol{B}\boldsymbol{y}.$$

It suffices to prove that $\boldsymbol{x}^T\boldsymbol{B}\boldsymbol{y}\geq 0$. Note that, for every $i\in\{1,2,\ldots,d\}$,

$$\sum_{j=1}^d b_{ij}=\sum_{j=1}^d a_{ij}-\sum_{j=1}^d\lambda_{ij}=1-1=0, \tag{4.1}$$

and, for every $j\in\{1,2,\ldots,d\}$,

$$\sum_{j=1}^d b_{ij}=\sum_{j=1}^d a_{ij}-\sum_{j=1}^d\lambda_{ij}=1-1=0, \tag{4.2}$$

so that we have, for every pair of indices $k,\ell\in\{1,2,\ldots,d\}$,

$$\begin{aligned}\sum_{i=1}^k\sum_{j=1}^\ell b_{ij}&=\sum_{i=1}^k\left(-\sum_{j=\ell+1}^d b_{ij}\right)=-\sum_{j=\ell+1}^d\left(\sum_{i=1}^k b_{ij}\right)=-\sum_{j=\ell+1}^d\left(-\sum_{i=k+1}^d b_{ij}\right)\\&=\sum_{i=k+1}^d\sum_{j=\ell+1}^d b_{ij}.\end{aligned}$$

Since $b_{ij}=a_{ij}\geq 0$ whenever $i+j\neq d+1$, at least one of the two sets $\{b_{ij};1\leq i\leq k,$ $1\leq j\leq\ell\}$ and $\{b_{ij};\,k+1\leq i\leq d,\ \ell+1\leq j\leq d\}$ contains only non-negative numbers (if

$k+\ell\leq d$, then all the elements in the first set are non-negative, if $k+\ell\geq d$, then all the elements in the second set are non-negative). Then we infer from the previous relation that, on the one hand,

$$\sum_{i=1}^{k}\sum_{j=1}^{\ell} b_{ij}\geq 0 \quad \text{for every pair of indices} \quad k,\ell\in\{1,2,\ldots,d\}.$$

On the other hand, denoting $\boldsymbol{e}_k:=(1,\ldots,1,0,\ldots,0)^T\in\mathbb{R}^d$ (the first k components of $\boldsymbol{e}_k$ are equal to 1), we deduce $\boldsymbol{B}\boldsymbol{e}_d=\boldsymbol{0}$ and $\boldsymbol{e}_d^T\boldsymbol{B}=\boldsymbol{0}$ respectively by relations (4.1) and (4.2), then that

$$\begin{aligned}\boldsymbol{x}^T\boldsymbol{B}\boldsymbol{y}&=\left(\sum_{k=1}^{d-1}(x_k-x_{k+1})\boldsymbol{e}_k+x_d\boldsymbol{e}_d\right)^T\boldsymbol{B}\left(\sum_{\ell=1}^{d-1}(y_\ell-y_{\ell+1})\boldsymbol{e}_\ell+y_d\boldsymbol{e}_d\right)\\&=\sum_{k=1}^{d-1}\sum_{\ell-1}^{d-1}(x_k-x_{k+1})(y_\ell-y_{\ell+1})\boldsymbol{e}_k^T\boldsymbol{B}\boldsymbol{e}_\ell.\end{aligned}$$

Therefore,

$$\boldsymbol{x}^T\boldsymbol{B}\boldsymbol{y}=\sum_{k=1}^{d-1}\sum_{l-1}^{d-1}(x_k-x_{k+1})(y_l-y_{l+1})\left(\sum_{i=1}^{k}\sum_{j=1}^{l}b_{ij}\right)\geq 0.$$

Part (b) of the lemma follows from part (a) as follows. Let

$$\boldsymbol{D_X}=\operatorname{diag}(x_1,x_2,\ldots,x_d),\quad \boldsymbol{D_Y}=\operatorname{diag}(y_1,y_2,\ldots,y_d)$$

denote the diagonal matrices formed by the eigenvalues of $\boldsymbol{X}$ and $\boldsymbol{Y}$. Then there exist orthogonal matrices $\boldsymbol{P}$ and $\boldsymbol{Q}$ such that $\boldsymbol{X}=\boldsymbol{P}^T\boldsymbol{D_X}\boldsymbol{P}$ and $\boldsymbol{Y}=\boldsymbol{Q}^T\boldsymbol{D_Y}\boldsymbol{Q}$. Consequently,

$$\boldsymbol{X}:\boldsymbol{Y}=\boldsymbol{P}^T\boldsymbol{D_X}\boldsymbol{P}:\boldsymbol{Q}^T\boldsymbol{D_Y}\boldsymbol{Q}=\boldsymbol{Q}\boldsymbol{P}^T\boldsymbol{D_X}\boldsymbol{P}\boldsymbol{Q}^T:\boldsymbol{D_Y}=\boldsymbol{R}^T\boldsymbol{D_X}\boldsymbol{R}:\boldsymbol{D_Y},$$

where $\boldsymbol{R}=\boldsymbol{P}\boldsymbol{Q}^T$, or equivalently ($[\cdot]_{ij}$ denotes the component at row i and column j of the matrix between the brackets),

$$\begin{aligned}\boldsymbol{X}:\boldsymbol{Y}&=\sum_{j=1}^{d}\left[\boldsymbol{R}^T\boldsymbol{D_X}\boldsymbol{R}\right]_{jj}y_j=\sum_{j=1}^{d}\left(\sum_{k=1}^{d}\sum_{i=1}^{d}[\boldsymbol{R}^T]_{jk}[\boldsymbol{D_X}]_{ki}[\boldsymbol{R}]_{ij}\right)y_j\\&=\sum_{j=1}^{d}\sum_{i=1}^{d}[\boldsymbol{R}^T]_{ji}x_i[\boldsymbol{R}]_{ij}y_j=\sum_{j=1}^{d}\sum_{i=1}^{d}([\boldsymbol{R}]_{ij})^2x_iy_j.\end{aligned}$$

Let $a_{ij} := ([\boldsymbol{R}]_{ij})^2$. Since the matrix $\boldsymbol{R}$ is orthogonal as a product of orthogonal matrices, we have $\boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{I}$, which implies in particular that

$$\sum_{j=1}^{d} a_{ij} = \sum_{j=1}^{d} ([\boldsymbol{R}]_{ij})^2 = 1, \quad \forall i \in \{1,2,\ldots,d\},$$
$$\sum_{i=1}^{d} a_{ij} = \sum_{i=1}^{d} ([\boldsymbol{R}]_{ij})^2 = 1, \quad \forall j \in \{1,2,\ldots,d\}.$$

Since in addition $a_{ij} \geq 0$ for all $i,j \in \{1,2,\ldots,d\}$, the matrix $\boldsymbol{A} = (a_{ij}) \in \mathbb{M}^d$ satisfies all the assumptions of part (a) of the lemma. Therefore,

$$\boldsymbol{X} : \boldsymbol{Y} = \sum_{i=1}^{d}\sum_{j=1}^{d} a_{ij} x_i y_j \geq x_1 y_d + x_2 y_{d-1} + \cdots + x_d y_1.$$

The proof is complete. □

We will also need Poincaré inequality, which are stated below (without proof) for clarity.

Lemma 4.2 (Poincaré)**.** *Given any domain $\Omega \subset \mathbb{R}^d$ and any non-empty relatively open subset Γ_0 of the boundary of Ω, there exists a constant $P(\Omega,\Gamma_0)$ such that, for all $\boldsymbol{u} \in H^1_{\Gamma_0}(\Omega;\mathbb{R}^d)$,*

$$\|\boldsymbol{u}\|_{L^2(\Omega)} \leq P(\Omega,\Gamma_0)\|\boldsymbol{\nabla}\boldsymbol{u}\|_{L^2(\Omega)}.$$

We are now in a position to prove the main result of this section, which gives an upper bound for the constant C_2 appearing in Korn's inequality (1.2) stated in the introduction.

Theorem 4.1 (Korn Inequality for Fields Satisfying Boundary Conditions)**.** *Given any domain Ω in $\mathbb{R}^d$, $d \geq 3$, and any non-empty relatively open subset $\Gamma_0 \subset \Gamma$ of the boundary $\Gamma := \partial\Omega$, let $x_0 := \fint_{\Gamma_0} x \,\mathrm{d}\Gamma_0$, let p_1 and p_2 denote the two smallest eigenvalues of the symmetric semi-positive definite matrix $\int_{\Gamma_0}(x-x_0)(x-x_0)^T \,\mathrm{d}x$, and $K = K(\Omega)$, $W = W(\Omega), P = P(\Omega,\Gamma_0)$ and $T = T(\Omega,\Gamma_0)$ be the constants appearing in Lemmas* 3.1, 3.2, 4.1 *and* 4.2*. Define the constant*

$$c_2 = c_2(\Omega,\Gamma_0) := (1+2K\sqrt{d})\left(1+T(1+W)\sqrt{\frac{d|\Omega|}{p_1+p_2}}\right).$$

Then, for all $\boldsymbol{u} \in H^1_{\Gamma_0}(\Omega;\mathbb{R}^d)$,

$$\|\boldsymbol{\nabla}\boldsymbol{u}\|_{L^2(\Omega)} \leq c_2\|\boldsymbol{\nabla}_s\boldsymbol{u}\|_{L^2(\Omega)}, \tag{4.3}$$
$$\|\boldsymbol{u}\|_{H^1(\Omega)} \leq c_2(1+P)\|\boldsymbol{\nabla}_s\boldsymbol{u}\|_{L^2(\Omega)}. \tag{4.4}$$

Proof. Inequality (4.4) is obtained by combining inequality (4.3) with Poincaré inequality (see Lemma 4.2).

Inequality (4.3) will be established below as a consequence of Korn's inequality established in Theorem 3.1, by using Lemmas 3.3, 4.1 and 4.2 to estimate the norm of the matrix appearing in this Korn inequality.

So let $\boldsymbol{u}\in H^1_{\Gamma_0}(\Omega;\mathbb{R}^d)$ and let $\boldsymbol{B}=\fint_\Omega \boldsymbol{\nabla}_{\mathrm{a}}\boldsymbol{u}\,\mathrm{d}x\in\mathbb{A}^d$ be the anti-symmetric matrix appearing in the first inequality of Theorem 3.1. Then Theorem 3.1 implies that

$$\|\boldsymbol{\nabla}\boldsymbol{u}-\boldsymbol{B}\|_{L^2(\Omega)}\le c_1\|\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}\|_{L^2(\Omega)},$$

where $c_1:=(1+2K\sqrt{d})$, then that

$$\|\boldsymbol{\nabla}\boldsymbol{u}\|_{L^2(\Omega)}\le c_1\|\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}\|_{L^2(\Omega)}+|\Omega|^{1/2}|\boldsymbol{B}|. \tag{4.5}$$

If $\boldsymbol{B}=\boldsymbol{0}$, then

$$\|\boldsymbol{\nabla}\boldsymbol{u}\|_{L^2(\Omega)}\le c_1\|\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}\|_{L^2(\Omega)},$$

and the first inequality of the theorem holds with $c_2=c_1$.

