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A Constructive Proof of Korn's Scaled Inequalities for Shells

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Abstract. One of Korn's scaled inequalities for shells asserts that the H^1-norm of a displacement field of a shell with thickness 2ε clamped on a portion of its lateral boundary, once scaled to a domain independent of ε, is bounded above by the L^2-norm of the corresponding scaled infinitesimal strain tensor field multiplied by a constant of order ε^{-1}. We give a constructive proof to this inequality, and to other two inequalities of this type, which is thus different from the original proof of Ciarlet et al. [Arch. Rational Mech. Anal. 136 (1996), 163–190].

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1 Introduction

The notation used, but not defined, in this introduction is defined in Section 2.

Given a domain ω in $\mathbb{R}^2$ and a non-empty relatively open subset γ_0 of the boundary of ω, one of Korn's scaled inequality for shells asserts that there exists two constants $\varepsilon_0 = \varepsilon_0(\omega,\boldsymbol{\theta}) > 0$ and $C_0 = C_0(\omega,\boldsymbol{\theta},\gamma_0)$ such that for all $0 < \varepsilon \leq \varepsilon_0$

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and for all vector fields $\boldsymbol{u}=(u_i)\in H^1(\Omega^1;\mathbb{R}^3), \Omega^1:=\omega\times(-1,1)$, that vanish on $\gamma_0\times(-1,1)$,

$$\|\boldsymbol{u}\|_{H^1(\Omega^1)}\leq\frac{C_0}{\varepsilon}\|\boldsymbol{e}(\varepsilon,\boldsymbol{u})\|_{L^2(\Omega^1)}, \tag{1.1}$$

where $\boldsymbol{e}(\varepsilon,\boldsymbol{u})=(e_{ij}(\varepsilon,\boldsymbol{u}))\in L^2(\Omega^1;\mathbb{S}^3)$ is the matrix field defined at each point $(y,x_3)\in\Omega^1, y=(y_\alpha)\in\omega$ by

$$\begin{aligned}
&e_{\alpha\beta}(\varepsilon,\boldsymbol{u}):=\frac{1}{2}\left(\frac{\partial u_\alpha}{\partial y_\beta}+\frac{\partial u_\beta}{\partial y_\alpha}\right)-\Gamma^k_{\alpha\beta}(\varepsilon)u_k,\\
&e_{\alpha3}(\varepsilon,\boldsymbol{u})=e_{3\alpha}(\varepsilon,\boldsymbol{u}):=\frac{1}{2}\left(\frac{\partial u_3}{\partial y_\alpha}+\frac{1}{\varepsilon}\frac{\partial u_\alpha}{\partial x_3}\right)-\Gamma^\beta_{\alpha3}(\varepsilon)u_\beta,\\
&e_{33}(\varepsilon,\boldsymbol{u}):=\frac{1}{\varepsilon}\frac{\partial u_3}{\partial x_3},
\end{aligned} \tag{1.2}$$

$\Gamma^k_{\alpha j}(\varepsilon)\in\mathcal{C}^0(\Omega^1)$ are the unique functions such that

$$\frac{\partial}{\partial y_\alpha}\big(\boldsymbol{g}_j(\varepsilon)\big)=\Gamma^k_{\alpha j}(\varepsilon)\boldsymbol{g}_k(\varepsilon),$$

where, for all $(y,x_3)\in\Omega^1$,

$$\begin{aligned}
&\boldsymbol{g}_\alpha(\varepsilon)(y,x_3):=\boldsymbol{a}_\alpha(y)+\varepsilon x_3\partial_\alpha\boldsymbol{a}_3(y),\\
&\boldsymbol{g}_3(\varepsilon)(y,x_3):=\boldsymbol{a}_3(y)
\end{aligned}$$

with

$$\boldsymbol{a}_\alpha:=\frac{\partial\boldsymbol{\theta}}{\partial y_\alpha}\in\mathcal{C}^2(\overline{\omega},\mathbb{R}^3),\quad \boldsymbol{a}_3:=\frac{\boldsymbol{a}_1\wedge\boldsymbol{a}_2}{|\boldsymbol{a}_1\wedge\boldsymbol{a}_2|}\in\mathcal{C}^2(\overline{\omega};\mathbb{R}^3).$$

This result is essential in shell theory to obtain two-dimensional shell models from the three-dimensional model of elasticity by means of convergence theorems when the thickness 2ε of the shell go to zero, see, e.g. Ciarlet [8].

Inequality (1.1) has been proved by Ciarlet *et al.* [10, Theorem 4.1] (see also Ciarlet [8, Theorem 5.3.1]) by a contradiction argument, which we briefly sketch below. Assume that no constants ε_0 and C_0 exist such that inequality (1.1) holds for all $0<\varepsilon\leq\varepsilon_0$ and for all vector fields $\boldsymbol{u}=(u_i)\in H^1(\Omega^1;\mathbb{R}^3)$ that vanish on $\gamma_0\times(-1,1)$. Then there exist sequences $\varepsilon_m>0$ and $\boldsymbol{u}^m=(u^m_i)\in H^1(\Omega^1;\mathbb{R}^3), m\in\mathbb{N}$ such that

$$\begin{aligned}
&\varepsilon_m \to 0 \quad \text{when} \quad m\to+\infty,\\
&\boldsymbol{u}^m=\boldsymbol{0} \quad \text{on } \gamma_0\times(-1,1),\quad \forall m\in\mathbb{N},\\
&\|\boldsymbol{u}^m\|_{H^1(\Omega)}=1,\qquad \forall m\in\mathbb{N},\\
&\frac{1}{\varepsilon_m}\boldsymbol{e}\big(\varepsilon_m,\boldsymbol{u}^m\big)\to 0 \quad \text{in } L^2(\Omega^1;\mathbb{S}^3)\quad \text{when}\quad m\to+\infty.
\end{aligned} \tag{1.3}$$

Since the space $H^1(\Omega^1;\mathbb{R}^3)$ is reflexive and the trace operator from this space into $L^2(\gamma_0\times(-1,1);\mathbb{R}^3)$ is linear and continuous, there exists a subsequence $\boldsymbol{u}^n, n\in\mathbb{N}$, of the sequence $\boldsymbol{u}^m$, $m\in\mathbb{N}$, and a vector field $\boldsymbol{u}=(u_i)\in H^1(\Omega^1;\mathbb{R}^3)$ such that $\boldsymbol{u}=\boldsymbol{0}$ on $\gamma_0\times(-1,1)$ and, when $n\to+\infty$,

$$\begin{aligned}\boldsymbol{u}^n &\rightharpoonup \boldsymbol{u} \quad \text{in } H^1(\Omega;\mathbb{R}^3),\\ \boldsymbol{u}^n &\to \boldsymbol{u} \quad \text{in } L^2(\Omega;\mathbb{R}^3).\end{aligned}$$

Noting that

$$\frac{\partial}{\partial x_3}\left(\frac{1}{\varepsilon_n}\boldsymbol{e}(\varepsilon_n,\boldsymbol{u}^n)\right) \rightharpoonup 0 \quad \text{in } H^{-1}(\Omega;\mathbb{S}^3) \quad \text{when} \quad n\to\infty,$$

and that the functions

$$\rho_{\alpha\beta}(\boldsymbol{u}^n):=\left(\frac{\partial^2\boldsymbol{u}^n}{\partial y_\alpha\partial y_\beta}-\Gamma^\sigma_{\alpha\beta}(0)\frac{\partial\boldsymbol{u}^n}{\partial y_\sigma}\right)\cdot\boldsymbol{g}_3(0)\in H^{-1}(\Omega^1)$$

satisfy the inequality

$$\begin{aligned}&\left\|\rho_{\alpha\beta}(\boldsymbol{u}^n)+\frac{1}{\varepsilon_n}\frac{\partial}{\partial x_3}\big(e_{\alpha\beta}(\varepsilon_n,\boldsymbol{u}^n)\big)\right\|_{H^{-1}(\Omega)}\\ &\le C_0\left(\sum_{i=1}^3\left\|e_{i3}(\varepsilon_n,\boldsymbol{u}^n)\right\|_{L^2(\Omega)}+\varepsilon_n\sum_{\alpha=1}^2\left\|u^n_\alpha\right\|_{L^2(\Omega)}+\varepsilon_n\left\|u^n_3\right\|_{H^1(\Omega)}\right)\end{aligned}$$

for some constant C_0 independent of n, one infers from (1.3) that

$$\rho_{\alpha\beta}(\boldsymbol{u}^n) \to 0 \quad \text{in } H^{-1}(\Omega) \quad \text{when} \quad n\to+\infty,$$

that the field $\boldsymbol{u}=(u_i)\in H^1(\Omega^1;\mathbb{R}^3)$ does not depend on the variable x_3, that the vector field

$$\overline{\boldsymbol{u}}:=(\overline{u}_i)\in H^1(\omega;\mathbb{R}^3),\quad \overline{u}_i(y):=\frac{1}{2}\int_{-1}^1 u_i(y,x_3)\,\mathrm{d}x_3 \quad \text{for a.e.} \quad y\in\omega$$

satisfies $\overline{u}_\alpha\in H^1(\omega), \overline{u}_3\in H^2(\omega)$ and $\overline{u}_\alpha=\overline{u}_3=\partial_\nu\overline{u}_3=0$ on γ_0, and finally that the functions

$$\begin{aligned}\gamma_{\alpha\beta}(\overline{\boldsymbol{u}})&:=\frac{1}{2}(\partial_\alpha\overline{\boldsymbol{u}}\cdot\partial_\beta\boldsymbol{\theta}+\partial_\beta\overline{\boldsymbol{u}}\cdot\partial_\alpha\boldsymbol{\theta}),\\ \rho_{\alpha\beta}(\overline{\boldsymbol{u}})&:=\left(\frac{\partial^2\overline{\boldsymbol{u}}}{\partial y_\alpha\partial y_\beta}-\Gamma^\sigma_{\alpha\beta}\frac{\partial\overline{\boldsymbol{u}}}{\partial y_\sigma}\right)\cdot\boldsymbol{a}_3,\end{aligned}$$

where $\Gamma^{\sigma}_{\alpha\beta} \in \mathcal{C}^1(\overline{\omega}) := \boldsymbol{a}^{\sigma} \cdot \partial \boldsymbol{a}_{\alpha} / \partial y_{\beta}$ and $\boldsymbol{a}^{\sigma}$ designate the dual basis of $\boldsymbol{a}_{\alpha}$, both vanish in ω.

That $\gamma_{\alpha\beta}(\overline{\boldsymbol{u}}) = \rho_{\alpha\beta}(\overline{\boldsymbol{u}}) = 0$ in ω mean that both the infinitesimal change of metric tensor and the infinitesimal change of curvature tensor associated with the displacement field $\overline{\boldsymbol{u}}$ of the surface $S = \boldsymbol{\theta}(\overline{\omega})$ vanish, so that $\overline{\boldsymbol{u}}$ is an infinitesimal rigid displacement field of S. Since in addition $\overline{\boldsymbol{u}}$ vanish on γ_0, a well-known infinitesimal rigid displacement lemma for surfaces in $\mathbb{R}^3$ (see, e.g. Ciarlet [8, 9]) shows that $\overline{\boldsymbol{u}} = \boldsymbol{0}$ in ω, on the one hand. On the other hand, since $\boldsymbol{u}$ is independent of x_3,

$$\sqrt{2}\|\overline{\boldsymbol{u}}\|_{H^1(\omega)} = \|\boldsymbol{u}\|_{H^1(\Omega)} = \lim_{n\to\infty} \|\boldsymbol{u}^n\|_{H^1(\Omega)} = 1$$

by (1.3). This is a absurd, so there exist two constants $\varepsilon_0 > 0$ and C_0 such that inequality (1.1) holds for all $0 < \varepsilon \leq \varepsilon_0$ and for all vector fields $\boldsymbol{u} = (u_i) \in H^1(\Omega^1; \mathbb{R}^3)$ that vanish on $\gamma_0 \times (-1,1)$.

The objective of this paper is to give a new, constructive proof to inequality (1.1). Note that our proof yields a slightly stronger inequality than (1.1) (cf. Theorem 4.1) and two other inequalities of similar type, and that it can be generalized to domains $\omega \subset \mathbb{R}^d$ in higher dimensions $d \geq 2$.

The paper is organised as follows. Section 2 states the notation and preliminary lemmas used in the paper. Section 3 establishes three inequalities of Korn's type in curvilinear coordinates on a domain dependent on the thickness $2\varepsilon > 0$ of the shell, with constants in their right-hand sides depending explicitly on ε, cf. Theorem 3.1. Section 4 establishes three inequalities of Korn's type in curvilinear coordinates on a domain independent of the thickness of the shell, cf. Theorem 4.1. Such inequalities are useful in elasticity theory for the asymptotic analysis when $\varepsilon \to 0$ of the three-dimensional model of elastic shells.

2 Preliminaries

Greek indices and exponents vary in the set $\{1,2\}$, Latin indices and exponents vary in the set $\{1,2,3\}$ unless stated otherwise, and the summation convention with respect to repeated indices and exponents are used in conjunction with these rules. Boldface letters denote vectors, matrices, vector fields and matrix fields to distinguish them from scalars and (scalar-valued) functions.

The Euclidean scalar product in $\mathbb{R}^3$ and the Frobenius scalar product in $\mathbb{R}^{k\times n}$, $k,n \geq 1$, are both denoted by $\cdot$ (a dot). The vector product in $\mathbb{R}^3$ is denoted by $\wedge$. The Euclidean norm of vectors, the Frobenius norm of matrices, Lebesgue's measure and Haussdorff's measure are all denoted by $|\cdot|$. In particular, if Ω is a domain in $\mathbb{R}^3$ and Γ_0 is a non-empty relatively open subset of its boundary $\Gamma := \partial\Omega$, then

$$|\Omega| = \int_\Omega \mathrm{d}x, \quad |\Gamma_0| = \int_{\Gamma_0} \mathrm{d}\Gamma.$$

Besides, $\mathbb{S}^n := \{\boldsymbol{S} \in \mathbb{R}^{n\times n}; \boldsymbol{S}^T = \boldsymbol{S}\}$ and $\mathbb{A}^n := \{\boldsymbol{A} \in \mathbb{R}^{n\times n}; \boldsymbol{A}^T = -\boldsymbol{A}\}$ respectively denote the space of all real symmetric matrices of order n and the space of all real anti-symmetric matrices of order n.

A subset $\Omega \subset \mathbb{R}^d, d \geq 2$, is called domain if it is open, connected, bounded, and has a Lipschitz-continuous boundary in the sense of Adams [2], the set Ω being then locally on only one side of $\Gamma := \partial\Omega$.

The gradient of a vector field $\boldsymbol{u} = (u_i) : \Omega \subset \mathbb{R}^d \to \mathbb{R}^k$ is the matrix field $\boldsymbol{\nabla u}$: $\Omega \to \mathbb{R}^{k\times d}$ with the partial derivative $\partial u_i / \partial x_j$ at its i-th row and j-th column. When $k = d$,

$$\begin{aligned} &\mathrm{div}(\boldsymbol{u}) := \mathrm{Tr}(\boldsymbol{\nabla u}), \\ &\boldsymbol{\nabla}_{\mathbf{s}}\boldsymbol{u} := \frac{1}{2}\left(\boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^T\right), \\ &\boldsymbol{\nabla}_{\mathbf{a}}\boldsymbol{u} := \frac{1}{2}\left(\boldsymbol{\nabla u} - (\boldsymbol{\nabla u})^T\right), \end{aligned}$$

where $(\boldsymbol{\nabla u})^T$ denote the transpose matrix of $\boldsymbol{\nabla u}$.

The notation $L^2(\Omega;\mathbb{R}^{k\times n})$ denotes the Lebesgue space of matrix fields from a domain $\Omega \subset \mathbb{R}^d$ into the space $\mathbb{R}^{k\times n}$ of $k \times n$ real matrices with components in the usual Lebesgue space $L^2(\Omega)$. The notation $H^1(\Omega;\mathbb{R}^k)$ denotes the Sobolev space of vector fields from a domain $\Omega \subset \mathbb{R}^d$ into $\mathbb{R}^k$ with components in $H^1(\Omega)$. The norms in these spaces are denoted and defined by

$$\begin{aligned} &\|\boldsymbol{A}\|_{L^2(\Omega)} := \|\,|\boldsymbol{A}|\,\|_{L^2(\Omega)}, && \forall \boldsymbol{A} = (a_{ij}) \in L^2(\Omega;\mathbb{R}^{k\times n}), \\ &\|\boldsymbol{u}\|_{H^1(\Omega)} := \left(\|\,|\boldsymbol{u}|\,\|^2_{L^2(\Omega)} + \|\,|\boldsymbol{\nabla u}|\,\|^2_{L^2(\Omega)}\right)^{1/2}, && \forall \boldsymbol{u} = (u_i) \in H^1(\Omega;\mathbb{R}^k), \end{aligned}$$

where

$$|\boldsymbol{A}| := \left(\sum_{i,j} |a_{ij}|^2\right)^{1/2}, \quad |\boldsymbol{u}| := \left(\sum_i |u_i|^2\right)^{1/2}, \quad |\boldsymbol{\nabla u}| := \left(\sum_{i,j} \left|\frac{\partial u_i}{\partial x_j}\right|^2\right)^{1/2}.$$

Given any domain $\Omega \subset \mathbb{R}^d$ and any non-empty relatively open subset $\Gamma_0 \subset \Gamma$, there exists a linear and continuous function (see, e.g. Adams [2])

$$\boldsymbol{u} \in H^1(\Omega;\mathbb{R}^k) \to \boldsymbol{u}|_{\Gamma_0} \in L^2\left(\Gamma_0;\mathbb{R}^k\right),$$

called trace operator, that extends the usual restriction operator $\boldsymbol{u}\in\mathcal{C}^1(\overline{\Omega};\mathbb{R}^k)\to\boldsymbol{u}|_{\Gamma_0}\in\mathcal{C}^0(\Gamma_0;\mathbb{R}^k)$ defined by $(\boldsymbol{u}|_{\Gamma_0})(x):=\boldsymbol{u}(x)$ for all $x\in\Gamma_0$. The kernel of this trace operator is denoted

$$H^1_{\Gamma_0}(\Omega;\mathbb{R}^k):=\{\boldsymbol{u}\in H^1(\Omega);\ \boldsymbol{u}|_{\Gamma_0}=0\}.$$

Since Ω is a domain, the space $\mathcal{D}(\Omega;\mathbb{R}^k)$ of infinitely differentiable fields from Ω into $\mathbb{R}^k$ with compact support contained in Ω is dense in $H^1_{\partial\Omega}(\Omega;\mathbb{R}^k)$. The dual of the space $H^1_0(\Omega):=H^1_{\partial\Omega}(\Omega)$ is denoted $H^{-1}(\Omega)$.

Convergences in normed vector spaces with respect to the corresponding strong and weak topologies are denoted respectively by $\to$ and $\rightharpoonup$.

We conclude this section by stating five classical lemmas that will be used in this paper.

Lemma 2.1 (Divergence Equation)**.** *Given any domain $\Omega\subset\mathbb{R}^d, d\geq 1$, there exists a constant $K(\Omega)$ with the following property: For every $f\in L^2(\Omega)$ satisfying $\int_\Omega f(x)\,\mathrm{d}x=0$, there exists a vector field $\boldsymbol{u}\in H^1(\Omega;\mathbb{R}^d)$ such that*

$$\begin{aligned} &\mathrm{div}(\boldsymbol{u})=f \quad \text{in } L^2(\Omega),\\ &\boldsymbol{u}|_{\partial\Omega}=\boldsymbol{0} \quad \text{in } L^2(\partial\Omega), \end{aligned}$$

and

$$\|\boldsymbol{\nabla u}\|_{L^2(\Omega)}\leq K(\Omega)\|f\|_{L^2(\Omega)}.$$

Proof. See, e.g. Acosta *et al.* [1], Amrouche and Girault [3], Bogovskii [4] or Borchers and Sohr [5], Bourgain and Brezis [6], Ciarlet [9], Dacorogna [11], Dacorogna [12], Galdi [16], Ladyzhenskaya [23], Temam [30]. □

Lemma 2.2 (Poincaré-Wirtinger)**.** *Given any domain $\Omega\subset\mathbb{R}^d$, there exists a constant $W(\Omega)$ such that, for all $\boldsymbol{u}\in H^1(\Omega;\mathbb{R}^d)$,*

$$\left\|\boldsymbol{u}-\frac{1}{|\Omega|}\int_\Omega \boldsymbol{u}(x)\,\mathrm{d}x\right\|_{L^2(\Omega)}\leq W(\Omega)\|\boldsymbol{\nabla u}\|_{L^2(\Omega)}.$$

Proof. See, e.g. Adams [2]. □

Lemma 2.3 (Poincaré)**.** *Given any domain $\Omega\subset\mathbb{R}^d, d\geq 2$, and any non-empty relatively open subset Γ_0 of the boundary of Ω, there exists a constant $P(\Omega,\Gamma_0)$ such that, for all $\boldsymbol{u}\in H^1_{\Gamma_0}(\Omega;\mathbb{R}^d)$,*

$$\|\boldsymbol{u}\|_{L^2(\Omega)}\leq P(\Omega,\Gamma_0)\|\boldsymbol{\nabla u}\|_{L^2(\Omega)}.$$

Proof. See, e.g. Adams [2]. □

Lemma 2.4 (Trace Operator)**.** *Given any domain $\Omega\subset\mathbb{R}^d$ and any non-empty relatively open subset Γ_0 of the boundary of Ω, there exists a constant $T(\Omega,\Gamma_0)$ such that, for all $\boldsymbol{u}\in H^1(\Omega;\mathbb{R}^d)$,*

$$\|\boldsymbol{u}|_{\Gamma_0}\|_{L^2(\Gamma_0)}\leq T(\Omega,\Gamma_0)\|\boldsymbol{u}\|_{H^1(\Omega)}.$$

Proof. See, e.g., Adams [2]. □

Lemma 2.5. *Let $\boldsymbol{X},\boldsymbol{Y}\in\mathbb{S}^d$ be two symmetric matrices with eigenvalues $-\infty<x_1\leq x_2\leq\cdots\leq x_d<\infty$ and $-\infty<y_1\leq y_2\leq\cdots\leq y_d<\infty$. Then*

$$\boldsymbol{X}:\boldsymbol{Y}\geq x_1y_d+x_2y_{d-1}+\cdots+x_dy_1.$$

Proof. See Mardare and Nguyen [25]. □

3 Inequalities of Korn's type in curvilinear coordinates for shells

The objective of this section is to establish inequalities similar to Korn's scaled inequality (1.1) mentioned in the introduction, but for vector fields defined on the "original" shell with thickness 2ε, instead of the "scaled one" in (1.1). Then inequality (1.1) will be recovered from one of these inequalities in Section 4.