If $\boldsymbol{B}\neq\boldsymbol{0}$, we estimate the norm $|\boldsymbol{B}|$ appearing in the right-hand side of (4.5) in the following way. Since the trace of $\boldsymbol{u}$ vanishes on Γ_0, we infer from Lemma 3.3 that, for every vector $\boldsymbol{a}\in\mathbb{R}^d$,

$$\begin{aligned}\|\boldsymbol{a}+\boldsymbol{B}(x-x_0)\|_{L^2(\Gamma_0)}&=\big\|\boldsymbol{u}(x)-\big(\boldsymbol{a}+\boldsymbol{B}(x-x_0)\big)\big\|_{L^2(\Gamma_0)}\\&\le T\big\|\boldsymbol{u}(x)-\big(\boldsymbol{a}+\boldsymbol{B}(x-x_0)\big)\big\|_{H^1(\Omega)}.\end{aligned}$$

In particular, for

$$\boldsymbol{a}=\fint_\Omega\big(\boldsymbol{u}(x)-\boldsymbol{B}(x-x_0)\big)\,\mathrm{d}x,$$

we have

$$\|\boldsymbol{a}+\boldsymbol{B}(x-x_0)\|_{L^2(\Gamma_0)}\le T\left\|\boldsymbol{u}-\fint_\Omega\boldsymbol{u}(x)\,\mathrm{d}x-\boldsymbol{B}\left(x-\fint_\Omega x\,\mathrm{d}x\right)\right\|_{H^1(\Omega)},$$

which combined with Poincaré-Wirtinger inequality (Lemma 3.2) and with the first inequality of Theorem 3.1, shows that

$$\|\boldsymbol{a}+\boldsymbol{B}(x-x_0)\|_{L^2(\Gamma_0)}\le T(1+W)\|\boldsymbol{\nabla}\boldsymbol{u}-\boldsymbol{B}\|_{L^2(\Omega)}\le T(1+W)c_1\|\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}\|_{L^2(\Omega)}.$$

Besides, the definition of x_0 (see the statement of Theorem 4.1) implies that $\int_{\Gamma_0}(x-x_0)\,\mathrm{d}\sigma=0$, so that the left-hand side of the above inequality satisfies

$$\|\boldsymbol{a}+\boldsymbol{B}(x-x_0)\|^2_{L^2(\Gamma_0)}=\|\boldsymbol{a}\|^2_{L^2(\Gamma_0)}+\|\boldsymbol{B}(x-x_0)\|^2_{L^2(\Gamma_0)}\ge\|\boldsymbol{B}(x-x_0)\|^2_{L^2(\Gamma_0)}.$$

Then we infer from the previous inequality that

$$\|\boldsymbol{B}(x-x_0)\|_{L^2(\Gamma_0)} \leq T(1+W)c_1\|\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}\|_{L^2(\Omega)}. \tag{4.6}$$

We now estimate the left-hand side of the above inequality from below. Firstly, we recast its square as

$$\begin{aligned}
\|\boldsymbol{B}(x-x_0)\|^2_{L^2(\Omega)} &= \int_{\Gamma_0} \boldsymbol{B}(x-x_0)\cdot\boldsymbol{B}(x-x_0)\,\mathrm{d}x \\
&= \int_{\Gamma_0} \boldsymbol{B}^T\boldsymbol{B}:(x-x_0)(x-x_0)^T\,\mathrm{d}x \\
&= \boldsymbol{B}^T\boldsymbol{B}:\boldsymbol{M},
\end{aligned}$$

where

$$\boldsymbol{M} := \int_{\Gamma_0} (x-x_0)(x-x_0)^T\,\mathrm{d}\Gamma_0 \in \mathbb{M}^d,$$

$(x-x_0)$ being considered here as a column-vector in $\mathbb{R}^d$.

Second, let $\lambda \geq 0$ denote the largest eigenvalue of the symmetric and semi-positive definite matrix $\boldsymbol{B}^T\boldsymbol{B}$. Using that the matrix $\boldsymbol{B}$ is anti-symmetric and non-zero, we deduce that $\boldsymbol{B}^T\boldsymbol{B}=-\boldsymbol{B}^2$, that $\lambda>0$, and that the multiplicity of λ is ≥ 2. To prove the last assertion, we note that if $\boldsymbol{v}\neq\boldsymbol{0}$ is an eigenvector of the matrix $\boldsymbol{B}^T\boldsymbol{B}$ associated with λ, then $\boldsymbol{B}\boldsymbol{v}$ is also an eigenvector associated with λ, and the family $\{\boldsymbol{v},\boldsymbol{B}\boldsymbol{v}\}$ is linearly independent. Indeed, since the matrix $\boldsymbol{B}$ is anti-symmetric, the relation $(\boldsymbol{B}^T\boldsymbol{B})\boldsymbol{v}=\lambda\boldsymbol{v}$ implies that $(\boldsymbol{B}^T\boldsymbol{B})(\boldsymbol{B}\boldsymbol{v})=\boldsymbol{B}(\boldsymbol{B}^T\boldsymbol{B})\boldsymbol{v}=\boldsymbol{B}(\lambda\boldsymbol{v})=\lambda(\boldsymbol{B}\boldsymbol{v})$. Besides, if for the sake of contradiction we assume that the family $\{\boldsymbol{v},\boldsymbol{B}\boldsymbol{v}\}$ were linearly dependent, then $\boldsymbol{B}\boldsymbol{v}=p\boldsymbol{v}$ for some scalar $p\in\mathbb{R}$, and then

$$\lambda\boldsymbol{v}=(\boldsymbol{B}^T\boldsymbol{B})\boldsymbol{v}=\boldsymbol{B}^T(p\boldsymbol{v})=-\boldsymbol{B}(p\boldsymbol{v})=-p(\boldsymbol{B}\boldsymbol{v})=-p^2\boldsymbol{v}.$$

Thus, $(\lambda+p^2)\boldsymbol{v}=0$, which is impossible since $\lambda>0$ and $\boldsymbol{v}\neq\boldsymbol{0}$.

Third, let $p_1\leq p_2\leq p_3\leq\cdots\leq p_d$ denote the eigenvalues of the symmetric matrix $\boldsymbol{M}\in\boldsymbol{S}^d$. Then we claim that $p_1\geq 0$ and $p_2>0$. That $p_1\geq 0$ is clear, since the matrix $\boldsymbol{M}$ is semi-positive definite by its definition. Assume for the sake of contradiction that $p_2=0$. Then $p_1=p_2=0$, so that there exist (at least) two linearly independent vectors $\boldsymbol{v}_1,\boldsymbol{v}_2\in\mathbb{R}^d$ such that $\boldsymbol{M}\boldsymbol{v}_1=\boldsymbol{M}\boldsymbol{v}_2=\boldsymbol{0}$. Consequently, using the definition of the matrix $\boldsymbol{M}$, we have, for each $i\in\{1,2\}$,

$$0=\boldsymbol{v}_i^T\boldsymbol{M}\boldsymbol{v}_i=\int_{\Gamma_0}\left|(x-x_0)^T\boldsymbol{v}_i\right|^2\mathrm{d}\sigma.$$

This implies that $(x-x_0)v_1=(x-x_0)v_2=0$ for all $x\in\Gamma_0$, which means that all the points of Γ_0 belong to the intersection of two distinct hyperplanes in $\mathbb{R}^d$ passing by x_0, which is impossible since Γ_0 is a relatively open subset of the boundary of a domain in $\mathbb{R}^d$.

Noting that the eigenvalues of both matrices $\boldsymbol{B}^T\boldsymbol{B}$ and $\boldsymbol{M}$ are all ≥ 0 (since both matrices are symmetric and semi-positive definite), we infer from the above three observations and from Lemma 4.1 that the left-hand side of (4.6) satisfies

$$\begin{aligned}\|\boldsymbol{B}(x-x_0)\|^2_{L^2(\Omega)}&=\boldsymbol{B}^T\boldsymbol{B}:\boldsymbol{M}\geq\lambda(p_1+p_2)\\&\geq\frac{\mathrm{Tr}(\boldsymbol{B}^T\boldsymbol{B})}{d}(p_1+p_2)\\&=\frac{|\boldsymbol{B}|^2}{d}(p_1+p_2).\end{aligned}$$

Consequently,

$$\left(\frac{p_1+p_2}{d}\right)^{1/2}|\boldsymbol{B}|\leq\|\boldsymbol{B}(x-x_0)\|_{L^2(\Gamma_0)}\leq T(1+W)c_1\|\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}\|_{L^2(\Omega)},$$

so that the norm of matrix $\boldsymbol{B}$ is bounded above by

$$|\boldsymbol{B}|\leq\left(\frac{d}{p_1+p_2}\right)^{1/2}T(1+W)c_1\|\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}\|_{L^2(\Omega)}.$$

Then we infer from (4.5) that

$$\begin{aligned}\|\boldsymbol{\nabla}\boldsymbol{u}\|_{L^2(\Omega)}&\leq c_1\|\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}\|_{L^2(\Omega)}+|\Omega|^{1/2}|\boldsymbol{B}|\\&\leq c_1\left(1+T(1+W)\sqrt{\frac{d|\Omega|}{p_1+p_2}}\right)\|\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}\|_{L^2(\Omega)}.\end{aligned}$$

This completes the proof of the theorem. □

5 Korn's inequalities for fields in $H^1(\Omega;\mathbb{R}^n)$

The objective of this section is to prove inequality (1.3) mentioned in the introduction, viz.,

$$\|\boldsymbol{u}\|_{H^1(\Omega)} \leq C_3 \|\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}\|_{L^2(\Omega)} + C_4 \|\boldsymbol{u}\|_{L^2(\Omega)}, \quad \forall \boldsymbol{u} \in H^1(\Omega;\mathbb{R}^d), \tag{5.1}$$

which is often referred to as the Korn inequality of the second kind, and to provide at the same time explicit constants $C_3 = C_3(\Omega,\Omega')$ and $C_4 = C_4(\Omega,\Omega')$ in terms of the constant $K(\Omega)$ defined in Lemma 3.1 (about the divergence equation) and of a given open subset Ω' of Ω. The novelty is that the shape of the set Ω' is adapted in this paper to the shape of the domain Ω in order to improve the classical estimates, e.g. Kondratev and Oleinik [14], where Ω' is a ball (which leads to non-optimal estimates for thin domains).

We will show that the above inequality can be deduced from the inequality (see Corollary 3.1)

$$\|\boldsymbol{u} - \boldsymbol{r}(\boldsymbol{u})\|_{H^1(\Omega)} \leq c_1(1+W)\|\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}\|_{L^2(\Omega)} \tag{5.2}$$

for all $\boldsymbol{u} \in H^1(\Omega;\mathbb{R}^d)$, where, for all $x \in \Omega$,

$$\boldsymbol{r}(\boldsymbol{u})(x) := \fint_{\Omega} \boldsymbol{u}(x)\,\mathrm{d}x + \left(\fint_{\Omega} \boldsymbol{\nabla}_{\mathrm{a}}\boldsymbol{u}(x)\,\mathrm{d}x\right)\left(x - \fint_{\Omega} x\,\mathrm{d}x\right) \tag{5.3}$$

combined with specific estimates of the norms of the vector $\fint_{\Omega}\boldsymbol{u}(x)\,\mathrm{d}x$ and of the matrix $\fint_{\Omega}\boldsymbol{\nabla}_{\mathrm{a}}\boldsymbol{u}(x)\,\mathrm{d}x$ in terms of the norms $\|\boldsymbol{u}\|_{L^2(\Omega)}$ and $\|\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}\|_{L^2(\Omega)}$. Since

$$\left|\fint_{\Omega} \boldsymbol{u}(x)\,\mathrm{d}x\right| \leq |\Omega|^{-1/2}\|\boldsymbol{u}\|_{L^2(\Omega)}, \quad \forall \boldsymbol{u} \in H^1(\Omega;\mathbb{R}^d),$$

it remains to estimate the norm of the matrix $\fint_{\Omega}\boldsymbol{\nabla}_{\mathrm{a}}\boldsymbol{u}(x)\,\mathrm{d}x$.

This is possible, but the result would not be optimal. This is why, instead of inequality (5.2) with the choice of $\boldsymbol{r}(\boldsymbol{u})$ given by (5.3), we will deduce (5.1) from the more general estimate (see Theorem 3.2)

$$\|\boldsymbol{u} - \boldsymbol{r}(\boldsymbol{u})\|_{H^1(\Omega)} \leq c_1(\Omega,\Omega')\big(1+W(\Omega,\Omega')\big)\|\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}\|_{L^2(\Omega)} \tag{5.4}$$

for all $\boldsymbol{u} \in H^1(\Omega;\mathbb{R}^d)$, where, for all $x \in \Omega$,

$$\boldsymbol{r}(\boldsymbol{u})(x) := \fint_{\Omega''} \boldsymbol{u} + \left(\fint_{\Omega'} \boldsymbol{\nabla}_{\mathrm{a}}\boldsymbol{u}\right)\left(x - \fint_{\Omega''} x\right), \tag{5.5}$$

by choosing Ω' in such a way that $|\fint_{\Omega'}\boldsymbol{\nabla}_{\mathrm{a}}\boldsymbol{u}(x)\,\mathrm{d}x|$ be bounded above by $\|\boldsymbol{u}\|_{L^2(\Omega)}$ multiplied by a constant. The essence of our argument to find such an upper bound, the details of which are given in the proofs of the following three theorems, is that, for all $\boldsymbol{u} = (u_i) \in \mathcal{C}^1(\overline{\Omega};\mathbb{R}^d)$, we have

$$\left|\int_{\Omega'} \boldsymbol{\nabla}_{\mathrm{a}}\boldsymbol{u}(x)\,\mathrm{d}x\right| = \frac{1}{2}\left|\int_{\partial\Omega'} \big(u_j(x)n_i(x) - u_i(x)n_j(x)\big)\,\mathrm{d}\sigma\right| \leq \int_{\partial\Omega'} |\boldsymbol{u}(x)|\,\mathrm{d}\sigma$$

for every domain $\Omega' \subset \Omega$, on the one hand. On the other hand, by Fubini's theorem, there exists a domain $\Omega' \subset \Omega$ such that

$$\fint_{\partial\Omega'} |\boldsymbol{u}(x)|\,\mathrm{d}\sigma \leq \fint_{\Omega} |\boldsymbol{u}(x)|\,\mathrm{d}x.$$

The shape of Ω' is a matter of choice, but to obtain better (meaning smaller) constants C_3 and C_4 in inequality (5.1), one needs to choose Ω' to mimic the shape of Ω. This is why we consider below three possible choices for Ω': a ball, a cylinder, or a curved cylinder, corresponding to three typical cases, whereby Ω is a bulky domain in all directions, a flat thin domain (like a plate with small thickness), or a curved thin domain (like a shell with small thickness), in view of their application in the mathematical theory of elasticity (which will be presented in a forthcoming paper [15]).