A shell is a three-dimensional domain $\hat{\Omega}^\varepsilon\subset\mathbb{R}^3$ that lies within a given distance $\varepsilon>0$ from a given (two-dimensional) surface $S\subset\mathbb{R}^3$. More specifically, this means that

$$\hat{\Omega}^\varepsilon:=\left\{\hat{x}=(\hat{x}_i)\in\mathbb{R}^3;\hat{x}=\boldsymbol{\theta}(y)+x_3^\varepsilon\boldsymbol{a}_3(y),y=(y_1,y_2)\in\omega,x_3^\varepsilon\in(-\varepsilon,\varepsilon)\right\},$$

where $S=\boldsymbol{\theta}(\omega)$ is a surface in $\mathbb{R}^3$ defined as the image of a domain $\omega\subset\mathbb{R}^2$ by an embedding $\boldsymbol{\theta}\in\mathcal{C}^3(\overline{\omega};\mathbb{R}^3)$ and

$$\boldsymbol{a}_3:=\frac{\boldsymbol{a}_1\wedge\boldsymbol{a}_2}{|\boldsymbol{a}_1\wedge\boldsymbol{a}_2|}\in\mathcal{C}^2(\overline{\omega};\mathbb{R}^3),\quad \boldsymbol{a}_\alpha:=\frac{\partial\boldsymbol{\theta}}{\partial y_\alpha}\in\mathcal{C}^2(\overline{\omega};\mathbb{R}^3).$$

Note that the assumption that $\boldsymbol{\theta}$ be an embedding, hence an immersion, from $\overline{\omega}$ into $\mathbb{R}^3$ implies that the two vector fields $\boldsymbol{a}_\alpha$ are linearly independent at every point of $\overline{\omega}$, which in turn implies that the vector field $\boldsymbol{a}_3:\omega\to\mathbb{R}^3$ is well defined and coincides with the positively-oriented unit normal vector field to the surface S. Note the difference of notation between the variable $x_3^\varepsilon\in(-\varepsilon,\varepsilon)$ used in this section and the variable $x_3\in(-1,1)$ used in Sections 1 and 4, and the difference between the sets $\hat{\Omega}^\varepsilon$ above and Ω^ε below.

Remark 3.1. We assumed for simplicity that the surface S is defined by a single local chart $\boldsymbol{\theta}$ and that the shell has constant thickness along S. The more general case of shells with variable thickness and with a middle surface S defined by several local charts can be considered as well by extending the arguments in this paper as in, e.g. Busse [7] and Mardare [26].

As proven by Ciarlet [8], the assumption that $\boldsymbol{\theta}$ is an embedding of class $\mathcal{C}^3$ from $\overline{\omega}$ into $\mathbb{R}^3$ implies that there exists $\varepsilon_0=\varepsilon_0(\omega,\boldsymbol{\theta})>0$ such that the mapping $\boldsymbol{\Theta}:\overline{\omega}\times\mathbb{R}\to\mathbb{R}^3$, defined by

$$\boldsymbol{\Theta}\big(y,x_3^\varepsilon\big):=\boldsymbol{\theta}(y)+x_3^\varepsilon \boldsymbol{a}_3(y)$$

for all $y=(y_1,y_2)\in\overline{\omega}$ and all $x_3^\varepsilon\in\mathbb{R}$, becomes an embedding when restricted to the subset $\overline{\omega}\times[-\varepsilon_0,\varepsilon_0]$ of $\overline{\omega}\times\mathbb{R}$. Since a shell must satisfy the impenetrability of matter axiom, we assume in all that follows that

$$0<\varepsilon\leq\varepsilon_0.$$

It follows that the restriction of $\boldsymbol{\Theta}$ to $\overline{\omega}\times[-\varepsilon,\varepsilon]$ is an embedding of class $\mathcal{C}^2$, so that, for each ε, the set

$$\hat{\Omega}^\varepsilon:=\boldsymbol{\Theta}(\Omega^\varepsilon),\quad \text{where}\quad \Omega^\varepsilon:=\omega\times(-\varepsilon,\varepsilon),$$

is a domain in $\mathbb{R}^3$, that is (the definition of a domain is given in Section 2), $\hat{\Omega}^\varepsilon$ is bounded, connected, open, and its boundary is Lipschitz-continuous. This implies that the following inequalities of Korn's type hold in $\hat{\Omega}^\varepsilon$.

Lemma 3.1 (Korn's Inequalities in Cartesian Coordinates)**.** *Given any domain $\omega\subset\mathbb{R}^2$, any embedding $\boldsymbol{\theta}\in\mathcal{C}^3(\overline{\omega};\mathbb{R}^3)$, and any $\varepsilon_0=\varepsilon_0(\omega,\boldsymbol{\theta})>0$ such that the restriction of $\boldsymbol{\Theta}$ to $\overline{\omega}\times[-\varepsilon_0,\varepsilon_0]$ be an embedding, define for each $0<\varepsilon\leq\varepsilon_0$ the set*

$$\hat{\Omega}^\varepsilon:=\big\{\boldsymbol{\theta}(y)+x_3^\varepsilon\boldsymbol{a}_3(y), y\in\omega, x_3^\varepsilon\in(-\varepsilon,\varepsilon)\big\}.$$

(a) For each $0<\varepsilon\leq\varepsilon_0$, there exists constants $C_1(\varepsilon),C_2(\varepsilon)$ and $C_3(\varepsilon)$ such that, for all $\hat{\boldsymbol{u}}\in H^1(\hat{\Omega}^\varepsilon;\mathbb{R}^3)$,

$$\|\hat{\boldsymbol{u}}\|_{H^1(\hat{\Omega}^\varepsilon)}\leq C_1(\varepsilon)\|\hat{\boldsymbol{\nabla}}_{\mathbf{s}}\hat{\boldsymbol{u}}\|_{L^2(\hat{\Omega}^\varepsilon)}+C_2(\varepsilon)\|\hat{\boldsymbol{u}}\|_{L^2(\hat{\Omega}^\varepsilon)},\tag{3.1}$$

$$\inf_{\hat{\boldsymbol{r}}\in\mathrm{Rig}(\hat{\Omega}^\varepsilon)}\|\hat{\boldsymbol{u}}-\hat{\boldsymbol{r}}\|_{H^1(\hat{\Omega}^\varepsilon)}\leq C_3(\varepsilon)\|\hat{\boldsymbol{\nabla}}_{\mathbf{s}}\hat{\boldsymbol{u}}\|_{L^2(\hat{\Omega}^\varepsilon)},\tag{3.2}$$

where $\hat{\boldsymbol{\nabla}}_{\mathbf{s}}\hat{\boldsymbol{u}}$ denotes the symmetric matrix field with components $(\partial\hat{u}_i/\partial\hat{x}_j+\partial\hat{u}_j/\partial\hat{x}_i)/2$ at its i-th row and j-th column, $\hat{x}_i$ denoting the Cartesian coordinates in the space $\mathbb{R}^3$ containing $\hat{\Omega}^\varepsilon$, and

$$\mathrm{Rig}(\hat{\Omega}^\varepsilon):=\big\{\hat{\boldsymbol{r}}\in H^1(\hat{\Omega}^\varepsilon;\mathbb{R}^3);\ \hat{\boldsymbol{\nabla}}_{\mathbf{s}}\hat{\boldsymbol{r}}=\mathbf{0}\ \textit{a.e. in}\ \hat{\Omega}^\varepsilon\big\}.$$

(b) Given in addition any non-empty relatively open subset γ_0 of the boundary of ω, for each $0<\varepsilon\leq\varepsilon_0$, there exists a constant $C_4(\varepsilon)$ such that

$$\|\hat{\boldsymbol{u}}\|_{H^1(\hat{\Omega}^\varepsilon)} \leq C_4(\varepsilon)\|\hat{\boldsymbol{\nabla}}_{\mathbf{s}}\hat{\boldsymbol{u}}\|_{L^2(\hat{\Omega}^\varepsilon)} \tag{3.3}$$

for all $\hat{\boldsymbol{u}}=(\hat{u}_i)\in H^1(\hat{\Omega}^\varepsilon;\mathbb{R}^3)$ that vanish on the subset

$$\hat{\Gamma}_0^\varepsilon := \left\{\boldsymbol{\theta}(y)+x_3^\varepsilon \boldsymbol{a}_3(y);\, y\in\gamma_0,\, x_3^\varepsilon\in(-\varepsilon,\varepsilon)\right\}$$

of the boundary of $\hat{\Omega}^\varepsilon$.

Proof. See, e.g. Duvaut and Lions [13], Fichera [14], Friedrichs [15], Gobert [17], Hlaváček [18], Hlaváček and Nečas [19], Horgan [20,21], Kontradev and Oleinik [22], Mardare and Nguyen [25], Miyoshi [27], Mosolov and Myasnikov [28], Nitsche [29], Temam [30]. □

Remark 3.2. A classical infinitesimal rigid displacement lemma for open sets (see, e.g. Ciarlet [8]) shows that

$$\mathrm{Rig}(\hat{\Omega}^\varepsilon)=\mathrm{i}(\mathbb{R}^3\times\mathbb{A}^3),$$

where $\mathrm{i}:\mathbb{R}^3\times\mathbb{A}^3\to H^1(\hat{\Omega}^\varepsilon;\mathbb{R}^3)$ is the function defined for each $(\boldsymbol{a},\boldsymbol{B})\in\mathbb{R}^3\times\mathbb{A}^3$ by $(\mathrm{i}(\boldsymbol{a},\boldsymbol{B}))(\hat{x}):=\boldsymbol{a}+\boldsymbol{B}\hat{x}$ for all $\hat{x}\in\hat{\Omega}^\varepsilon$, so that $\mathrm{Rig}(\hat{\Omega}^\varepsilon)$ is a finite-dimensional subspace of $H^1(\hat{\Omega}^\varepsilon;\mathbb{R}^3)$.

Note that inequalities (3.1)-(3.3) of Lemma 3.1 estimate the partial derivatives of the Cartesian components $\hat{u}_i$ of $\hat{\boldsymbol{u}}$ with respect to the Cartesian coordinates $\hat{x}_j$ of a point $\hat{x}\in\hat{\Omega}^\varepsilon$. This is not good enough in shell theory since the relevant unknowns are defined in the curvilinear coordinates along the middle surface S of the shell and across its thickness, and since the (covariant) components of the symmetric part of the gradient are of different orders of magnitude with respect to the thickness of the shell depending on whether they are tangential or normal to S. So the objective of this section (which is achieved in Theorem 3.1 below) is twofold: to show that inequalities of Korn's type similar to those in Lemma 3.1 hold as well in curvilinear coordinates, and to estimate the order of magnitude of the constants in these new inequalities with respect to ε. These are precisely the inequalities used in shell theory to model the behavior of elastic shells.

The curvilinear coordinates used to describe the shell $\hat{\Omega}^\varepsilon$ are thus

$$(y,x_3^\varepsilon)\in\Omega^\varepsilon:=\omega\times(-\varepsilon,\varepsilon),$$

where $y=(y_1,y_2)$ are the curvilinear coordinates along the surface $S=\boldsymbol{\theta}(\omega)$ and $x_3^\varepsilon\in(-\varepsilon,\varepsilon)$ is the abscissa along the normal line to S each point $\boldsymbol{\theta}(y)\in S$.

Let $(y,x_3^\varepsilon)\in\overline{\Omega^\varepsilon}$ be any point in $\overline{\Omega^\varepsilon}$. Since $\boldsymbol{\Theta}:\overline{\Omega^\varepsilon}\to\mathbb{R}^3$ is an embedding (remember that $\varepsilon\leq\varepsilon_0$ as mentioned above), the three vectors

$$\begin{aligned}
\boldsymbol{g}_\alpha(y,x_3^\varepsilon) &:= \frac{\partial\boldsymbol{\Theta}}{\partial y_\alpha}(y,x_3^\varepsilon) = \boldsymbol{a}_\alpha(y) + x_3^\varepsilon\frac{\partial\boldsymbol{a}_3}{\partial y_\alpha}(y),\\
\boldsymbol{g}_3(y,x_3^\varepsilon) &:= \frac{\partial\boldsymbol{\Theta}}{\partial x_3^\varepsilon}(y,x_3^\varepsilon) = \boldsymbol{a}_3(y)
\end{aligned}$$

form a basis of $\mathbb{R}^3$, the three vectors $\boldsymbol{g}^i(y,x_3^\varepsilon), i\in\{1,2,3\}$, defined by the nine relations

$$\boldsymbol{g}^i(y,x_3^\varepsilon)\cdot\boldsymbol{g}_j(y,x_3^\varepsilon) = \delta_j^i,$$

where δ_j^i denotes Kronecker's symbol, form its dual basis in $\mathbb{R}^3$, and the nine matrices

$$\boldsymbol{g}^i(y,x_3^\varepsilon)\otimes\boldsymbol{g}^j(y,x_3^\varepsilon) := \boldsymbol{g}^i(y,x_3^\varepsilon)\left[\boldsymbol{g}^j(y,x_3^\varepsilon)\right]^T$$

form a basis in $\mathbb{R}^{3\times3}$ (Note that $\boldsymbol{g}^i(y,x_3^\varepsilon)$ is a column vector in $\boldsymbol{R}^3$. Consequently, $\boldsymbol{g}^i(y,x_3^\varepsilon)[\boldsymbol{g}^j(y,x_3^\varepsilon)]^T$ is a 3×3 matrix). The basis formed by the vectors $\boldsymbol{g}_i(y,x_3^\varepsilon)$ is called the covariant basis, while the basis formed by the vectors $\boldsymbol{g}^i(y,x_3^\varepsilon)$ is called the contravariant basis, of $\mathbb{R}^3$ induced by $\boldsymbol{\Theta}$ at the point (y,x_3^ε).

The (covariant) components of the vector field $\hat{\boldsymbol{u}}\in H^1(\hat{\Omega}^\varepsilon;\mathbb{R}^3)$ associated with these curvilinear coordinates are the functions

$$u_i := \boldsymbol{u}\cdot\boldsymbol{g}_i \in H^1(\Omega^\varepsilon),$$

where

$$\boldsymbol{u} := \hat{\boldsymbol{u}}\circ\boldsymbol{\Theta}\in H^1(\Omega^\varepsilon;\mathbb{R}^3),$$

the (covariant) components of the matrix field $\hat{\boldsymbol{\nabla}}\hat{\boldsymbol{u}}$ are the functions

$$u_i|_j := \frac{\partial\boldsymbol{u}}{\partial x_j^\varepsilon}\cdot\boldsymbol{g}_i = \frac{\partial u_i}{\partial x_j^\varepsilon} - \Gamma_{ij}^k u_k \in L^2(\Omega^\varepsilon),$$

where $x_\alpha^\varepsilon := y_\alpha$ and

$$\Gamma_{ij}^k := \frac{\partial^2\boldsymbol{\Theta}}{\partial x_i^\varepsilon\partial x_j^\varepsilon}\cdot\boldsymbol{g}^k\in C^0(\overline{\Omega^\varepsilon}),$$

and the (covariant) components of the matrix field $\hat{\boldsymbol{\nabla}}_{\mathbf{s}}\hat{\boldsymbol{u}}$ are the functions

$$e_{ij}(\boldsymbol{u}) := \frac{1}{2}\left(\frac{\partial\boldsymbol{u}}{\partial x_j^\varepsilon}\cdot\boldsymbol{g}_i + \frac{\partial\boldsymbol{u}}{\partial x_i^\varepsilon}\cdot\boldsymbol{g}_j\right)$$

$$=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j^\varepsilon}+\frac{\partial u_j}{\partial x_i^\varepsilon}\right)-\Gamma_{ij}^k u_k$$
$$=\frac{1}{2}(u_i|_j+u_j|_i)\in L^2(\Omega^\varepsilon).$$

Note that the above covariant components of the fields $\hat{\boldsymbol{u}}$, $\hat{\boldsymbol{\nabla}}\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{\nabla}}_{\mathbf{s}}\hat{\boldsymbol{u}}$ are related to their Cartesian components with respect to a Cartesian basis $\hat{\boldsymbol{e}}_i, i\in\{1,2,3\}$, in $\mathbb{R}^3$ by the following relations: At almost all points of Ω^ε,

$$u_i\boldsymbol{g}^i=(\hat{u}_i\circ\boldsymbol{\Theta})\hat{\boldsymbol{e}}_i,$$
$$u_i|_j\boldsymbol{g}^i\otimes\boldsymbol{g}^j=\left(\frac{\partial\hat{u}_i}{\partial\hat{x}_j}\circ\boldsymbol{\Theta}\right)\hat{\boldsymbol{e}}_i\otimes\hat{\boldsymbol{e}}_j,$$
$$e_{ij}(\boldsymbol{u})\boldsymbol{g}^i\otimes\boldsymbol{g}^j=\frac{1}{2}\left(\frac{\partial\hat{u}_i}{\partial\hat{x}_j}\circ\boldsymbol{\Theta}+\frac{\partial\hat{u}_j}{\partial\hat{x}_i}\circ\boldsymbol{\Theta}\right)\hat{\boldsymbol{e}}_i\otimes\hat{\boldsymbol{e}}_j.$$

We are now in a position to state the main result of this section. Note that the assumption about $\boldsymbol{\theta}(\gamma_0)$ can be omitted in the statement of the theorem below at the expense of replacing $1/\varepsilon$ by $1/\varepsilon^2$ in the right-hand side of the last inequality (3.7).

Theorem 3.1 (Korn's Unscaled Inequalities for Shells in Curvilinear Coordinates)**.**
(a) Given any domain $\omega\subset\mathbb{R}^2$ and any embedding $\boldsymbol{\theta}\in\mathcal{C}^3(\overline{\omega};\mathbb{R}^3)$, there exist two constants $\varepsilon_0=\varepsilon_0(\omega,\boldsymbol{\theta})>0$ and $C_0=C_0(\omega,\boldsymbol{\theta})$ such that, for all $0<\varepsilon\le\varepsilon_0$ and for all $(u_i)\in H^1(\Omega^\varepsilon;\mathbb{R}^3)$, $\Omega^\varepsilon:=\omega\times(-\varepsilon,\varepsilon)$,

$$\|(u_i)\|_{H^1(\Omega^\varepsilon)}\le C_0\left(\|(u_i)\|_{L^2(\Omega^\varepsilon)}+\frac{1}{\varepsilon}\left\|\left(e_{ij}(\boldsymbol{u})\right)\right\|_{L^2(\Omega^\varepsilon)}\right), \tag{3.4}$$

$$\inf_{(r_i)\in\mathrm{Rig}(\Omega^\varepsilon)}\|(u_i)-(r_i)\|_{H^1(\Omega^\varepsilon)}\le\frac{C_0}{\varepsilon}\left\|\left(e_{ij}(\boldsymbol{u})\right)\right\|_{L^2(\Omega^\varepsilon)}, \tag{3.5}$$

where

$$\begin{aligned}e_{ij}(\boldsymbol{u})&:=\frac{1}{2}(u_i|_j+u_j|_i)=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j^\varepsilon}+\frac{\partial u_j}{\partial x_i^\varepsilon}\right)-\Gamma_{ij}^k u_k,\\ \mathrm{Rig}(\Omega^\varepsilon)&:=\left\{(r_i)\in H^1(\Omega^\varepsilon;\mathbb{R}^3);\ e_{ij}(\boldsymbol{r})=0 \text{ a.e. in } \Omega^\varepsilon\right\}.\end{aligned} \tag{3.6}$$

(b) Given any domain $\omega\subset\mathbb{R}^2$, any embedding $\boldsymbol{\theta}\in\mathcal{C}^3(\overline{\omega};\mathbb{R}^3)$ and any non-empty relatively open subset γ_0 of the boundary of ω such that $\boldsymbol{\theta}(\gamma_0)$ is not contained in a straight line, there exist two constants $\varepsilon_0=\varepsilon_0(\omega,\boldsymbol{\theta})>0$ and $C_0=C_0(\omega,\gamma_0,\boldsymbol{\theta})$ such that, for all $0<\varepsilon\le\varepsilon_0$ and for all $(u_i)\in H^1(\Omega^\varepsilon;\mathbb{R}^3)$ that vanish on $\Gamma_0^\varepsilon:=\gamma_0\times(-\varepsilon,\varepsilon)$,

$$\|(u_i)\|_{H^1(\Omega^\varepsilon)}\le\frac{C_0}{\varepsilon}\left\|\left(e_{ij}(\boldsymbol{u})\right)\right\|_{L^2(\Omega^\varepsilon)}. \tag{3.7}$$

Remark 3.3. A classical infinitesimal rigid displacement lemma in curvilinear coordinates for open sets (see, e.g. Ciarlet [8]) shows that

$$\mathrm{Rig}(\Omega^\varepsilon) := \big\{ (r_i) \in H^1(\Omega^\varepsilon;\mathbb{R}^3); r_i \boldsymbol{g}^i = \boldsymbol{a} + \boldsymbol{B}\boldsymbol{\Theta}|_{\Omega^\varepsilon} \text{ for some } \boldsymbol{a} \in \mathbb{R}^3 \text{ and } \boldsymbol{B} \in \mathbb{A}^3 \big\},$$

so that $\mathrm{Rig}(\Omega^\varepsilon)$ is a finite-dimensional subspace of $H^1(\Omega^\varepsilon;\mathbb{R}^3)$.

Proof. The proof is divided for clarity into five parts, numbered (i) to (v).