Theorem 5.1. *Given any domain Ω in $\mathbb{R}^d$, let $K = K(\Omega)$ denote the constant appearing in Lemma* 3.1. *Let $R > 0$ be any constant such that there exists an open ball B_R contained in Ω and let*

$$\begin{aligned} &C_3(\Omega, B_R) := 1 + C(d)K\frac{|\Omega|^{1/2}}{|B_R|^{1/2}}, \quad C(d) := d^{1/2}2^{2+d/2}, \\ &C_4(\Omega, B_R) := 1 + \frac{3d}{R}\frac{|\Omega|^{1/2}}{|B_R|^{1/2}}. \end{aligned} \tag{5.6}$$

Then, for all vector fields $\boldsymbol{u} \in H^1(\Omega;\mathbb{R}^d)$,

$$\|\boldsymbol{u}\|_{H^1(\Omega)} \leq C_3(\Omega, B_R)\|\boldsymbol{u}\|_{L^2(\Omega)} + C_4(\Omega, B_R)\|\boldsymbol{\nabla}_s \boldsymbol{u}\|_{L^2(\Omega)}. \tag{5.7}$$

In particular, for all $\boldsymbol{u} \in H^1(\Omega;\mathbb{R}^d)$,

$$\|\boldsymbol{u}\|_{H^1(\Omega)} \leq \left(1 + C(d)(R^{-1} + K)\frac{|\Omega|^{1/2}}{|B_R|^{1/2}}\right)\left(\|\boldsymbol{u}\|_{L^2(\Omega)} + \|\boldsymbol{\nabla}_s \boldsymbol{u}\|_{L^2(\Omega)}\right), \tag{5.8}$$

where $C(d)$ is a constant depending only on the dimension d of Ω.

Proof. The proof is broken for clarity into three parts, numbered (i) to (iii).

(i) Let $B_R := B(x_0, R) \subset \Omega$. Given any continuous function $f : \overline{B_R} \to \mathbb{R}$, we have

$$\int_{B_R \setminus B_{R/2}} f\,\mathrm{d}x = \int_{R/2}^{R} \left(\int_{S_r} f dS_r\right) dr,$$

where $S_r := \partial B_R$ and dS_r is the measure induced on S_r by the Lebesgue measure in $\mathbb{R}^d$. Let $g:[R/2,R]\to\mathbb{R}$ de the function defined by $g(r) := \fint_{S_r} f dS_r$ for all $r\in[R/2,R]$, so that

$$\fint_{B_R\setminus B_{R/2}} f\,\mathrm{d}x = \frac{1}{|B_R\setminus B_{R/2}|}\int_{R/2}^{R} g(r)|S_r|dr.$$

Since

$$\int_{R/2}^{R}|S_r|dr = \int_{R/2}^{R}\int_{S_r} dS_r dr = \int_{B_R\setminus B_{R/2}}\mathrm{d}x,$$

the previous relations implies that

$$\inf_{r\in[R/2,R]} g \le \fint_{B_R\setminus B_{R/2}} f\,\mathrm{d}x \le \sup_{r\in[R/2,R]} g(r).$$

Since the function g is continuous, there exists $r'\in[R/2,R]$ such that

$$\fint_{B_R\setminus B_{R/2}} f\,\mathrm{d}x = \fint_{S_{r'}} f dS_{r'}.$$

(ii) Let $B_R := B(x_0,R)\subset\Omega$. Given any vector field $\boldsymbol{u}\in\mathcal{C}^1(\overline{\Omega};\mathbb{R}^d)$, there exists (cf. part (i) above) $r'=r'(\boldsymbol{u})\in[R/2,R]$ such that

$$\fint_{B_R\setminus B_{R/2}} |\boldsymbol{u}|^2\,\mathrm{d}x = \fint_{S_{r'}} |\boldsymbol{u}|^2 dS_{r'}. \tag{5.9}$$

Let $\Omega' := B_{r'}$. Then

$$\int_{\Omega'} \boldsymbol{\nabla}_{\mathrm{a}}\boldsymbol{u}\,\mathrm{d}x = \int_{B_{r'}} \boldsymbol{\nabla}_{\mathrm{a}}\boldsymbol{u}\,\mathrm{d}x = \frac{1}{2}\int_{B_{r'}}(\partial_i u_j - \partial_j u_i)\,\mathrm{d}x = \frac{1}{2}\int_{S_{r'}}(n_i u_j - n_j u_i) dS_{r'},$$

where $S_{r'}$ denotes the boundary of $B_{r'}$ and $n_i(x) = (x_i - x_{0,i})/|x-x_0|$ denotes the i-th component of the outer unit normal vector at x to the boundary of Ω', so that

$$\begin{aligned}
\left|\fint_{\Omega'} \boldsymbol{\nabla}_{\mathrm{a}}\boldsymbol{u}\,\mathrm{d}x\right| &= \frac{|S_{r'}|}{2|B_{r'}|}\left|\fint_{S_{r'}}(n_i u_j - n_j u_i)\right| \\
&= \frac{|S_{r'}|}{|B_{r'}|}\fint_{S_{r'}}|(n_i u_j)| dS_{r'} \\
&\le \frac{|S_{r'}|}{|B_{r'}|}\left(\fint_{S_{r'}}|(n_i u_j)|^2 dS_{r'}\right)^{1/2} \\
&= \frac{|S_{r'}|}{|B_{r'}|}\left(\fint_{S_{r'}}|\boldsymbol{u}|^2 dS_{r'}\right)^{1/2}.
\end{aligned}$$

Combined with relation (5.9), this implies that the $L^2(\Omega)$-norm of the constant vector field $\fint_{\Omega'} \nabla_{\mathrm{a}} \boldsymbol{u} \, \mathrm{d}x$ is bounded above by

$$\begin{aligned}\left\| \fint_{\Omega'} \nabla_{\mathrm{a}} \boldsymbol{u} \, \mathrm{d}x \right\|_{L^2(\Omega)} &= |\Omega|^{1/2} \left| \fint_{\Omega'} \nabla_{\mathrm{a}} \boldsymbol{u} \, \mathrm{d}x \right| \\ &\le |\Omega|^{1/2} \frac{|S_{r'}|}{|B_{r'}|} \left(\fint_{S_{r'}} |\boldsymbol{u}|^2 dS_{r'} \right)^{1/2} \\ &\le |\Omega|^{1/2} \frac{|S_{r'}|}{|B_{r'}|} \left(\fint_{B_R \setminus B_{R/2}} |\boldsymbol{u}|^2 \, \mathrm{d}x \right)^{1/2}.\end{aligned}$$

Furthermore, using that $|B_r| = r^d |B_1|$, $|S_r| = r^{d-1}|S_1|$, $|B_r \setminus B_{r/2}| = r^d(1-2^{-d})|B_1|$ for all $r > 0$, we deduce that

$$\begin{aligned}\left\| \fint_{\Omega'} \nabla_{\mathrm{a}} \boldsymbol{u} \, \mathrm{d}x \right\|_{L^2(\Omega)} &\le |\Omega|^{1/2} \frac{|S_{r'}|}{|B_{r'}|} \frac{1}{|B_R \setminus B_{R/2}|^{1/2}} \left(\int_{B_R \setminus B_{R/2}} |\boldsymbol{u}|^2 \, \mathrm{d}x \right)^{1/2} \\ &\le |\Omega|^{1/2} \frac{|S_1|}{r' |B_1| |B_1|^{1/2} R^{d/2} (1-2^{-d})^{1/2}} \|\boldsymbol{u}\|_{L^2(B_R \setminus B_{R/2})}.\end{aligned}$$

Finally, using that $r' \ge R/2, d \ge 1, |S_1| = d|B_1|$ and $B_R \subset \Omega$, we obtain that

$$\left\| \fint_{\Omega'} \nabla_{\mathrm{a}} \boldsymbol{u} \, \mathrm{d}x \right\|_{L^2(\Omega)} \le \frac{|\Omega|^{1/2}}{|B_1|^{1/2}} \frac{3d}{R^{1+d/2}} \|\boldsymbol{u}\|_{L^2(\Omega)}. \tag{5.10}$$

(iii) Let $\boldsymbol{u} \in \mathcal{C}^1(\overline{\Omega}; \mathbb{R}^d)$. Then

$$\begin{aligned}\|\boldsymbol{u}\|_{H^1(\Omega)} \le & \|\boldsymbol{u}\|_{L^2(\Omega)} + \|\nabla_{\mathrm{s}} \boldsymbol{u}\|_{L^2(\Omega)} \\ & + \left\| \nabla_{\mathrm{a}} \boldsymbol{u} \, \mathrm{d}x - \fint_{\Omega'} \nabla_{\mathrm{a}} \boldsymbol{u} \, \mathrm{d}x \right\|_{L^2(\Omega)} + \left\| \fint_{\Omega'} \nabla_{\mathrm{a}} \boldsymbol{u} \, \mathrm{d}x \right\|_{L^2(\Omega)},\end{aligned}$$

so that, by inequality (5.10) established above, we have

$$\begin{aligned}\|\boldsymbol{u}\|_{H^1(\Omega)} \le & \left(1 + \frac{|\Omega|^{1/2}}{|B_1|^{1/2}} \frac{3d}{R^{1+d/2}} \right) \|\boldsymbol{u}\|_{L^2(\Omega)} + \|\nabla_{\mathrm{s}} \boldsymbol{u}\|_{L^2(\Omega)} \\ & + \left\| \nabla_{\mathrm{a}} \boldsymbol{u} \, \mathrm{d}x - \fint_{\Omega'} \nabla_{\mathrm{a}} \boldsymbol{u} \, \mathrm{d}x \right\|_{L^2(\Omega)}.\end{aligned}$$

Then we conclude by applying Theorem 3.2 (see inequality (3.2)) that

$$\begin{aligned} \|\boldsymbol{u}\|_{H^1(\Omega)} &\leq C_3\|\boldsymbol{u}\|_{L^2(\Omega)} + \left(1+4K\sqrt{\frac{d|\Omega|}{|\Omega'|}}\right)\|\boldsymbol{\nabla}_s\boldsymbol{u}\|_{L^2(\Omega)} \\ &\leq C_3\|\boldsymbol{u}\|_{L^2(\Omega)} + C_4\|\boldsymbol{\nabla}_s\boldsymbol{u}\|, \end{aligned} \tag{5.11}$$

where

$$C_3 := 1+\frac{3d}{R}\frac{|\Omega|^{1/2}}{|B_R|^{1/2}}$$

and (remember that $\Omega' = B_{r'}$ with $r' \in [R/2, R]$, cf. part (ii) of the proof)

$$C_4 := 1+d^{1/2}2^{2+d/2}K\frac{|\Omega|^{1/2}}{|B_R|^{1/2}}.$$

Since Ω is a domain by the assumptions of the theorem, so in particular its boundary is Lipschitz-continuous, the set $\mathcal{C}^1(\overline{\Omega};\mathbb{R}^d)$ is dense in the Sobolev space $H^1(\Omega;\mathbb{R}^d)$. Thus, inequality (5.7) of the theorem holds with the same constants C_3 and C_4 for all vector fields $\boldsymbol{u} \in H^1(\Omega;\mathbb{R}^d)$.