(i) That $\boldsymbol{\theta}$ is an embedding implies that there exists a constant $\varepsilon_0 = \varepsilon_0(\omega,\boldsymbol{\theta}) > 0$ such that, for every $0 < \varepsilon \le \varepsilon_0$, the set $\hat{\Omega}^\varepsilon := \boldsymbol{\Theta}(\Omega^\varepsilon)$ is a domain in $\mathbb{R}^3$. Then Lemma 3.1 shows that there exit constants $C_1(\varepsilon),\ldots,C_4(\varepsilon)$ such that, for all $\hat{\boldsymbol{u}} \in H^1(\hat{\Omega}^\varepsilon;\mathbb{R}^3)$,

$$\|\hat{\boldsymbol{u}}\|_{H^1(\hat{\Omega}^\varepsilon)} \le C_1(\varepsilon)\|\hat{\boldsymbol{\nabla}}_{\mathbf{s}}\hat{\boldsymbol{u}}\|_{L^2(\hat{\Omega}^\varepsilon)} + C_2(\varepsilon)\|\hat{\boldsymbol{u}}\|_{L^2(\hat{\Omega}^\varepsilon)}, \tag{3.8}$$

$$\inf_{\hat{\boldsymbol{r}} \in \mathrm{Rig}(\hat{\Omega}^\varepsilon)} \|\hat{\boldsymbol{u}} - \hat{\boldsymbol{r}}\|_{H^1(\hat{\Omega}^\varepsilon)} \le C_3(\varepsilon)\|\hat{\boldsymbol{\nabla}}_{\mathbf{s}}\hat{\boldsymbol{u}}\|_{L^2(\hat{\Omega}^\varepsilon)}, \tag{3.9}$$

and that, for all $\hat{\boldsymbol{u}} = (\hat{u}_i) \in H^1_{\hat{\Gamma}^\varepsilon_0}(\hat{\Omega}^\varepsilon;\mathbb{R}^3)$,

$$\|\hat{\boldsymbol{u}}\|_{H^1(\hat{\Omega}^\varepsilon)} \le C_4(\varepsilon)\|\hat{\boldsymbol{\nabla}}_{\mathbf{s}}\hat{\boldsymbol{u}}\|_{L^2(\hat{\Omega}^\varepsilon)}. \tag{3.10}$$

Besides, the above inequalities hold with the following explicit constants in their right-hand sides (cf. Mardare and Nguyen [25]):

$$\begin{aligned}
&C_1(\varepsilon) = 1 + (AB + 4\sqrt{3}K_\varepsilon)\sqrt{\frac{|\boldsymbol{\Theta}(\Omega^\varepsilon)|}{|\boldsymbol{\Theta}(B_{R,\varepsilon})|}},\\
&C_2(\varepsilon) = 1 + A^{1/2}B\left(C^{1/2} + 8D^{1/2}E^{1/2}\frac{1}{R}\right)\sqrt{\frac{|\boldsymbol{\Theta}(\Omega^\varepsilon)|}{|\boldsymbol{\Theta}(B_{R,\varepsilon})|}},\\
&C_3(\varepsilon) = (1 + W_\varepsilon)(1 + 2\sqrt{3}K_\varepsilon),\\
&C_4(\varepsilon) = (1 + P_\varepsilon)(1 + 2\sqrt{3}K_\varepsilon)\left(1 + T_\varepsilon(1 + W_\varepsilon)\sqrt{\frac{3|\boldsymbol{\Theta}(\Omega^\varepsilon)|}{p_1^\varepsilon + p_2^\varepsilon}}\right),
\end{aligned} \tag{3.11}$$

where the constants $K_\varepsilon := K(\hat{\Omega}^\varepsilon), W_\varepsilon := W(\hat{\Omega}^\varepsilon), P_\varepsilon := P(\hat{\Omega}^\varepsilon, \hat{\Gamma}^\varepsilon_0)$ and $T_\varepsilon := T(\hat{\Omega}^\varepsilon, \hat{\Gamma}^\varepsilon_0)$ are those appearing in Lemmas 2.1-2.4, the constants p_1^ε and p_2^ε denote the two smallest eigenvalues of the symmetric matrix

$$\boldsymbol{M}^\varepsilon := \int_{\hat{\Gamma}^\varepsilon_0} (\hat{x} - \hat{x}^\varepsilon_0)(\hat{x} - \hat{x}^\varepsilon_0)^T \,\mathrm{d}\hat{\Gamma}^\varepsilon, \quad \hat{x}^\varepsilon_0 := \frac{1}{|\hat{\Gamma}^\varepsilon_0|}\int_{\hat{\Gamma}^\varepsilon_0} \hat{x}\,\mathrm{d}\hat{\Gamma}^\varepsilon,$$

$R=R(\omega)>0$ is the radius of any fixed open ball ω_R contained in ω, $B_{R,\varepsilon}:=\omega_R\times(-\varepsilon,\varepsilon)$, and A,B,C,D and E are any constants such that, at every point of $\overline{\Omega^{\varepsilon_0}}$,

$$\begin{aligned}
&|\boldsymbol{\nabla}\boldsymbol{\Theta}|^2\leq A, && |(\boldsymbol{\nabla}\boldsymbol{\Theta})^{-1}|^2\leq B,\\
&\sum_{i,j}\left|(\boldsymbol{\nabla}\boldsymbol{\Theta})^{-1}\frac{\partial^2\boldsymbol{\Theta}}{\partial x_i^\varepsilon\partial x_j^\varepsilon}\right|^2\leq C, && D^{-1}\leq|\det\boldsymbol{\nabla}\boldsymbol{\Theta}|\leq E.
\end{aligned}$$

It is clear from the above definitions that the constants A,B,C,D,E and R are independent of ε and that the quotient $|\boldsymbol{\Theta}(\Omega^\varepsilon)|/|\boldsymbol{\Theta}(B_{R,\varepsilon})|$ is also independent of ε.

Since $\hat{\Omega}^\varepsilon=\boldsymbol{\Theta}(\omega\times(-\varepsilon,\varepsilon))$ is a (curved) cylinder with thickness 2ε, there exists a constant $C_0=C_0(\omega,\boldsymbol{\theta})$ such that, for all $0<\varepsilon\leq\varepsilon_0$, the constants appearing in Lemmas 2.1-2.4 satisfy (see, e.g. [2,16,24])

$$K_\varepsilon\leq\frac{C_0}{\varepsilon},\quad W_\varepsilon\leq C_0,\quad P_\varepsilon\leq C_0,\quad T(\varepsilon)\leq C_0.$$

Therefore, there exists a constant $C_0=C_0(\omega,\boldsymbol{\theta})$ such that, for all $0<\varepsilon\leq\varepsilon_0$,

$$C_1(\varepsilon)\leq\frac{C_0}{\varepsilon},\quad C_2(\varepsilon)\leq C_0,\quad C_3(\varepsilon)\leq\frac{C_0}{\varepsilon}. \tag{3.12}$$

The remaining constant $C_4(\varepsilon)$ is estimated below.

(ii) Since p_1^ε and p_2^ε are eigenvalues of the matrix $\boldsymbol{M}^\varepsilon$, which is symmetric, there exist two unit orthogonal vectors $\boldsymbol{v}_1\in\mathbb{R}^3$ and $\boldsymbol{v}_2\in\mathbb{R}^3$ such that

$$\boldsymbol{M}^\varepsilon\boldsymbol{v}_\alpha=p_\alpha^\varepsilon\boldsymbol{v}_\alpha.$$

Let

$$\hat{x}_0:=\frac{1}{|\boldsymbol{\theta}(\gamma_0)|}\int_{\boldsymbol{\theta}(\gamma_0)}\hat{x}\,\mathrm{d}\hat{\gamma}\in\mathbb{R}^3,$$

where $\mathrm{d}\hat{\gamma}$ is the unit length along the curve $\hat{\gamma}:=\boldsymbol{\theta}(\gamma),\gamma:=\partial\omega$, and $|\boldsymbol{\theta}(\gamma_0)|:=\int_{\boldsymbol{\theta}(\gamma_0)}\mathrm{d}\hat{\gamma}$. Note that $\hat{x}_0$ is independent of ε, by contrast to $\hat{x}_0^\varepsilon$ defined in part (i) above.

Since the set $\boldsymbol{\theta}(\gamma_0)$ is not contained in a straight line, there exists a unit vector $\boldsymbol{v}:=\lambda_1\boldsymbol{v}_1+\lambda_2\boldsymbol{v}_2\in\mathbb{R}^3,\lambda_1,\lambda_2\in\mathbb{R}$, and a point $\hat{x}_1\in\boldsymbol{\theta}(\gamma_0)$ such that $(\hat{x}_1-\hat{x}_0)\cdot\boldsymbol{v}\neq0$. Consequently, since the function $\boldsymbol{\theta}$ is continuous and $\delta:=|(\hat{x}_1-\hat{x}_0)\cdot\boldsymbol{v}|>0$, there exists a non-empty relatively open subset $\gamma_{0,1}$ of γ_0 such that

$$|(\hat{x}-\hat{x}_0)\cdot \boldsymbol{v}| \geq \frac{\delta}{2}, \quad \forall \hat{x} \in \boldsymbol{\theta}(\gamma_{0,1}).$$

Let $\Gamma^\varepsilon_{0,1} := \gamma_{0,1}\times(-\varepsilon,\varepsilon)$ and $\hat{\Gamma}^\varepsilon_{0,1} := \boldsymbol{\Theta}(\Gamma^\varepsilon_{0,1})$. Then

$$\begin{aligned}
\boldsymbol{v}^T\boldsymbol{M}^\varepsilon \boldsymbol{v} &= \int_{\hat{\Gamma}^\varepsilon_0} \boldsymbol{v}^T\left(\hat{x}-\hat{x}^\varepsilon_0\right)\left(\hat{x}-\hat{x}^\varepsilon_0\right)^T \boldsymbol{v}\, \mathrm{d}\hat{\Gamma}^\varepsilon = \int_{\hat{\Gamma}^\varepsilon_0} \left|\left(\hat{x}-\hat{x}^\varepsilon_0\right)\cdot \boldsymbol{v}\right|^2 \mathrm{d}\hat{\Gamma}^\varepsilon \\
&\geq \frac{1}{2}\int_{\hat{\Gamma}^\varepsilon_0} \left|\left(\hat{x}-\hat{x}_0\right)\cdot \boldsymbol{v}\right|^2 \mathrm{d}\hat{\Gamma}^\varepsilon - \int_{\hat{\Gamma}^\varepsilon_0} \left|\left(\hat{x}^\varepsilon_0-\hat{x}_0\right)\cdot \boldsymbol{v}\right|^2 \mathrm{d}\hat{\Gamma}^\varepsilon \\
&\geq \frac{\delta^2}{8}\left|\hat{\Gamma}^\varepsilon_{0,1}\right| - \left|\hat{\Gamma}^\varepsilon_0\right|\left|\hat{x}^\varepsilon_0-\hat{x}_0\right|^2,
\end{aligned}$$

so that, using that

$$\begin{aligned}
&\left|\hat{\Gamma}^\varepsilon_{0,1}\right| = |\boldsymbol{\theta}(\gamma_{0,1})|\left(2\varepsilon+\varepsilon o(1)\right) \quad \text{when} \quad \varepsilon \to 0, \\
&\left|\hat{\Gamma}^\varepsilon_0\right| = |\boldsymbol{\theta}(\gamma_0)|\left(2\varepsilon+\varepsilon o(1)\right) \quad \text{when} \quad \varepsilon \to 0, \\
&\left|\hat{x}^\varepsilon_0-\hat{x}_0\right| = \left|\frac{1}{|\hat{\Gamma}^\varepsilon_0|}\int_{\hat{\Gamma}^\varepsilon_0} \hat{x}\, \mathrm{d}\hat{\Gamma}^\varepsilon - \frac{1}{|\boldsymbol{\theta}(\gamma_0)|}\int_{\boldsymbol{\theta}(\gamma_0)} \hat{x}\, \mathrm{d}\hat{\gamma}\right| = o(1) \to 0 \quad \text{when} \quad \varepsilon \to 0,
\end{aligned}$$

there exists an $\varepsilon_0 = \varepsilon_0(\omega,\boldsymbol{\theta}) > 0$ such that, for all $0<\varepsilon\leq\varepsilon_0$,

$$\boldsymbol{v}^T\boldsymbol{M}^\varepsilon\boldsymbol{v} \geq \frac{\delta^2|\boldsymbol{\theta}(\gamma_{0,1})|}{8}\varepsilon,$$

on the one hand.