This inequality implies in particular that, for some constant $C(d)$ depending only on the dimension d of the domain Ω, the following inequality holds for all $\boldsymbol{u} \in H^1(\Omega;\mathbb{R}^d)$:

$$\|\boldsymbol{u}\|_{H^1(\Omega)} \leq \left(1+C(d)[R^{-1}+K]\frac{|\Omega|^{1/2}}{|B_R|^{1/2}}\right)\left(\|\boldsymbol{u}\|_{L^2(\Omega)} + \|\boldsymbol{\nabla}_s\boldsymbol{u}\|_{L^2(\Omega)}\right).$$

This completes the proof of the theorem. □

Note that the larger R, the sharper the above inequality. Thus, the best choice of R is the radius of the largest open ball B_R contained in Ω. This means that the above constants C_3 and C_4 are suitable for "bulky" domains (where the size of B_R is "of the same order" as the size of the diameter of Ω), but not for "thin" domains Ω (where the ratio $R/\mathrm{diam}(\Omega)$ is close to zero). We will show in the remainder of this section how to obtain sharper Korn inequalities for such domains, simply by adapting the choice of the subset Ω' to the shape of the domain Ω. We consider for conciseness only two examples, a "plate-like" domain Ω in the next theorem, and a "shell-like" in Theorem 5.3.

Theorem 5.2. *Given any domain Ω in $\mathbb{R}^d$, let $K = K(\Omega)$ denote the constant appearing in Lemma* 3.1. *Let $R > 0$ and $h > 0$ be any two constants such that there exists a cylinder $B_{R,h} := b_R \times I_h$, $b_R := \{y \in \mathbb{R}^{d-1};\ |y-y_0| < R\}$ and $I_h := (z_0-h, z_0+h) \subset \mathbb{R}$ such that $B_{R,h} \subset \Omega$. Define the constants*

$$\begin{aligned} C_3(\Omega,B_{R,h}) &:= 1+4(d-1)R^{-1}\frac{|\Omega|^{1/2}}{|B_{R,h}|^{1/2}},\\ C_4(\Omega,B_{R,h}) &:= 1+(1+4K\sqrt{d})\frac{|\Omega|^{1/2}}{|B_{R,h}|^{1/2}}. \end{aligned} \tag{5.12}$$

Then, for all vector fields $\boldsymbol{u}\in H^1(\Omega;\mathbb{R}^d)$,

$$\|\boldsymbol{u}\|_{H^1(\Omega)} \le C_3(\Omega,B_{R,h})\|\boldsymbol{u}\|_{L^2(\Omega)} + C_4(\Omega,B_{R,h})\|\boldsymbol{\nabla}_s\boldsymbol{u}\|_{L^2(\Omega)}. \tag{5.13}$$

In particular, there exists a constant $C(d)$ *depending only on the dimension* d *such that*

$$\|\boldsymbol{u}\|_{H^1(\Omega)} \le \left\{1+C(d)(1+R^{-1}+K)\frac{|\Omega|^{1/2}}{|B_{R,h}|^{1/2}}\right\}\left(\|\boldsymbol{u}\|_{L^2(\Omega)}+\|\boldsymbol{\nabla}_s u\|_{L^2(\Omega)}\right). \tag{5.14}$$

Proof. Let $\Omega_R := B_{R,h}$ be the cylinder defined in the statement of the theorem (we drop for conciseness the dependence on h for conciseness). We assume without losing in generality that it is centered at the origin, so $(y_0,z_0)=0$.

The proof is broken for clarity into three parts, numbered (i) to (iii).

(i) Let $\Sigma_r := \partial b_r \times (-h,h), r\in[R/2,R]$, denote the lateral boundary of the cylinder Ω_r. Then for any $f\in\mathcal{C}^0(\overline{\Omega_R})$, we have

$$\int_{\Omega_R\setminus\Omega_{R/2}} f\,\mathrm{d}x = \int_{R/2}^{R}\left(\int_{\Sigma_r} f\,\mathrm{d}\Sigma_r\right)dr,$$

from which wet deduce, by using an argument similar to the one used in the proof of Theorem 5.1, that there exists $r'\in[R/2,R]$ such that

$$\fint_{\Omega_R\setminus\Omega_{R/2}} f\,\mathrm{d}x = \fint_{\Sigma_{r'}} f\,\mathrm{d}\Sigma_{r'}.$$

(ii) Given any vector field $\boldsymbol{u}\in\mathcal{C}^1(\overline{\Omega},\mathbb{R}^d)$, part (i) of the proof shows that there exists $r'=r'(\boldsymbol{u})\in[R/2,R]$ such that

$$\fint_{\Omega_R\setminus\Omega_{R/2}} |\boldsymbol{u}|^2\,\mathrm{d}x = \fint_{\Sigma_{r'}} |\boldsymbol{u}|^2\,\mathrm{d}\Sigma_{r'}. \tag{5.15}$$

Let $\Omega' := \Omega_{r'}$. Using that

$$[\boldsymbol{\nabla}_{\mathrm{a}}\boldsymbol{u}]_{ij} = \frac{1}{2}(\partial_j u_i - \partial_i u_j) = \partial_j u_i - \frac{1}{2}(\partial_j u_i + \partial_i u_j),$$

and that the components $n_j, 1\leq j\leq d$, of the outer unit normal vector field on the two bases $b_{r'}\times\{+h\}$ and $b_{r'}\times\{-h\}$ of the cylinder Ω' vanish unless $j=d$, we deduce that, for each pair (i,j) of indices that satisfy $1\leq j<i\leq d$,

$$\begin{aligned}\int_{\Omega'}[\boldsymbol{\nabla}_{\mathrm{a}}\boldsymbol{u}]_{ij}\,\mathrm{d}x&=\int_{\Omega'}\partial_j u_i\,\mathrm{d}x-\int_{\Omega'}[\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}]_{ij}\,\mathrm{d}x\\&=\int_{\Sigma_{r'}}u_i n_j\,\mathrm{d}\Sigma_{r'}-\int_{\Omega'}[\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}]_{ij}\,\mathrm{d}x,\end{aligned}$$

where $[\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}]_{ij}$ denotes the component of the matrix field $\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}$ at its i-th row and j-th column. Consequently,

$$\int_{\Omega'}\boldsymbol{\nabla}_{\mathrm{a}}\boldsymbol{u}\,\mathrm{d}x=\int_{\Sigma_{r'}}\boldsymbol{A}\,\mathrm{d}\Sigma_{r'}-\int_{\Omega'}\boldsymbol{B}\,\mathrm{d}x,$$

where $\boldsymbol{A}$ and $\boldsymbol{B}$ respectively denote the anti-symmetric matrix fields with components $[\boldsymbol{A}]_{ij}=u_i n_j$ and $[\boldsymbol{B}]_{ij}=[\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}]_{ij}$ for all $1\leq j<i\leq d$.

Using in particular Cauchy-Schwarz inequality, we deduce that the Frobenius norm of the matrix in the left-hand side of the above relation satisfies the inequality

$$\begin{aligned}\left|\int_{\Omega'}\boldsymbol{\nabla}_{\mathrm{a}}\boldsymbol{u}\,\mathrm{d}x\right|&\leq\left|\int_{\Sigma_{r'}}\boldsymbol{A}\,\mathrm{d}\Sigma_{r'}\right|+\left|\int_{\Omega'}\boldsymbol{B}\,\mathrm{d}x\right|\\&\leq|\Sigma_{r'}|^{1/2}\left(\int_{\Sigma_{r'}}|\boldsymbol{A}|^2\,\mathrm{d}\Sigma_{r'}\right)^{1/2}+|\Omega'|^{1/2}\left(\int_{\Omega'}|\boldsymbol{B}|^2\,\mathrm{d}x\right)^{1/2}\\&=\left(2|\Sigma_{r'}|\int_{\Sigma_{r'}}\sum_{1\leq j<i\leq d}(u_i n_j)^2\,\mathrm{d}\Sigma_{r'}\right)^{1/2}\\&\quad+\left(2|\Omega'|\int_{\Omega'}\sum_{1\leq j<i\leq d}([\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}]_{ij})^2\,\mathrm{d}x\right)^{1/2}.\end{aligned}$$

Since $\sum_{j=1}^d(n_j)^2=1$ on $\Sigma_{r'}$ and since the matrix filed $\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}$ is symmetric, we deduce from the above inequality that

$$\left|\int_{\Omega'}\boldsymbol{\nabla}_{\mathrm{a}}\boldsymbol{u}\,\mathrm{d}x\right|\leq\left(2|\Sigma_{r'}|\int_{\Sigma_{r'}}|\boldsymbol{u}|^2\,\mathrm{d}\Sigma_{r'}\right)^{1/2}+\left(|\Omega'|\int_{\Omega'}|\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}|^2\,\mathrm{d}x\right)^{1/2},$$

then, by using inequality (5.15) in the right-hand side above, that

$$\left|\int_{\Omega'}\boldsymbol{\nabla}_{\mathrm{a}}\boldsymbol{u}\,\mathrm{d}x\right|\leq\sqrt{2}|\Sigma_{r'}|\left(⨏_{\Omega_R\setminus\Omega_{R/2}}|\boldsymbol{u}|^2\,\mathrm{d}x\right)^{1/2}+|\Omega'|^{1/2}\|\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}\|_{L^2(\Omega')}$$

$$=\frac{\sqrt{2}|\Sigma_{r'}|}{|\Omega_R\setminus\Omega_{R/2}|^{1/2}}\|\boldsymbol{u}\|_{L^2(\Omega_R\setminus\Omega_{R/2})}+|\Omega'|^{1/2}\|\boldsymbol{\nabla}_{\rm s}\boldsymbol{u}\|_{L^2(\Omega')},$$

and finally, since $\Omega'=\Omega_{r'}\subset\Omega_R\subset\Omega$, that

$$\begin{aligned}\left\|\fint_{\Omega'}\boldsymbol{\nabla}_{\rm a}\boldsymbol{u}\,{\rm d}x\right\|_{L^2(\Omega)}&=|\Omega|^{1/2}\left|\fint_{\Omega'}\boldsymbol{\nabla}_{\rm a}\boldsymbol{u}\,{\rm d}x\right|\\&\le\frac{|\Omega|^{1/2}}{|\Omega'|}\left(\frac{\sqrt{2}|\Sigma_{r'}|}{|\Omega_R\setminus\Omega_{R/2}|^{1/2}}\|\boldsymbol{u}\|_{L^2(\Omega)}+|\Omega'|^{1/2}\|\boldsymbol{\nabla}_{\rm s}\boldsymbol{u}\|_{L^2(\Omega)}\right).\end{aligned}$$

(iii) Let $\boldsymbol{u}\in\mathcal{C}^1(\overline{\Omega},\mathbb{R}^d)$. Then

$$\begin{aligned}\|\boldsymbol{u}\|_{H^1(\Omega)}\le\|\boldsymbol{u}\|_{L^2(\Omega)}+\|\boldsymbol{\nabla}_{\rm s}\boldsymbol{u}\|_{L^2(\Omega)}+\left\|\boldsymbol{\nabla}_{\rm a}\boldsymbol{u}-\fint_{\Omega'}\boldsymbol{\nabla}_{\rm a}\boldsymbol{u}\,{\rm d}x\right\|_{L^2(\Omega)}\\+\left\|\fint_{\Omega'}\boldsymbol{\nabla}_{\rm a}\boldsymbol{u}\,{\rm d}x\right\|_{L^2(\Omega)},\end{aligned}$$

which combined with the previous inequality yields

$$\begin{aligned}\|\boldsymbol{u}\|_{H^1(\Omega)}\le\|\boldsymbol{u}\|_{L^2(\Omega)}+\|\boldsymbol{\nabla}_{\rm s}\boldsymbol{u}\|_{L^2(\Omega)}+\frac{|\Omega|^{1/2}}{|\Omega'|}\frac{\sqrt{2}|\Sigma_{r'}|}{|\Omega_R\setminus\Omega_{R/2}|^{1/2}}\|\boldsymbol{u}\|_{L^2(\Omega)}\\+\frac{|\Omega|^{1/2}}{|\Omega'|}|\Omega'|^{1/2}\|\boldsymbol{\nabla}_{\rm s}\boldsymbol{u}\|_{L^2(\Omega)}+\left\|\boldsymbol{\nabla}_{\rm a}\boldsymbol{u}-\fint_{\Omega'}\boldsymbol{\nabla}_{\rm a}\boldsymbol{u}\,{\rm d}x\right\|_{L^2(\Omega)}.\end{aligned}$$