Since $\boldsymbol{v} = \lambda_1\boldsymbol{v}_1 + \lambda_2\boldsymbol{v}_2$ and $(\lambda_1)^2+(\lambda_2)^2 = 1$, we have, for all $0<\varepsilon\leq\varepsilon_0$,

$$\begin{aligned}
\boldsymbol{v}^T\boldsymbol{M}^\varepsilon\boldsymbol{v} &= (\lambda_1\boldsymbol{v}_1+\lambda_2\boldsymbol{v}_2)^T\left(\lambda_1 p^\varepsilon_1\boldsymbol{v}_1 + \lambda_2 p^\varepsilon_2\boldsymbol{v}_2\right) \\
&= p^\varepsilon_1\lambda_1^2 + p^\varepsilon_2\lambda_2^2 \leq p^\varepsilon_2\left(\lambda_1^2+\lambda_2^2\right) = p^\varepsilon_2,
\end{aligned}$$

on the other hand. Therefore, for all $0<\varepsilon\leq\varepsilon_0$,

$$p^\varepsilon_2 \geq \boldsymbol{v}^T\boldsymbol{M}^\varepsilon\boldsymbol{v} \geq \frac{\delta^2|\boldsymbol{\theta}(\gamma_{0,1})|}{2}\varepsilon.$$

Note that the above estimate implies that, for all $0<\varepsilon\leq\varepsilon_0$,

$$\frac{|\hat{\Omega}^\varepsilon|}{p^\varepsilon_1+p^\varepsilon_2} \leq \frac{\left|\int_{\Omega^\varepsilon}\sqrt{g(x)}\,\mathrm{d}x\right|}{p^\varepsilon_2} \leq \frac{4E|\omega|}{\delta^2|\boldsymbol{\theta}(\gamma_{0,1})|}.$$

Consequently, using in addition the estimates for the constants $K_\varepsilon, W_\varepsilon, P_\varepsilon$ and T_ε obtained in part (i), we deduce that there exist two constants $\varepsilon_0 = \varepsilon_0(\omega,\boldsymbol{\theta}) > 0$ and $C_0 = C_0(\omega,\gamma_0,\boldsymbol{\theta})$ such that, for all $0 < \varepsilon \leq \varepsilon_0$,

$$C_4(\varepsilon) \leq \frac{C_0}{\varepsilon}. \tag{3.13}$$

As a consequence of the estimates (3.12) and (3.13) of the constants appearing in inequalities (3.8)-(3.10), we just proved that, for all $\hat{\boldsymbol{u}} \in H^1(\hat{\Omega}^\varepsilon;\mathbb{R}^3)$,

$$\|\hat{\boldsymbol{u}}\|_{H^1(\hat{\Omega}^\varepsilon)} \leq C_0 \left(\|\hat{\boldsymbol{u}}\|_{L^2(\hat{\Omega}^\varepsilon)} + \frac{1}{\varepsilon} \|\hat{\boldsymbol{\nabla}}_s \hat{\boldsymbol{u}}\|_{L^2(\Omega^\varepsilon)} \right), \tag{3.14}$$

$$\inf_{\hat{\boldsymbol{r}} \in \mathrm{Rig}(\hat{\Omega}^\varepsilon)} \|\hat{\boldsymbol{u}} - \hat{\boldsymbol{r}}\|_{H^1(\hat{\Omega}^\varepsilon)} \leq \frac{C_0}{\varepsilon} \|\hat{\boldsymbol{\nabla}}_s \hat{\boldsymbol{u}}\|_{L^2(\hat{\Omega}^\varepsilon)}, \tag{3.15}$$

and that, for all $\hat{\boldsymbol{u}} \in H^1(\hat{\Omega}^\varepsilon;\mathbb{R}^3)$ that vanish on $\hat{\Gamma}_0^\varepsilon$,

$$\|\hat{\boldsymbol{u}}\|_{H^1(\hat{\Omega}^\varepsilon)} \leq \frac{C_0}{\varepsilon} \|\hat{\boldsymbol{\nabla}}_s \hat{\boldsymbol{u}}\|_{L^2(\hat{\Omega}^\varepsilon)}. \tag{3.16}$$

It remains to estimate the left-hand sides and the right-hand sides of these inequalities, respectively from below and from above, in terms of the covariant components (u_i) and $(e_{ij}(\boldsymbol{u}))$ of $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{\nabla}}_{\mathbf{s}} \hat{\boldsymbol{u}}$.

(iii) Let $g_{ij} := \boldsymbol{g}_i \cdot \boldsymbol{g}_j \in \mathcal{C}^1(\overline{\Omega^{\varepsilon_0}})$ and $g^{ij} := \boldsymbol{g}^i \cdot \boldsymbol{g}^j \in\in \mathcal{C}^1(\overline{\Omega^{\varepsilon_0}})$ respectively denote the covariant and contravariant components of the metric tensor field associated with the immersion $\boldsymbol{\Theta} : \overline{\Omega^{\varepsilon_0}} \to \mathbb{R}^3$, and let $\boldsymbol{C} := (g_{ij}) \in \mathcal{C}^1(\overline{\Omega^{\varepsilon_0}},\mathbb{S}^3)$. Then $\boldsymbol{C}^{-1} = (g^{ij})$ and $g := \det(\boldsymbol{C}) = |\det \boldsymbol{\nabla\Theta}|^2$.

Given any $\hat{\boldsymbol{u}} \in H^1(\hat{\Omega}^\varepsilon;\mathbb{R}^3)$, let $\boldsymbol{u} := \hat{\boldsymbol{u}} \circ \boldsymbol{\Theta} \in H^1(\Omega^\varepsilon;\mathbb{R}^3)$ and let $u_i := \boldsymbol{u} \cdot \boldsymbol{g}_i \in H^1(\Omega^\varepsilon)$. Then $\boldsymbol{u} = u_i \boldsymbol{g}^i$ in Ω^ε, so that

$$\begin{aligned}
\|\hat{\boldsymbol{u}}\|^2_{L^2(\hat{\Omega}^\varepsilon)} &= \int_{\hat{\Omega}^\varepsilon} |\hat{\boldsymbol{u}}|^2 \,\mathrm{d}\hat{x} = \int_{\Omega^\varepsilon} (u_i \boldsymbol{g}^i) \cdot (u_j \boldsymbol{g}^j) \sqrt{g} \,\mathrm{d}x \\
&= \int_{\Omega^\varepsilon} g^{ij} u_i u_j \sqrt{g} \,\mathrm{d}x \\
&= \int_{\Omega^\varepsilon} (\boldsymbol{C}^{-1} \boldsymbol{u}) \cdot \boldsymbol{u} \sqrt{g} \,\mathrm{d}x.
\end{aligned}$$

Consequently,

$$\frac{1}{\lambda_3 D} \|(u_i)\|^2_{L^2(\Omega^\varepsilon)} \leq \|\hat{\boldsymbol{u}}\|^2_{L^2(\hat{\Omega}^\varepsilon)} \leq \frac{E}{\lambda_1} \|(u_i)\|^2_{L^2(\Omega^\varepsilon)}, \tag{3.17}$$

where

$$\lambda_1 := \inf_{x\in\Omega^\varepsilon} \lambda_1\big(\boldsymbol{C}(x)\big), \quad \lambda_3 := \inf_{x\in\Omega^\varepsilon} \lambda_3\big(\boldsymbol{C}(x)\big),$$

$\lambda_1(\boldsymbol{C}(x))$ and $\lambda_3(\boldsymbol{C}(x))$ denoting respectively the smallest and the largest eigenvalues of the matrix $\boldsymbol{C}(x)$ (remember that $0<D<E$ are constants such that $D^{-1}\le |\det(\boldsymbol{\nabla\Theta}(x))| \le E$ for all $x\in\overline{\Omega^{\varepsilon_0}}$).

Let $u_i|_j := \partial u_i/\partial x_j^\varepsilon - \Gamma_{ij}^k u_k \in L^2(\Omega^\varepsilon)$ denote the covariant derivatives of the functions u_i. Then $(\hat{\boldsymbol{\nabla}}_{\mathbf{s}}\hat{\boldsymbol{u}})\circ\boldsymbol{\Theta} = u_i|_j \boldsymbol{g}^i\otimes\boldsymbol{g}^j$ in Ω^ε, so that

$$\begin{aligned}\|\hat{\boldsymbol{\nabla}}\hat{\boldsymbol{u}}\|^2_{L^2(\hat{\Omega}^\varepsilon)} &= \int_{\hat{\Omega}^\varepsilon} |\hat{\boldsymbol{\nabla}}_{\mathbf{s}}\hat{\boldsymbol{u}}|^2 \,\mathrm{d}\hat{x} = \int_{\Omega^\varepsilon} \big(u_i|_k \boldsymbol{g}^i\otimes\boldsymbol{g}^k\big)\cdot\big(u_j|_l \boldsymbol{g}^j\otimes\boldsymbol{g}^l\big)\sqrt{g}\,\mathrm{d}x \\ &= \int_{\Omega^\varepsilon} g^{ij}g^{kl}u_i|_k u_j|_l \sqrt{g}\,\mathrm{d}x.\end{aligned}$$

Thus,

$$\|\hat{\boldsymbol{\nabla}}\hat{\boldsymbol{u}}\|^2_{L^2(\hat{\Omega}^\varepsilon)} = \int_{\Omega^\varepsilon} (\boldsymbol{C}^{-1}\cdot\boldsymbol{S})\sqrt{g}\,\mathrm{d}x,$$

where $\boldsymbol{C}^{-1} = (g^{ij}) \in \mathcal{C}^1(\overline{\Omega^{\varepsilon_0}}, \mathbb{S}^3)$ and $\boldsymbol{S}\in L^2(\Omega^\varepsilon;\mathbb{S}^3)$ is the matrix field with components $[\boldsymbol{S}]_{ij} := u_i|_k g^{kl} u_j|_l$ at its i-th row and j-th column. Note that $\boldsymbol{S} = \boldsymbol{U}\boldsymbol{C}^{-1}\boldsymbol{U}^T$, where $\boldsymbol{U}\in L^2(\Omega^\varepsilon;\mathbb{R}^{3\times 3})$ denotes the matrix field with components $[\boldsymbol{U}]_{ij} = u_i|_j$ at its i-th row and j-th column.