The last term of the right-hand side above is bounded above by Korn's inequality (3.2). This yields the inequality

$$\|\boldsymbol{u}\|_{H^1(\Omega)}\le C_3(r')\|\boldsymbol{\nabla}_{\rm s}\boldsymbol{u}\|_{L^2(\Omega)}+C_4(r')\|\boldsymbol{u}\|_{L^2(\Omega)},$$

where

$$\begin{aligned}C_3(r')&:=1+\frac{|\Omega|^{1/2}}{|\Omega'|}|\Omega'|^{1/2}+4K\sqrt{\frac{d|\Omega|}{|\Omega'|}}=1+\frac{|\Omega|^{1/2}}{|\Omega'|^{1/2}}(1+4K\sqrt{d}),\\C_4(r')&:=1+\frac{|\Omega|^{1/2}}{|\Omega'|}\frac{\sqrt{2}|\Sigma_{r'}|}{|\Omega_R\setminus\Omega_{R/2}|^{1/2}}.\end{aligned}$$

Since $\Omega'=\Omega_{r'}=B_{r',h}:=b_{r'}\times(-h,h)$, where $b_{r'}:=\{y\in\mathbb{R}^{d-1};|y|<r'\},d\ge2$, and $R/2\le r'\le R$, we have

$$\frac{|\Sigma_{r'}|}{|\Omega'|}=\frac{d-1}{r'}\le\frac{2(d-1)}{R},$$

$$
\begin{aligned}
&|\Omega_R\setminus\Omega_{R/2}|=(1-2^{1-d})|B_{R,h}|\leq\frac{1}{2}|B_{R,h}|,\\
&|\Omega'|\leq|B_{R,h}|,
\end{aligned}
$$

so that, for all $r'\in[R/2,R]$,

$$
\begin{aligned}
C_3(r')&\leq C_4(\Omega,B_{R,h}):=1+(1+4K\sqrt{d})\frac{|\Omega|^{1/2}}{|B_{R,h}|^{1/2}},\\
C_3(r')&\leq C_3(\Omega,B_{R,h}):=1+4(d-1)R^{-1}\frac{|\Omega|^{1/2}}{|B_{R,h}|^{1/2}}.
\end{aligned}
$$

Since Ω is a domain (the definition of a domain is given in Section 2) by the assumptions of the theorem, so in particular its boundary is Lipschitz-continuous, the set $\mathcal{C}^1(\overline{\Omega};\mathbb{R}^d)$ is dense in the Sobolev space $H^1(\Omega;\mathbb{R}^d)$. Thus, the inequality

$$
\|\boldsymbol{u}\|_{H^1(\Omega)}\leq C_3(\Omega,B_{R,h})\|\boldsymbol{u}\|_{L^2(\Omega)}+C_4(\Omega,B_{R,h})\|\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}\|_{L^2(\Omega)},
$$

holds for all vector fields $\boldsymbol{u}\in H^1(\Omega;\mathbb{R}^d)$. This is precisely inequality (5.13) announced in the statement of the theorem.

This inequality implies in particular that, for some explicit constant $C(d)$ depending only on the dimension d, the following inequality holds for all $\boldsymbol{u}\in H^1(\Omega;\mathbb{R}^d)$:

$$
\|\boldsymbol{u}\|_{H^1(\Omega)}\leq\left[1+C(d)(1+R^{-1}+K)\frac{|\Omega|^{1/2}}{|B_{R,h}|^{1/2}}\right]\big(\|\boldsymbol{u}\|_{L^2(\Omega)}+\|\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}\|_{L^2(\Omega)}\big).
$$

This is precisely inequality (5.14) of the theorem. The proof is complete. $\square$

Note that inequality (5.14) of Theorem 5.2 is sharper than inequality (5.8) of Theorem 5.2, since it replaces $|\Omega|/|B_R|$ by $|\Omega|/|B_{R,h}|$ in its right-hand side. This is an advantage since larger (up to the multiplication by a fixed constant) cylinders than balls fit inside a domain, as is the case for instance of thin domains such as $\Omega:=\omega\times(-h,h)$ or $\Omega:=B(0,R)\times(0,1)$ with $h\to0^+$ and $R\to0^+$.

The next theorem further improve this inequality by replacing the "straight" cylinder $B_{R,h}$ by a "curved" one, which are for instance needed for domains Ω that are thin open neighbourhoods of hypersurfaces in $\mathbb{R}^d$. Note that the constants D and E appearing in the statement of the theorem satisfy $D\leq(B/d)^{d/2}$ and $E\leq(A/d)^{d/2}$, so they can be replaced by these right-hand sides for a simpler statement.

Theorem 5.3. *Given any domain Ω in $\mathbb{R}^d, d \geq 2$, let $K = K(\Omega)$ denote the constant appearing in Lemma* 3.1. *Given any numbers $0 < R \leq R_0$ and $0 < h \leq h_0$ and any embedding $\boldsymbol{\Theta} \in \mathcal{C}^2(\overline{B_{R_0,h_0}}; \mathbb{R}^d)$ such that $\boldsymbol{\Theta}(B_{R_0,h_0}) \subset \Omega$, where*

$$B_{R,h} := \left\{ (y', y_d) \in \mathbb{R}^{d-1} \times \mathbb{R};\ |y'| < R, |y_d| < h \right\},$$

define the constants

$$\begin{aligned} C_3(\Omega, \boldsymbol{\Theta}, B_{R,h}) &:= 1 + (AB + 4K\sqrt{d}) \frac{|\Omega|^{1/2}}{|\boldsymbol{\Theta}(B_{R,h})|^{1/2}}, \\ C_4(\Omega, \boldsymbol{\Theta}, B_{R,h}) &:= 1 + A^{1/2} B \left(C^{1/2} + 2^{(d+1)/2} (d-1) D^{1/2} E^{1/2} \frac{1}{R} \right) \frac{|\Omega|^{1/2}}{|\boldsymbol{\Theta}(B_{R,h})|^{1/2}}, \end{aligned} \tag{5.16}$$

where A, B, C, D and E are constants such that

$$\begin{aligned} &|\boldsymbol{\nabla}\boldsymbol{\Theta}|^2 \leq A, && |(\boldsymbol{\nabla}\boldsymbol{\Theta})^{-1}|^2 \leq B, \\ &\sum_{i \neq j = 1}^{d} \left| \boldsymbol{\nabla}\boldsymbol{\Theta}^{-1} \frac{\partial^2 \boldsymbol{\Theta}}{\partial y_i \partial y_j} \right|^2 \leq C, && D^{-1} \leq |\det \boldsymbol{\nabla}\boldsymbol{\Theta}| \leq E \quad \text{in } B_{R_0,h_0}. \end{aligned}$$

Then, for all vector fields $\boldsymbol{u} \in H^1(\Omega; \mathbb{R}^d)$,

$$\|\boldsymbol{u}\|_{H^1(\Omega)} \leq C_3(\Omega, \boldsymbol{\Theta}, B_{R,h}) \|\boldsymbol{u}\|_{L^2(\Omega)} + C_4(\Omega, \boldsymbol{\Theta}, B_{R,h}) \|\boldsymbol{\nabla}_s \boldsymbol{u}\|_{L^2(\Omega)}. \tag{5.17}$$

In particular, there exists a constant $C(\boldsymbol{\Theta}, d)$ depending only on the dimension d and on the mapping $\boldsymbol{\Theta}$ such that

$$\|\boldsymbol{u}\|_{H^1(\Omega)} \leq \left\{ 1 + C(\boldsymbol{\Theta}, d)(1 + R^{-1} + K) \frac{|\Omega|^{1/2}}{|B_{R,h}|^{1/2}} \right\} \left(\|\boldsymbol{u}\|_{L^2(\Omega)} + \|\boldsymbol{\nabla}_s u\|_{L^2(\Omega)} \right). \tag{5.18}$$

Proof. For conciseness, we let $\Omega_R := B_{R,h}$ denote the cylinder defined in the statement of the theorem (so we drop the dependence on h in the notation) and let $\widetilde{\Omega} := \overline{B_{R_0,h_0}}$ denote the closure of the cylinder B_{R_0,h_0}.

With the embedding $\boldsymbol{\Theta}$ defined in the statement of the theorem, we associate the vector fields $\boldsymbol{g}_j \in \mathcal{C}^1(\widetilde{\Omega}; \mathbb{R}^d)$ and $\boldsymbol{g}^i \in \mathcal{C}^1(\widetilde{\Omega}; \mathbb{R}^d)$ defined by, for all $i, j \in \{1, \ldots, d\}$,

$$\boldsymbol{g}_j = \frac{\partial \boldsymbol{\Theta}}{\partial y_j}, \quad \boldsymbol{g}^i \cdot \boldsymbol{g}_j = \delta^i_j \quad \text{in } \widetilde{\Omega}.$$

Note that the vectors fields $\boldsymbol{g}_j, 1 \leq j \leq d$, are linearly independent at every point of $\widetilde{\Omega}$, so they form a basis in $\mathbb{R}^d$, since $\boldsymbol{\Theta}$ is an embedding (hence immersion) by assumption, which in turn imply that the vector fields $\boldsymbol{g}^i, 1 \le i \le j$, are uniquely defined and of class $\mathcal{C}^1$ in $\widetilde{\Omega}$.

Then we define the matrix field

$$\boldsymbol{\nabla\Theta} = (\boldsymbol{g}_1|\ldots|\boldsymbol{g}_d) \in \mathcal{C}^1(\widetilde{\Omega};\mathbb{M}^d)$$

with $\boldsymbol{g}_j$ as its j-th column vector, its inverse matrix field $\boldsymbol{\nabla\Theta} = (\boldsymbol{g}^1|\ldots|\boldsymbol{g}^d)^T \in \mathcal{C}^1(\widetilde{\Omega};\mathbb{M}^d)$ with $\boldsymbol{g}^i$ as its i-th row vector, the symmetric matrix field

$$\boldsymbol{C} := (\boldsymbol{\nabla\Theta})^T \boldsymbol{\nabla\Theta} \in \mathcal{C}^1(\widetilde{\Omega};\mathbb{S}^d),$$

the function

$$g := |\det(\boldsymbol{\nabla\Theta})|^2 = \det(\boldsymbol{C}) \in \mathcal{C}^1(\widetilde{\Omega}),$$

and the functions

$$\Gamma^k_{ij} = \Gamma^k_{ji} := \frac{\partial \boldsymbol{g}_i}{\partial y_j} \cdot \boldsymbol{g}^k \in \mathcal{C}^0(\widetilde{\Omega}), \quad i,j,k \in \{1,\ldots,d\}.$$

All the functions defined above are continuous on $\widetilde{\Omega}$, which is a compact set, and $g(y) > 0$ for all $y \in \widetilde{\Omega}$, so there exist constants $A = A(\boldsymbol{\Theta}), B = B(\boldsymbol{\Theta}), C = C(\boldsymbol{\Theta})$, $D = D(\boldsymbol{\Theta})$ and $E = E(\boldsymbol{\Theta})$ such that, for all $y \in \widetilde{\Omega}$,

$$\begin{aligned}
&\operatorname{Tr}\big(\boldsymbol{C}(y)\big) = |\boldsymbol{\nabla\Theta}(y)|^2 \le A, \\
&\operatorname{Tr}\big(\boldsymbol{C}^{-1}(y)\big) = |\boldsymbol{\nabla\Theta}^{-1}(y)|^2 \le B, \\
&D^{-1} \le \sqrt{g(y)} = |\det(\boldsymbol{\nabla\Theta})| \le E,
\end{aligned}$$

and

$$\sum_{1 \le i \ne j \le d} \sum_{k=1}^{d} |\Gamma^k_{ij}|^2 = \sum_{i \ne j = 1}^{d} \left| \boldsymbol{\nabla\Theta}^{-1} \frac{\partial^2 \boldsymbol{\Theta}}{\partial y_i \partial y_j} \right|^2 \le C.$$

Finally, given any vector field $\boldsymbol{u} = (u_i) \in \mathcal{C}^1(\overline{\Omega};\mathbb{R}^d)$, we define $\widetilde{\boldsymbol{u}} = \widetilde{u}_i \boldsymbol{g}^i \in \mathcal{C}^1(\widetilde{\Omega};\mathbb{R}^d)$ by $\widetilde{\boldsymbol{u}}(y) = \boldsymbol{u}(x)$ for all $x = \boldsymbol{\Theta}(y), y \in \widetilde{\Omega}$, and the functions

$$\widetilde{u}_i|_j := \frac{\partial \widetilde{u}_i}{\partial y_j} - \Gamma^k_{ij} \widetilde{u}_k \in \mathcal{C}^0(\widetilde{\Omega}).$$

The proof is broken for clarity into three parts, numbered (i) to (iii).