The matrix field $\boldsymbol{C}^{-1}$ is symmetric and positive definite, so its eigenvalues are positive. The matrix fields $\boldsymbol{S}$ and $\boldsymbol{U}^T\boldsymbol{U}$ are both symmetric and semi-positive definite, so their eigenvalues are non-negative. Then Lemma 2.5 implies that, at almost all points of Ω^ε,

$$\begin{aligned}\boldsymbol{C}^{-1}\cdot\boldsymbol{S} &\ge \frac{1}{\lambda_3(\boldsymbol{C})}\mathrm{Tr}(\boldsymbol{S}) = \frac{1}{\lambda_3(\boldsymbol{C})}\mathrm{Tr}(\boldsymbol{U}\boldsymbol{C}^{-1}\boldsymbol{U}^T) \\ &= \frac{1}{\lambda_3(\boldsymbol{C})}\mathrm{Tr}(\boldsymbol{C}^{-1}\boldsymbol{U}^T\boldsymbol{U}) = \frac{1}{\lambda_3(\boldsymbol{C})}\boldsymbol{C}^{-1}\cdot(\boldsymbol{U}^T\boldsymbol{U}) \\ &\ge \frac{1}{\big(\lambda_3(\boldsymbol{C})\big)^2}\mathrm{Tr}(\boldsymbol{U}^T\boldsymbol{U}) = \frac{1}{\big(\lambda_3(\boldsymbol{C})\big)^2}|\boldsymbol{U}|^2 \\ &\ge \frac{1}{(\lambda_3)^2}|\boldsymbol{U}|^2 = \frac{1}{(\lambda_3)^2}\sum_{i,j}(u_i|_j)^2,\end{aligned}$$

which combined with the previous in equality yields

$$\|\hat{\boldsymbol{\nabla}}\hat{\boldsymbol{u}}\|^2_{L^2(\Omega^\varepsilon)} \ge \frac{1}{(\lambda_3)^2}\int_{\Omega^\varepsilon}\sum_{i,j}(u_i|_j)^2\sqrt{g}\,\mathrm{d}x$$

$$
\begin{aligned}
&\geq \frac{D^{-1}}{(\lambda_3)^2}\sum_{i,j}\int_{\Omega^\varepsilon}(u_i|_j)^2\,\mathrm{d}x\\
&\geq \frac{D^{-1}}{(\lambda_3)^2}\sum_{i,j}\int_{\Omega^\varepsilon}\left\{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j^\varepsilon}\right)^2-\left(\Gamma_{ij}^k u_k\right)^2\right\}\mathrm{d}x.
\end{aligned}
$$

Since $\Gamma_{ij}^k = \boldsymbol{g}^k\cdot\partial^2\boldsymbol{\Theta}/(\partial x_i^\varepsilon \partial x_j^\varepsilon)$ in $\overline{\Omega^{\varepsilon_0}}$, we have

$$
\sum_{i,j}\sum_k\left(\Gamma_{ij}^k\right)^2=\sum_{i,j}\left|(\boldsymbol{\nabla}\boldsymbol{\Theta})^{-1}\frac{\partial^2\boldsymbol{\Theta}}{\partial x_i^\varepsilon\partial x_j^\varepsilon}\right|^2\leq C
$$

in $\overline{\Omega^{\varepsilon_0}}$ (thanks to the definition of the constant C in part (i) of the proof). Then the above inequality implies that

$$
\|\hat{\boldsymbol{\nabla}}\hat{\boldsymbol{u}}\|^2_{L^2(\Omega^\varepsilon)}\geq\frac{D^{-1}}{2(\lambda_3)^2}\left\|\left(\frac{\partial u_i}{\partial x_j^\varepsilon}\right)\right\|^2_{L^2(\Omega^\varepsilon)}-\frac{CD^{-1}}{(\lambda_3)^2}\|(u_k)\|^2_{L^2(\Omega^\varepsilon)},
$$

then that, in view of inequalities (3.17),

$$
\begin{aligned}
\|(u_i)\|^2_{H^1(\Omega^\varepsilon)}&=\|(u_i)\|^2_{L^2(\Omega^\varepsilon)}+\left\|\left(\frac{\partial u_i}{\partial x_j^\varepsilon}\right)\right\|^2_{L^2(\Omega^\varepsilon)}\\
&\leq\|(u_i)\|^2_{L^2(\Omega^\varepsilon)}+2D(\lambda_3)^2\|\hat{\boldsymbol{\nabla}}\hat{\boldsymbol{u}}\|^2_{L^2(\hat{\Omega}^\varepsilon)}+2C\|(u_k)\|^2_{L^2(\Omega^\varepsilon)}\\
&\leq(1+2C)\lambda_3 D\|\hat{\boldsymbol{u}}\|^2_{L^2(\hat{\Omega}^\varepsilon)}+2D(\lambda_3)^2\|\hat{\boldsymbol{\nabla}}\hat{\boldsymbol{u}}\|^2_{L^2(\hat{\Omega}^\varepsilon)}\\
&\leq F\|\hat{\boldsymbol{u}}\|^2_{H^1(\hat{\Omega}^\varepsilon)},
\end{aligned}
\tag{3.18}
$$

where

$$
F:=\lambda_3 D\max\left\{1+2C,\,2\lambda_3\right\}.
$$

(iv) Given any $\hat{\boldsymbol{u}}\in H^1(\hat{\Omega}^\varepsilon;\mathbb{R}^3)$, let

$$
\begin{aligned}
&\boldsymbol{u}:=\hat{\boldsymbol{u}}\circ\boldsymbol{\Theta}\in H^1(\Omega^\varepsilon;\mathbb{R}^3), && u_i:=\boldsymbol{u}\cdot\boldsymbol{g}_i\in H^1(\Omega^\varepsilon),\\
&u_i|_j:=\frac{\partial u_i}{\partial x_j^\varepsilon}-\Gamma_{ij}^k u_k\in L^2(\Omega^\varepsilon), && e_{ij}(\boldsymbol{u}):=\frac{1}{2}(u_i|_j+u_j|_i).
\end{aligned}
$$

Then $(\hat{\boldsymbol{\nabla}}_{\mathbf{s}}\hat{\boldsymbol{u}})\circ\boldsymbol{\Theta}=e_{ij}(\boldsymbol{u})\boldsymbol{g}^i\otimes\boldsymbol{g}^j$ in Ω^ε, so that, by a series of computations similar to those in part (iii) of the proof, we obtain that

$$
\|\hat{\boldsymbol{\nabla}}_{\mathbf{s}}\hat{\boldsymbol{u}}\|^2_{L^2(\hat{\Omega}^\varepsilon)}=\int_{\Omega^\varepsilon}g^{ij}g^{kl}e_{ik}(\boldsymbol{u})e_{jl}(\boldsymbol{u})\sqrt{g}\,\mathrm{d}x
$$

$$
\begin{aligned}
&\leq \frac{1}{\lambda_1}\int_{\Omega^\varepsilon} g^{kl} e_{ik}(\boldsymbol{u}) e_{il}(\boldsymbol{u})\sqrt{g}\,\mathrm{d}x \\
&\leq \frac{1}{(\lambda_1)^2}\int_{\Omega^\varepsilon} \sum_{i,k}|e_{ik}(\boldsymbol{u})|^2\sqrt{g}\,\mathrm{d}x \\
&\leq \frac{E}{(\lambda_1)^2}\big\|\big(e_{ij}(\boldsymbol{u})\big)\big\|^2_{L^2(\Omega^\varepsilon)},
\end{aligned}
\tag{3.19}
$$

where E is the constant defined in part (i) of the proof.

(v) First, using estimates (3.17)-(3.19) in inequality (3.14), we deduce that, for all $(u_i)\in H^1(\Omega^\varepsilon;\mathbb{R}^3)$,

$$
\begin{aligned}
\|(u_i)\|_{H^1(\Omega^\varepsilon)} &\leq F^{1/2}\|\hat{\boldsymbol{u}}\|_{H^1(\hat{\Omega}^\varepsilon)} \\
&\leq F^{1/2}C_0\Big(\|\hat{\boldsymbol{u}}\|_{L^2(\hat{\Omega}^\varepsilon)}+\varepsilon^{-1}\|\hat{\boldsymbol{\nabla}}_{\mathrm{s}}\hat{\boldsymbol{u}}\|_{L^2(\hat{\Omega}^\varepsilon)}\Big) \\
&\leq \frac{F^{1/2}C_0E^{1/2}}{\lambda_1}\Big((\lambda_1)^{1/2}\|(u_i)\|_{L^2(\Omega^\varepsilon)}+\varepsilon^{-1}\big\|\big(e_{ij}(\boldsymbol{u})\big)\big\|_{L^2(\Omega^\varepsilon)}\Big),
\end{aligned}
$$

which is precisely inequality (3.4) of the theorem.

Second, using estimate (3.19) in inequality (3.15), we deduce that, for all $(u_i)\in H^1(\Omega^\varepsilon;\mathbb{R}^3)$,

$$
\inf_{\boldsymbol{r}\in \mathrm{Rig}(\hat{\Omega}^\varepsilon)}\|\hat{\boldsymbol{u}}-\hat{\boldsymbol{r}}\|_{H^1(\hat{\Omega}^\varepsilon)} \leq \frac{C_0}{\varepsilon}\|\hat{\boldsymbol{\nabla}}_{\mathrm{s}}\hat{\boldsymbol{u}}\|_{L^2(\Omega^\varepsilon)} \leq \frac{C_0E^{1/2}}{\lambda_1}\varepsilon^{-1}\big\|\big(e_{ij}(\boldsymbol{u})\big)\big\|_{L^2(\Omega^\varepsilon)},
$$

on the one hand.

The space $\mathrm{Rig}(\hat{\Omega}^\varepsilon)$ being finite-dimensional, there exists $\hat{\boldsymbol{q}}=\hat{\boldsymbol{q}}(\hat{\boldsymbol{u}})\in \mathrm{Rig}(\hat{\Omega}^\varepsilon)$ such that

$$
\inf_{\boldsymbol{r}\in \mathrm{Rig}(\hat{\Omega}^\varepsilon)}\|\hat{\boldsymbol{u}}-\hat{\boldsymbol{r}}\|_{H^1(\hat{\Omega}^\varepsilon)} = \|\hat{\boldsymbol{u}}-\hat{\boldsymbol{q}}\|_{H^1(\hat{\Omega}^\varepsilon)}.
$$