(i) For each $r \in [R/2, R]$, let

$$\Sigma_r := \{(y,z) \in \mathbb{R}^{d-1} \times \mathbb{R}; |y| = r, |z| < h\},$$

denote the lateral face of the cylinder Ω_r. Then, for any $f \in \mathcal{C}^0(\overline{\Omega})$, we have

$$\int_{\Theta\left(\Omega_R\setminus\Omega_{R/2}\right)} f\,\mathrm{d}x = \int_{\Omega_R\setminus\Omega_{R/2}} f\circ\Theta\sqrt{g}\,\mathrm{d}y = \int_{R/2}^{R}\left(\int_{\Sigma_r} f\circ\Theta\sqrt{g}\,\mathrm{d}\Sigma_r\right)dr$$
$$= \int_{R/2}^{R} h(r)|\Sigma_r|dr,$$

where $h(r) := \fint_{\Sigma_r} f\circ\Theta\sqrt{g}\,\mathrm{d}\Sigma_r$, on the one hand.

On the other hand, the relation

$$\int_{R/2}^{R}|\Sigma_r|dr = \int_{R/2}^{R}\left(\int_{\Sigma_r}\mathrm{d}\Sigma_r\right)dr = \int_{\Omega_R\setminus\Omega_{R/2}}\mathrm{d}y = |\Omega_R\setminus\Omega_{R/2}|$$

implies that

$$\min_{[R/2,R]} h \le \frac{1}{|\Omega_R\setminus\Omega_{R/2}|}\int_{R/2}^{R} h(r)|\Sigma_r|dr \le \max_{[R/2,R]} h.$$

Since the function h is continuous, it follows that there exists $r' \in [R/2,R]$ such that

$$h(r') = \frac{1}{|\Omega_R\setminus\Omega_{R/2}|}\int_{R/2}^{R} h(r)|\Sigma_r|dr,$$

or equivalently, that

$$\fint_{\Sigma_{r'}} f\circ\Theta\sqrt{g}\,\mathrm{d}\Sigma_{r'} = \frac{1}{|\Omega_R\setminus\Omega_{R/2}|}\int_{\Theta(\Omega_R\setminus\Omega_{R/2})} f\,\mathrm{d}x.$$

(ii) Let $\boldsymbol{u}\in\mathcal{C}^1(\overline{\Omega};\mathbb{R}^d)$ and $\widetilde{\boldsymbol{u}} := \boldsymbol{u}\circ\Theta\in\mathcal{C}^1(\widetilde{\Omega};\mathbb{R}^d)$. Then applying the above relation to $f := |\boldsymbol{u}|^2$ shows that there exists $r' = r'(\boldsymbol{u}) \in [R/2,R]$ such that

$$\int_{\Sigma_{r'}} |\widetilde{\boldsymbol{u}}|^2\sqrt{g}\,\mathrm{d}\Sigma_{r'} = \frac{|\Sigma_{r'}|}{|\Omega_R\setminus\Omega_{R/2}|}\int_{\Theta\left(\Omega_R\setminus\Omega_{R/2}\right)} |\boldsymbol{u}|^2\,\mathrm{d}x. \tag{5.19}$$

Define $\Omega' := \Theta(\Omega_{r'})$ a subset of Ω that is key to the ensuing proof.

Since, for all $y\in\widetilde{\Omega}$, the vectors $\boldsymbol{g}^1(y),\ldots,\boldsymbol{g}^d(y)$ form a basis in $\mathbb{R}^d$, the matrices $\boldsymbol{g}^i(y)\otimes\boldsymbol{g}^j(y) := \boldsymbol{g}^i(y)(\boldsymbol{g}^j(y))^T, i,j\in\{1,\ldots,d\}$, form a basis in $\mathbb{M}^d$. Then the definition of the functions $\widetilde{u}_i|_j$ and Γ_{ij}^k in the preamble of the proof imply that

$$\frac{\partial\boldsymbol{g}_j}{\partial y_i} = \Gamma_{ij}^k\boldsymbol{g}_k, \quad \frac{\partial\boldsymbol{g}^k}{\partial y_i} = -\Gamma_{ij}^k\boldsymbol{g}^j \quad \text{in } \widetilde{\Omega},$$

that

$$\frac{\partial\widetilde{\boldsymbol{u}}}{\partial y_j} = \widetilde{u}_i|_j\boldsymbol{g}^i,$$

and that

$$\begin{aligned}
(\boldsymbol{\nabla} \boldsymbol{u}) \circ \boldsymbol{\Theta} &= \widetilde{u}_i|_j \boldsymbol{g}^i \otimes \boldsymbol{g}^j, \\
(\boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{u}) \circ \boldsymbol{\Theta} &= \frac{1}{2}(\widetilde{u}_i|_j + \widetilde{u}_j|_i) \boldsymbol{g}^i \otimes \boldsymbol{g}^j, \\
(\boldsymbol{\nabla}_{\mathrm{a}} \boldsymbol{u}) \circ \boldsymbol{\Theta} &= \frac{1}{2}(\widetilde{u}_i|_j - \widetilde{u}_j|_i) \boldsymbol{g}^i \otimes \boldsymbol{g}^j.
\end{aligned}$$

In order to estimate the Frobenius norm of the anti-symmetric matrix $\fint_{\Omega'} \boldsymbol{\nabla}_{\mathrm{a}} \boldsymbol{u}\, \mathrm{d}x$, we first have

$$\begin{aligned}
\left| \int_{\Omega'} \boldsymbol{\nabla}_{\mathrm{a}} \boldsymbol{u}\, \mathrm{d}x \right| &= \left| \int_{\Omega_{r'}} \sum_{i \neq j} \frac{\widetilde{u}_i|_j - \widetilde{u}_j|_i}{2} \boldsymbol{g}^i \otimes \boldsymbol{g}^j \sqrt{g}\, \mathrm{d}y \right| \\
&= \left| \int_{\Omega_{r'}} \left[\sum_{j<i} \left(\widetilde{u}_i|_j - \frac{\widetilde{u}_i|_j + \widetilde{u}_j|_i}{2} \right) \boldsymbol{g}^i \otimes \boldsymbol{g}^j \right.\right. \\
&\qquad \left.\left. + \sum_{j>i} \left(\frac{\widetilde{u}_i|_j + \widetilde{u}_j|_i}{2} - \widetilde{u}_j|_i \right) \boldsymbol{g}^i \otimes \boldsymbol{g}^j \right] \sqrt{g}\, \mathrm{d}y \right| \\
&\leq \left| \int_{\Omega_{r'}} \sum_{j<i} \widetilde{u}_i|_j (\boldsymbol{g}^i \otimes \boldsymbol{g}^j - \boldsymbol{g}^j \otimes \boldsymbol{g}^i) \sqrt{g}\, \mathrm{d}y \right| \\
&\quad + \left| \int_{\Omega_{r'}} \sum_{j<i} \frac{\widetilde{u}_i|_j + \widetilde{u}_j|_i}{2} (\boldsymbol{g}^j \otimes \boldsymbol{g}^i - \boldsymbol{g}^i \otimes \boldsymbol{g}^j) \sqrt{g}\, \mathrm{d}y \right|. \qquad (5.20)
\end{aligned}$$

Noting that the components $n_j, 1 \leq j \leq d$, of the outer unit normal vector field on the two bases $b_{r'} \times \{+h\}$ and $b_{r'} \times \{-h\}$ of the cylinder Ω' vanish unless $j = d$, and that $\partial_k(\sqrt{g}) = (\sum_{j=1}^d \Gamma_{kj}^j)\sqrt{g}$, so that $\sum_{k=1}^d \partial_k(\boldsymbol{g}^k \sqrt{g}) = \boldsymbol{0}$, we deduce that, for each pair (i,j) of indices that satisfy $1 \leq j < i \leq d$,

$$\begin{aligned}
&\int_{\Omega_{r'}} \widetilde{u}_i|_j (\boldsymbol{g}^i \otimes \boldsymbol{g}^j - \boldsymbol{g}^j \otimes \boldsymbol{g}^i) \sqrt{g}\, \mathrm{d}y \\
&= \int_{\Omega_{r'}} \partial_j \widetilde{u}_i (\boldsymbol{g}^i \otimes \boldsymbol{g}^j - \boldsymbol{g}^j \otimes \boldsymbol{g}^i) \sqrt{g}\, \mathrm{d}y - \int_{\Omega_{r'}} \left(\sum_{k=1}^d \Gamma_{ij}^k \widetilde{u}_k \right) (\boldsymbol{g}^i \otimes \boldsymbol{g}^j - \boldsymbol{g}^j \otimes \boldsymbol{g}^i) \sqrt{g}\, \mathrm{d}y \\
&= \int_{\Sigma_{r'}} \widetilde{u}_i n_j (\boldsymbol{g}^i \otimes \boldsymbol{g}^j - \boldsymbol{g}^j \otimes \boldsymbol{g}^i) \sqrt{g}\, \mathrm{d}y - \int_{\Omega_{r'}} \widetilde{u}_i \left((\partial_j \boldsymbol{g}^i) \otimes \boldsymbol{g}^j - \boldsymbol{g}^j \otimes (\partial_j \boldsymbol{g}^i) \right) \sqrt{g}\, \mathrm{d}y \\
&\quad - \int_{\Omega_{r'}} \left(\sum_{k=1}^d \widetilde{u}_k \Gamma_{ij}^k \right) (\boldsymbol{g}^i \otimes \boldsymbol{g}^j - \boldsymbol{g}^j \otimes \boldsymbol{g}^i) \sqrt{g}\, \mathrm{d}y.
\end{aligned}$$

Besides,

$$
\begin{aligned}
&-\sum_{j<i}\widetilde{u}_i\big((\partial_j\boldsymbol{g}^i)\otimes\boldsymbol{g}^j-\boldsymbol{g}^j\otimes(\partial_j\boldsymbol{g}^i)\big)\\
&=\sum_{j<i}\widetilde{u}_i\sum_{k=1}^{d}\Gamma^i_{jk}\big(\boldsymbol{g}^k\otimes\boldsymbol{g}^j-\boldsymbol{g}^j\otimes\boldsymbol{g}^k\big)\\
&=\sum_{k=1}^{d}\sum_{j=1}^{d-1}\left(\sum_{i=j+1}^{d}\widetilde{u}_i\Gamma^i_{jk}\right)\big(\boldsymbol{g}^k\otimes\boldsymbol{g}^j-\boldsymbol{g}^j\otimes\boldsymbol{g}^k\big)\\
&=\sum_{j<k}\left(\sum_{i=j+1}^{d}\widetilde{u}_i\Gamma^i_{jk}-\sum_{i=k+1}^{d}\widetilde{u}_i\Gamma^i_{kj}\right)\big(\boldsymbol{g}^k\otimes\boldsymbol{g}^j-\boldsymbol{g}^j\otimes\boldsymbol{g}^k\big)\\
&=\sum_{j<k}\left(\sum_{i=j+1}^{k}\widetilde{u}_i\Gamma^i_{jk}\right)\big(\boldsymbol{g}^k\otimes\boldsymbol{g}^j-\boldsymbol{g}^j\otimes\boldsymbol{g}^k\big)\\
&=\sum_{j<i}\left(\sum_{k=j+1}^{i}\widetilde{u}_k\Gamma^k_{ji}\right)\big(\boldsymbol{g}^i\otimes\boldsymbol{g}^j-\boldsymbol{g}^j\otimes\boldsymbol{g}^i\big).
\end{aligned}
$$

Therefore, by combining the last two relation, we have

$$
\begin{aligned}
&\int_{\Omega'}\sum_{j<i}\widetilde{u}_i|_j\big(\boldsymbol{g}^i\otimes\boldsymbol{g}^j-\boldsymbol{g}^j\otimes\boldsymbol{g}^i\big)\sqrt{g}\,\mathrm{d}y\\
=&\int_{\Sigma_{r'}}\sum_{j<i}\widetilde{u}_i n_j\big(\boldsymbol{g}^i\otimes\boldsymbol{g}^j-\boldsymbol{g}^j\otimes\boldsymbol{g}^i\big)\sqrt{g}\,\mathrm{d}y\\
&+\int_{\Omega_{r'}}\sum_{j<i}\left(\sum_{k=j+1}^{i}\widetilde{u}_k\Gamma^k_{ji}-\sum_{k=1}^{d}\widetilde{u}_k\Gamma^k_{ij}\right)\big(\boldsymbol{g}^i\otimes\boldsymbol{g}^j-\boldsymbol{g}^j\otimes\boldsymbol{g}^i\big)\sqrt{g}\,\mathrm{d}y.
\end{aligned}
$$

Then we infer from inequality (5.20) that

$$
\begin{aligned}
\left|\int_{\Omega'}\boldsymbol{\nabla}_{\mathrm{a}}\boldsymbol{u}\,\mathrm{d}x\right|\le&\left|\int_{\Sigma_{r'}}\sum_{j<i}\widetilde{u}_i n_j\big(\boldsymbol{g}^i\otimes\boldsymbol{g}^j-\boldsymbol{g}^j\otimes\boldsymbol{g}^i\big)\sqrt{g}\,\mathrm{d}y\right|\\
&+\left|\int_{\Omega_{r'}}\sum_{j<i}\left(\sum_{k\notin\{j+1,\dots,i\}}\widetilde{u}_k\Gamma^k_{ij}\right)\big(\boldsymbol{g}^i\otimes\boldsymbol{g}^j-\boldsymbol{g}^j\otimes\boldsymbol{g}^i\big)\sqrt{g}\,\mathrm{d}y\right|\\
&+\left|\int_{\Omega_{r'}}\sum_{j<i}\frac{\widetilde{u}_i|_j+\widetilde{u}_j|_i}{2}\big(\boldsymbol{g}^i\otimes\boldsymbol{g}^j-\boldsymbol{g}^j\otimes\boldsymbol{g}^i\big)\sqrt{g}\,\mathrm{d}y\right|. \qquad (5.21)
\end{aligned}
$$

To the right-hand side of this inequality is expressed in terms of anti-symmetric matrix fields of type $\boldsymbol{Y}:=\sum_{i<j}\widetilde{Y}_{ij}(\boldsymbol{g}^i\otimes\boldsymbol{g}^j-\boldsymbol{g}^j\otimes\boldsymbol{g}^i)$ defined over $\Omega_{r'}$, or over $\Sigma_{r'}$. Let Y_{kl} denote the Cartesian components of $\boldsymbol{Y}$, i.e $\boldsymbol{Y}=\sum_{l<k}Y_{kl}\left(\boldsymbol{e}_k\otimes\boldsymbol{e}_l-\boldsymbol{e}_l\otimes\boldsymbol{e}_k\right)$, where $\boldsymbol{e}_k:=(\delta_{kj})_{j=1}^d$, $k\in\{1,\dots,d\}$, denote the vectors of the Cartesian basis in $\mathbb{R}^d$. Then the matrix field $\widetilde{\boldsymbol{Y}}:=(\widetilde{Y}_{ij})$ satisfies $\widetilde{\boldsymbol{Y}}=(\boldsymbol{\nabla\Theta})^T\boldsymbol{Y}\boldsymbol{\nabla\Theta}$ and therefore the following estimates hold at all points of $\widetilde{\Omega}$ (the constants A and B are defined in the preamble to the proof):

$$
\begin{aligned}
|\widetilde{\boldsymbol{Y}}|&:=\left[\sum_{i,j}(Y_{ij})^2\right]^{1/2}\le|\boldsymbol{\nabla\Theta}|^2|\boldsymbol{Y}|=\mathrm{Tr}(\boldsymbol{C})|\boldsymbol{Y}|\le A|\boldsymbol{Y}|,\\
|\boldsymbol{Y}|&:=\left[\sum_{i,j}(\widetilde{Y}_{ij})^2\right]^{1/2}\le|\boldsymbol{\nabla\Theta}^{-1}|^2|\widetilde{\boldsymbol{Y}}|=\mathrm{Tr}(\boldsymbol{C}^{-1})|\widetilde{\boldsymbol{Y}}|\le B|\widetilde{\boldsymbol{Y}}|.
\end{aligned}
$$

Furthermore, with U denoting either the set $\Sigma_{r'}$ or the set $\Omega_{r'}$, we have

$$
\begin{aligned}
\left|\int_U\boldsymbol{Y}\sqrt{g}\,\mathrm{d}U\right|&\le\left(\int_U\sqrt{g}\,\mathrm{d}U\right)^{1/2}\left(\int_U|\boldsymbol{Y}|^2\sqrt{g}\,\mathrm{d}U\right)^{1/2}\\
&\le B\left(\int_U\sqrt{g}\,\mathrm{d}U\right)^{1/2}\left[\int_U|\widetilde{\boldsymbol{Y}}|^2\sqrt{g}\,\mathrm{d}U\right]^{1/2}.
\end{aligned}
$$

Using this inequality in (5.21) gives

$$
\begin{aligned}
\left|\int_{\Omega'}\boldsymbol{\nabla}_{\mathrm{a}}\boldsymbol{u}\,\mathrm{d}x\right|&\le B\left(\int_{\Sigma_{r'}}\sqrt{g}\,\mathrm{d}\Sigma_{r'}\right)^{1/2}\left[\int_{\Sigma_{r'}}2\sum_{j<i}(\widetilde{u}_in_j)^2\sqrt{g}\,\mathrm{d}\Sigma_{r'}\right]^{1/2}\\
&\quad+B\left(\int_{\Omega_{r'}}\sqrt{g}\,\mathrm{d}y\right)^{1/2}\left[\int_{\Omega_{r'}}2\sum_{j<i}\left(\sum_{k\notin\{j+1,\dots,i\}}\widetilde{u}_k\Gamma_{ij}^k\right)^2\sqrt{g}\,\mathrm{d}y\right]^{1/2}\\
&\quad+B\left(\int_{\Omega_{r'}}\sqrt{g}\,\mathrm{d}y\right)^{1/2}\left[\int_{\Omega_{r'}}\sum_{j\neq i}\left(\frac{\widetilde{u}_i|_j+\widetilde{u}_j|_i}{2}\right)^2\sqrt{g}\,\mathrm{d}y\right]^{1/2},
\end{aligned}\tag{5.22}
$$

on the one hand.

On the other hand, since $\sum_{j=1}^d(n_j)^2=1$ on $\Sigma_{r'}$, we have

$$
\begin{aligned}
\sum_{j<i}(\widetilde{u}_in_j)^2&\le\left(\sum_i(\widetilde{u}_i)^2\right)\left(\sum_j(n_j)^2\right)=\sum_i(\boldsymbol{u}\cdot\boldsymbol{g}_i)^2\le\sum_i|\boldsymbol{u}|^2|\boldsymbol{g}_i|^2\\
&=|\boldsymbol{u}|^2|\boldsymbol{\nabla\Theta}|^2=|\boldsymbol{u}|^2\mathrm{Tr}(\boldsymbol{C})\le A|\boldsymbol{u}|^2,
\end{aligned}
$$

$$
\begin{aligned}
2\sum_{j<i}\left(\sum_{k\notin\{j+1,\dots,i\}}\widetilde{u}_k\Gamma_{ij}^k\right)^2 &\leq 2\sum_{j<i}\left(\sum_{k\notin\{j+1,\dots,i\}}(\widetilde{u}_k)^2\right)\left(\sum_{k\notin\{j+1,\dots,i\}}(\Gamma_{ij}^k)^2\right)\\
&\leq\left(\sum_{k=1}^d(\widetilde{u}_k)^2\right)\left(2\sum_{j<i}\left(\sum_{k\notin\{j+1,\dots,i\}}(\Gamma_{ij}^k)^2\right)\right)\\
&\leq|\widetilde{\boldsymbol{u}}|^2\left(\sum_{j\neq i}\sum_{k=1}^d(\Gamma_{ij}^k)^2\right)\leq AC|\boldsymbol{u}|^2,
\end{aligned}
$$

$$
\sum_{j\neq i}\left(\frac{\widetilde{u}_i|_j+\widetilde{u}_j|_i}{2}\right)^2\leq\sum_{i,j}\left(\frac{\widetilde{u}_i|_j+\widetilde{u}_j|_i}{2}\right)^2\leq A^2|\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}|^2.
$$

Then we infer from inequality (5.22) that

$$
\begin{aligned}
\left|\int_{\Omega'}\boldsymbol{\nabla}_{\mathrm{a}}\boldsymbol{u}\,\mathrm{d}x\right|\leq{}& B\sqrt{2A}\left(\int_{\Sigma_{r'}}\sqrt{g}\,\mathrm{d}\Sigma_{r'}\right)^{1/2}\left(\int_{\Sigma_{r'}}|\boldsymbol{u}|^2\sqrt{g}\,\mathrm{d}\Sigma_{r'}\right)^{1/2}\\
&+B\sqrt{AC}\left(\int_{\Omega_{r'}}\sqrt{g}\,\mathrm{d}y\right)^{1/2}\left(\int_{\Omega_{r'}}|\boldsymbol{u}|^2\sqrt{g}\,\mathrm{d}y\right)^{1/2}\\
&+AB\left(\int_{\Omega_{r'}}\sqrt{g}\,\mathrm{d}y\right)^{1/2}\left(\int_{\Omega_{r'}}|\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}|^2\sqrt{g}\,\mathrm{d}y\right)^{1/2}.
\end{aligned}
$$

Using the relations

$$
\int_{\Sigma_{r'}}\sqrt{g}\,\mathrm{d}\Sigma_{r'}\leq E|\Sigma_{r'}|,\quad \int_{\Omega_{r'}}\sqrt{g}\,\mathrm{d}y=\int_{\Omega'}\mathrm{d}x=|\Omega'|,
$$

and replacing the integral $\int_{\Sigma_{r'}}|\boldsymbol{u}|^2\sqrt{g}\,\mathrm{d}\Sigma_{r'}$ by relation (5.19) in the right-hand side of the above inequality yields

$$
\begin{aligned}
\left|\int_{\Omega'}\boldsymbol{\nabla}_{\mathrm{a}}\boldsymbol{u}\,\mathrm{d}x\right|\leq{}&\frac{B\sqrt{2AE}|\Sigma_{r'}|}{|\Omega_R\backslash\Omega_{R/2}|^{1/2}}\left(\int_{\boldsymbol{\Theta}(\Omega_R\backslash\Omega_{R/2})}|\boldsymbol{u}|^2\,\mathrm{d}x\right)^{1/2}\\
&+B\sqrt{AC}|\Omega'|^{1/2}\left(\int_{\Omega_{r'}}|\boldsymbol{u}|^2\sqrt{g}\,\mathrm{d}y\right)^{1/2}\\
&+AB|\Omega'|^{1/2}\left(\int_{\Omega_{r'}}|\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}|^2\sqrt{g}\,\mathrm{d}y\right)^{1/2},
\end{aligned}
$$

from which, by using in particular the change of variables $x=\boldsymbol{\Theta}(y), y\in\Omega_{r'}$, in the integrals above, we deduce that

$$
\begin{aligned}
\left|\fint_{\Omega'} \boldsymbol{\nabla}_{\mathrm{a}} \boldsymbol{u}\, \mathrm{d}x\right| \leq & \frac{B\sqrt{2AE}|\Sigma_{r'}|}{|\Omega'||\Omega_R \backslash \Omega_{R/2}|^{1/2}} \left(\int_{\boldsymbol{\Theta}\left(\Omega_R \backslash \Omega_{R/2}\right)} |\boldsymbol{u}|^2 \mathrm{d}x\right)^{1/2} \\
& + \frac{B\sqrt{AC}}{|\Omega'|^{1/2}} \left(\int_{\Omega'} |\boldsymbol{u}|^2 \mathrm{d}x\right)^{1/2} + \frac{AB}{|\Omega'|^{1/2}} \left(\int_{\Omega'} |\boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{u}|^2 \mathrm{d}x\right)^{1/2}.
\end{aligned}
$$

Finally, using that $\Omega' = \boldsymbol{\Theta}(\Omega_{r'}) \subset \boldsymbol{\Theta}(\Omega_R) \subset \Omega$, we obtain the estimate

$$
\left\|\fint_{\Omega'} \boldsymbol{\nabla}_{\mathrm{a}} \boldsymbol{u}\, \mathrm{d}x\right\|_{L^2(\Omega)} \leq \alpha \|\boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{u}\|_{L^2(\Omega)} + \|\boldsymbol{u}\|_{L^2(\Omega)},
$$

where

$$
\begin{aligned}
\alpha &:= \frac{AB|\Omega|^{1/2}}{|\Omega'|^{1/2}}, \\
\beta &:= \frac{B\sqrt{2AE}|\Sigma_{r'}||\Omega|^{1/2}}{|\Omega'||\Omega_R \backslash \Omega_{R/2}|^{1/2}} + \frac{B\sqrt{AC}|\Omega|^{1/2}}{|\Omega'|^{1/2}}.
\end{aligned}
$$

(iii) Let $\boldsymbol{u} \in \mathcal{C}^1(\overline{\Omega}, \mathbb{R}^d)$. Then

$$
\begin{aligned}
\|\boldsymbol{u}\|_{H^1(\Omega)} \leq & \|\boldsymbol{u}\|_{L^2(\Omega)} + \|\boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{u}\|_{L^2(\Omega)} + \left\|\fint_{\Omega'} \boldsymbol{\nabla}_{\mathrm{a}} \boldsymbol{u}\, \mathrm{d}x\right\|_{L^2(\Omega)} \\
& + \left\|\boldsymbol{\nabla}_{\mathrm{a}} \boldsymbol{u} - \fint_{\Omega'} \boldsymbol{\nabla}_{\mathrm{a}} \boldsymbol{u}\, \mathrm{d}x\right\|_{L^2(\Omega)},
\end{aligned}
$$

which combined with the previous inequality yields

$$
\begin{aligned}
\|\boldsymbol{u}\|_{H^1(\Omega)} \leq & \|\boldsymbol{u}\|_{L^2(\Omega)} + \|\boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{u}\|_{L^2(\Omega)} + \alpha \|\boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{u}\|_{L^2(\Omega)} + \beta \|\boldsymbol{u}\|_{L^2(\Omega)} \\
& + \left\|\boldsymbol{\nabla}_{\mathrm{a}} \boldsymbol{u} - \fint_{\Omega'} \boldsymbol{\nabla}_{\mathrm{a}} \boldsymbol{u}\, \mathrm{d}x\right\|_{L^2(\Omega)}.
\end{aligned}
$$

Furthermore, using inequality (3.2) established in Section 3 to estimate the last term of the right-hand side above, we have

$$
\|\boldsymbol{u}\|_{H^1(\Omega)} \leq \left(1 + \alpha + 4K\sqrt{d} \frac{|\Omega|^{1/2}}{|\Omega'|^{1/2}}\right) \|\boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{u}\|_{L^2(\Omega)} + (1+\beta) \|\boldsymbol{u}\|_{L^2(\Omega)}. \tag{5.23}
$$

Remember that $\Omega' := \boldsymbol{\Theta}(\Omega_{r'})$, where $r' \in [R/2, R]$ and $\Omega_R := B_{R,h}$ is a cylinder with radius R and height $2h$ such that $\boldsymbol{\Theta}(\Omega_R) \subset \Omega$, and Σ_R is the lateral face of the cylinder Ω_R. Then $|\Omega_R| = R^{d-1}|\Omega_1|$ and $|\Sigma_R| = ((d-1)/R)|\Omega_R|$, so that

$$
\begin{aligned}
&|\Omega'| = |\boldsymbol{\Theta}(\Omega_{r'})| = \int_{\Omega_{r'}} \sqrt{g}\, \mathrm{d}y \leq E|\Omega_{r'}| \leq E|\Omega_R|, \\
&|\Omega'| = |\boldsymbol{\Theta}(\Omega_{r'})| = \int_{\Omega_{r'}} \sqrt{g}\, \mathrm{d}y \geq \frac{|\Omega_{r'}|}{D} \geq \frac{|\Omega_{R/2}|}{D} = \frac{|\Omega_R|}{2^{d-1}D},
\end{aligned}
$$

and

$$
\frac{|\Sigma_{r'}|}{|\Omega'|^{1/2}|\Omega_R \backslash \Omega_{R/2}|^{1/2}} \leq \frac{|\Sigma_R|}{|\Omega_R|}\left(\frac{2^{d-1}D}{1-2^{d-1}}\right)^{1/2} \leq \frac{(d-1)\sqrt{2^d D}}{R}.
$$

Consequently, the coefficients of inequality (5.23) are bounded above by

$$
\begin{aligned}
1+\alpha+4K\sqrt{d}\frac{|\Omega|^{1/2}}{|\Omega'|^{1/2}} &= 1+(AB+4K\sqrt{d})\frac{|\Omega|^{1/2}}{|\boldsymbol{\Theta}(\Omega_R)|^{1/2}} =: C_3(\Omega,\boldsymbol{\Theta},B_{R,h}), \\
1+\beta &= 1+\left(\frac{B\sqrt{2AE}|\Sigma_{r'}|}{|\Omega'|^{1/2}|\Omega_R\backslash\Omega_{R/2}|^{1/2}}+B\sqrt{AC}\right)\frac{|\Omega|^{1/2}}{|\Omega'|^{1/2}} \\
&\leq 1+B\Big((d-1)\sqrt{2^{d+1}ADE}\frac{1}{R}+\sqrt{AC}\Big)\frac{|\Omega|^{1/2}}{|\boldsymbol{\Theta}(\Omega_R)|^{1/2}} =: C_4(\Omega,\boldsymbol{\Theta},B_{R,h}),
\end{aligned}
$$

so that we finally have

$$
\|\boldsymbol{u}\|_{H^1(\Omega)} \leq C_3(\Omega,\boldsymbol{\Theta},B_{R,h})\|\boldsymbol{u}\|_{L^2(\Omega)} + C_4(\Omega,\boldsymbol{\Theta},B_{R,h})\|\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}\|_{L^2(\Omega)}.
$$

Note that the constants in the right-hand side are independent of the radius $r' = r'(\boldsymbol{u})$ used in the proof, so the inequality holds for all $\boldsymbol{u} \in \mathcal{C}^1(\overline{\Omega};\mathbb{R}^d)$.

Since Ω is a domain (the definition of a domain is given in Section 2) by the assumptions of the theorem, so in particular its boundary is Lipschitz-continuous, the set $\mathcal{C}^1(\overline{\Omega};\mathbb{R}^d)$ is dense in the Sobolev space $H^1(\Omega;\mathbb{R}^d)$. Thus, the inequality

$$
\|\boldsymbol{u}\|_{H^1(\Omega)} \leq C_3(\Omega,\boldsymbol{\Theta},B_{R,h})\|\boldsymbol{u}\|_{L^2(\Omega)} + C_4(\Omega,\boldsymbol{\Theta},B_{R,h})\|\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}\|_{L^2(\Omega)}
$$

holds for all vector fields $\boldsymbol{u} \in H^1(\Omega;\mathbb{R}^d)$. This is precisely inequality (5.17) announced in the statement of the theorem.

This inequality implies in particular that, for all $\boldsymbol{u} \in H^1(\Omega;\mathbb{R}^d)$,

$$
\|\boldsymbol{u}\|_{H^1(\Omega)} \leq \left(1+C(\boldsymbol{\Theta},d)[1+R^{-1}+K]\frac{|\Omega|^{1/2}}{|\boldsymbol{\Theta}(B_{R,h})|^{1/2}}\right)\left(\|\boldsymbol{u}\|_{L^2(\Omega)}+\|\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}\|_{L^2(\Omega)}\right),
$$

where $C(\boldsymbol{\Theta},d)$ is a constant depending only on the dimension d and on the mapping $\boldsymbol{\Theta}: B_{R_0,h_0} \to \mathbb{R}^d$. The proof is complete. $\square$

6 Concluding remarks

Given any bounded and connected open subset Ω of $\mathbb{R}^d$ with a Lipschitz-continuous boundary, and any non-empty relatively open subset Γ_0 of the boundary of Ω, we showed how the inequalities

$$\begin{aligned}
&\inf_{\boldsymbol{r}\in\mathrm{Rig}(\Omega)}\|\boldsymbol{u}-\boldsymbol{r}\|_{H^1(\Omega)}\le C_1\|\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}\|_{L^2(\Omega)}, &&\forall \boldsymbol{u}\in H^1(\Omega;\mathbb{R}^d),\\
&\|\boldsymbol{u}\|_{H^1(\Omega)}\le C_2\|\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}\|_{L^2(\Omega)}, &&\forall \boldsymbol{u}\in H^1(\Omega;\mathbb{R}^d)\quad \text{that vanish on } \Gamma_0,\\
&\|\boldsymbol{u}\|_{H^1(\Omega)}\le C_3\|\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}\|_{L^2(\Omega)}+C_4\|\boldsymbol{u}\|_{L^2(\Omega)}, &&\forall \boldsymbol{u}\in H^1(\Omega;\mathbb{R}^d)
\end{aligned}$$

(which coincide with inequalities (1.1)-(1.3) stated in the introduction) can be derived from the existence of a linear and continuous inverse for the divergence operator $\mathrm{div}: H_0^1(\Omega;\mathbb{R}^d)\to L_0^2(\Omega)$, and we established estimates for the constants appearing in these inequalities that are sharper than those available in the literature, at least in the case of thin domains. More specifically, the constants appearing in these inequalities are given by

$$\begin{aligned}
&C_1=\big(1+2d^{1/2}K\big)(1+W),\\
&C_2=\big(1+2d^{1/2}K\big)(1+P)\left(1+T(1+W)\left(\frac{d|\Omega|}{p_1+p_2}\right)^{1/2}\right),\\
&C_3=1+(AB+4K\sqrt{d})\frac{|\Omega|^{1/2}}{|\boldsymbol{\Theta}(B_{R,h})|^{1/2}},\\
&C_4=1+A^{1/2}B\left(C^{1/2}+2^{(d+1)/2}(d-1)D^{1/2}E^{1/2}\frac{1}{R}\right)\frac{|\Omega|^{1/2}}{|\boldsymbol{\Theta}(B_{R,h})|^{1/2}}
\end{aligned}$$

in terms of the following parameters associated with the set Ω, and also with Γ_0 in the case of C_2. Note that the parameters of interest for asymptotic problems are K,R and h, since they would give the order of magnitude of these constants (the other ones being either independent of Ω and Γ_0, or of lower order of magnitude).

$K=K(\Omega)$ is a constant such that (see Lemma 3.1), for all $f\in L_0^2(\Omega)$, there exists $\boldsymbol{v}\in H_0^1(\Omega;\mathbb{R}^d)$ such that

$$\mathrm{div}\boldsymbol{v}=f \quad \text{in } L^2(\Omega), \quad \|\boldsymbol{\nabla v}\|_{L^2(\Omega)}\le K\|f\|_{L^2(\Omega)};$$

$W=W(\Omega)$ is a constant such that (see Lemma 3.2), for all $f\in H^1(\Omega)$,

$$\left\|f-\fint_\Omega f\right\|_{L^2(\Omega)}\le W\|\nabla f\|_{L^2(\Omega)};$$

$P = P(\Omega,\Gamma_0)$ is a constant such that (see Lemma 4.1), for all $f \in H^1_{\Gamma_0}(\Omega)$,

$$\|f\|_{L^2(\Omega)} \leq P\|\nabla f\|_{L^2(\Omega)};$$

$T = T(\Omega,\Gamma_0)$ is a constant such that (see Lemma 4.2), for all $f \in H^1(\Omega)$,

$$\|f\|_{L^2(\Gamma_0)} \leq T\|f\|_{H^1(\Omega)};$$

$p_1 = p_1(\Omega,\Gamma_0)$ and $p_2 = p_2(\Omega,\Gamma_0)$ are the two smallest eigenvalues of the matrix

$$\int_{\Gamma_0} \left(x - \fint_{\Gamma_0} x \, \mathrm{d}\Gamma_0\right)\left(x - \fint_{\Gamma_0} x \, \mathrm{d}\Gamma_0\right)^T \mathrm{d}\Gamma_0;$$

$B_{R,h} := \{(y,z) \in \mathbb{R}^{d-1} \times \mathbb{R}; |y| < R, |z| < h\}$ and $\boldsymbol{\Theta} \in \mathcal{C}^2(\overline{B_{R_0,h_0}};\mathbb{R}^d)$ are respectively any given cylinder and embedding that satisfy $0 < R \leq R_0, 0 < h \leq h_0$, and $\boldsymbol{\Theta}(\overline{B_{R_0,h_0}}) \subset \overline{\Omega}$;

A,B,C,D and E are constants such that, for all $y \in B_{R_0,h_0}$,

$$\begin{aligned}
&|\boldsymbol{\nabla\Theta}(y)|^2 \leq A,\\
&|(\boldsymbol{\nabla\Theta})^{-1}(y)|^2 \leq B,\\
&\sum_{i \neq j=1}^{d} \left|\boldsymbol{\nabla\Theta}^{-1}(y)\partial_{ij}\boldsymbol{\Theta}(y)\right|^2 \leq C,\\
&D^{-1} \leq |\det \boldsymbol{\nabla\Theta}(y)| \leq E.
\end{aligned}$$

Note that $R = R_0$ and $h = h_0$ is optimal, but it is convenient to distinguish them so that the constants A,B,C,D and E be independent of R and h. This is useful in asymptotic problems with respect to the domain Ω, when R or h would go to zero or to infinity.

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