Then estimate (3.18) implies that

$$
\inf_{\boldsymbol{r}\in \mathrm{Rig}(\hat{\Omega}^\varepsilon)}\|\hat{\boldsymbol{u}}-\hat{\boldsymbol{r}}\|_{H^1(\hat{\Omega}^\varepsilon)} = \|\hat{\boldsymbol{u}}-\hat{\boldsymbol{q}}\|_{H^1(\hat{\Omega}^\varepsilon)} \geq F^{-1/2}\|(u_i-q_i)\|_{H^1(\Omega^\varepsilon)},
$$

where $u_i:=\boldsymbol{u}\cdot\boldsymbol{g}_i\in H^1(\Omega^\varepsilon)$, $\boldsymbol{u}:=\hat{\boldsymbol{u}}\circ\boldsymbol{\Theta}$, and $q_i:=\boldsymbol{q}\cdot\boldsymbol{g}_i\in H^1(\Omega^\varepsilon)$, $\boldsymbol{q}:=\hat{\boldsymbol{q}}\circ\boldsymbol{\Theta}$. Noting that $\hat{\boldsymbol{q}}\in \mathrm{Rig}(\hat{\Omega}^\varepsilon)$ implies that $\hat{\boldsymbol{\nabla}}_{\mathrm{s}}\hat{\boldsymbol{q}}=\boldsymbol{0}$ in $\hat{\Omega}^\varepsilon$, and that $(\hat{\boldsymbol{\nabla}}_{\mathrm{s}}\hat{\boldsymbol{q}})\circ\boldsymbol{\Theta}=e_{ij}(\boldsymbol{q})\boldsymbol{g}^i\otimes\boldsymbol{g}^j$ in Ω^ε, we deduce that

$$
e_{ij}(\boldsymbol{q})=0 \quad \text{in } \Omega^\varepsilon,
$$

which next implies that $(q_i)\in \mathrm{Rig}(\Omega^\varepsilon)$. Hence,

$$\begin{aligned}\inf_{\boldsymbol{r}\in \mathrm{Rig}(\hat{\Omega}^\varepsilon)}\|\hat{\boldsymbol{u}}-\hat{\boldsymbol{r}}\|_{H^1(\hat{\Omega}^\varepsilon)} &\geq F^{-1/2}\|(u_i-q_i)\|_{H^1(\Omega^\varepsilon)}\\ &\geq F^{-1/2}\inf_{(r_i)\in \mathrm{Rig}(\Omega^\varepsilon)}\|(u_i-r_i)\|_{H^1(\Omega^\varepsilon)},\end{aligned}$$

on the other hand. Therefore, for all $(u_i)\in H^1(\Omega^\varepsilon;\mathbb{R}^3)$,

$$F^{-1/2}\inf_{(r_i)\in \mathrm{Rig}(\Omega^\varepsilon)}\|(u_i)-(r_i)\|_{H^1(\Omega^\varepsilon)}\leq \frac{C_0E^{1/2}}{\lambda_1}\varepsilon^{-1}\left\|\left(e_{ij}(\boldsymbol{u})\right)\right\|_{L^2(\Omega^\varepsilon)},$$

which is precisely inequality (3.5) of the theorem.

Third, using estimates (3.18) and (3.19) in inequality (3.16), we deduce that, for all $(u_i)\in H^1(\Omega^\varepsilon;\mathbb{R}^3)$ that vanish on Γ_0^ε,

$$\begin{aligned}\|(u_i)\|_{H^1(\Omega^\varepsilon)}\leq F^{1/2}\|\hat{\boldsymbol{u}}\|_{H^1(\hat{\Omega}^\varepsilon)}&\leq F^{1/2}\frac{C_0}{\varepsilon}\|\hat{\boldsymbol{\nabla}}_s\hat{\boldsymbol{u}}\|_{L^2(\Omega^\varepsilon)}\\ &\leq \frac{F^{1/2}C_0E^{1/2}}{\lambda_1}\varepsilon^{-1}\left\|\left(e_{ij}(\boldsymbol{u})\right)\right\|_{L^2(\Omega^\varepsilon)},\end{aligned}$$

which is precisely inequality (3.7) of the theorem. □

4 Korn's scaled inequalities in curvilinear coordinates for shells

We proved in the previous section that the constants appearing in Korn's inequalities in curvilinear coordinates on a shell-like domain with thickness $2\varepsilon>0$ are of order $1/\varepsilon$. But the norms appearing in these inequalities are defined by means of integrals defined over the domain $\Omega^\varepsilon:=\omega\times(-\varepsilon,\varepsilon)$, so they themselves depend on ε. This is why it is necessary, in view of their applications in shell theory, to transform these inequalities into inequalities defined over a fixed domain (that is, a domain independent of ε).

To this end, assume without losing in generality that the constant ε_0 appearing in Theorem 4.1 is equal to one, let

$$\Omega^1:=\omega\times(-1,1),$$

and, for each $\varepsilon>0$, let $\boldsymbol{\pi}_\varepsilon:\Omega^1\to\Omega^\varepsilon$ be the function defined by

$$\boldsymbol{\pi}_\varepsilon(y,x_3)=(y,\varepsilon x_3),\quad \forall(y,x_3)\in\Omega^1.$$

Generic points in Ω^ε and Ω^1 are respectively denoted $x^\varepsilon=(x_i^\varepsilon)\in\Omega^\varepsilon$ and $x=(x_i)\in\Omega^1$, so that $x^\varepsilon=\boldsymbol{\pi}^\varepsilon(x)$ if and only if there exists $y=(y_\alpha)\in\omega$ such that

$$x_\alpha^\varepsilon=x_\alpha=y_\alpha,\quad x_3^\varepsilon=\varepsilon x_3.$$

Note that Ω^1 is a domain independent of ε and that $\boldsymbol{\pi}_\varepsilon$ allows to associate with each function or field defined on Ω^ε a function or field defined on Ω^1 by function composition. Thus, given any $\hat{\boldsymbol{u}}\in H^1(\hat{\Omega}^\varepsilon;\mathbb{R}^3)$, we associate with the vector field

$$\boldsymbol{u}:=\hat{\boldsymbol{u}}\circ\boldsymbol{\Theta}\in H^1(\Omega^\varepsilon;\mathbb{R}^3)$$

the "scaled" vector field

$$\boldsymbol{u}(\varepsilon):=\boldsymbol{u}\circ\boldsymbol{\pi}^\varepsilon\in H^1(\Omega^1;\mathbb{R}^3);$$

with the covariant components

$$u_i:=\boldsymbol{u}\cdot\boldsymbol{g}_i\in H^1(\Omega^\varepsilon),\quad \boldsymbol{g}_i:=\frac{\partial\boldsymbol{\Theta}}{\partial x_i^\varepsilon},$$

of $\hat{\boldsymbol{u}}$, we associate the "scaled" covariant components

$$u_i(\varepsilon):=u_i\circ\boldsymbol{\pi}^\varepsilon\in H^1(\Omega^1);$$

and, with the covariant components

$$\begin{aligned}&e_{ij}(\boldsymbol{u}):=\frac{1}{2}\left(\frac{\partial\boldsymbol{u}}{\partial x_j^\varepsilon}\cdot\boldsymbol{g}_i+\frac{\partial\boldsymbol{u}}{\partial x_i^\varepsilon}\cdot\boldsymbol{g}_j\right)=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j^\varepsilon}+\frac{\partial u_j}{\partial x_i^\varepsilon}\right)-\Gamma_{ij}^k u_k\in L^2(\Omega^\varepsilon),\\&\Gamma_{ij}^k:=\frac{\partial\boldsymbol{g}_i}{\partial x_j^\varepsilon}\cdot\boldsymbol{g}^k\end{aligned}$$

of $\hat{\boldsymbol{\nabla}}_{\mathbf{s}}\hat{\boldsymbol{u}}$, we associate the "scaled" covariant components

$$e_{ij}(\varepsilon,\boldsymbol{u}):=e_{ij}(\boldsymbol{u})\circ\boldsymbol{\pi}_\varepsilon\in H^1(\Omega^1).$$

Note that the above definition of the functions $e_{ij}(\varepsilon,\boldsymbol{u})$ is equivalent to the following relations:

$$\begin{aligned}e_{\alpha\beta}(\varepsilon,\boldsymbol{u})&=\frac{1}{2}\left(\frac{\partial\boldsymbol{u}(\varepsilon)}{\partial y_\alpha}\cdot\boldsymbol{g}_\beta(\varepsilon)+\frac{\partial\boldsymbol{u}(\varepsilon)}{\partial y_\beta}\cdot\boldsymbol{g}_\alpha(\varepsilon)\right)\\&=\frac{1}{2}\left(\frac{\partial u_\alpha(\varepsilon)}{\partial y_\beta}+\frac{\partial u_\beta(\varepsilon)}{\partial y_\alpha}\right)-\Gamma_{\alpha\beta}^k(\varepsilon)u_k(\varepsilon),\end{aligned}$$

$$\begin{aligned}
e_{\alpha 3}(\varepsilon,\boldsymbol{u}) = e_{3\alpha}(\varepsilon,\boldsymbol{u}) &= \frac{1}{2}\left(\frac{\partial \boldsymbol{u}(\varepsilon)}{\partial y_\alpha}\cdot \boldsymbol{g}_3(\varepsilon) + \frac{1}{\varepsilon}\frac{\partial \boldsymbol{u}(\varepsilon)}{\partial x_3}\cdot \boldsymbol{g}_\alpha(\varepsilon)\right) \\
&= \frac{1}{2}\left(\frac{\partial u_3(\varepsilon)}{\partial y_\alpha} + \frac{1}{\varepsilon}\frac{\partial u_\alpha(\varepsilon)}{\partial x_3}\right) - \Gamma^\beta_{\alpha 3}(\varepsilon) u_\beta(\varepsilon), \\
e_{33}(\varepsilon,\boldsymbol{u}) &= \frac{1}{\varepsilon}\frac{\partial \boldsymbol{u}(\varepsilon)}{\partial x_3}\cdot \boldsymbol{g}_3(\varepsilon) = \frac{1}{\varepsilon}\frac{\partial u_3(\varepsilon)}{\partial x_3},
\end{aligned}$$

where $\boldsymbol{g}_i(\varepsilon) := \boldsymbol{g}_i \circ \boldsymbol{\pi}_\varepsilon$ and $\Gamma^k_{ij}(\varepsilon) := \Gamma^k_{ij} \circ \boldsymbol{\pi}_\varepsilon$.

We are now in a position to establish inequalities of Korn's type similar to those of Theorem 3.1, but this time with norms independent of ε:

Theorem 4.1 (Korn's Scaled Inequalities for Shells in Curvilinear Coordinates)**.**

(a) Given any domain $\omega \subset \mathbb{R}^2$ and any immersion $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega};\mathbb{R}^3)$, there exist two constants $\varepsilon_0 = \varepsilon_0(\omega,\boldsymbol{\theta})$ and $C_0 = C_0(\omega,\boldsymbol{\theta})$ independent of ε such that, for all $0 < \varepsilon \le \varepsilon_0$ and for all $\boldsymbol{u} \in H^1(\Omega^1;\mathbb{R}^3)$, $\Omega^1 := \omega \times (-1,1)$,

$$\|(u_i)\|_{H^1(\Omega^1)} + \frac{1}{\varepsilon}\left\|\frac{\partial (u_i)}{\partial x_3}\right\|_{L^2(\Omega^1)} \le C_0\left(\|(u_i)\|_{L^2(\Omega^1)} + \frac{1}{\varepsilon}\left\|\left(e_{ij}(\varepsilon,\boldsymbol{u})\right)\right\|_{L^2(\Omega^1)}\right), \tag{4.1}$$

$$\inf_{(r_i)\in \mathrm{Rig}(\Omega^1)}\left(\|(u_i - r_i)\|_{H^1(\Omega^1)} + \frac{1}{\varepsilon}\left\|\frac{\partial (u_i - r_i)}{\partial x_3}\right\|_{L^2(\Omega^1)}\right) \le \frac{C_0}{\varepsilon}\left\|\left(e_{ij}(\varepsilon,\boldsymbol{u})\right)\right\|_{L^2(\Omega^1)}, \tag{4.2}$$

where

$$\begin{aligned}
e_{\alpha\beta}(\varepsilon,\boldsymbol{u}) &:= \frac{1}{2}\left(\frac{\partial u_\alpha}{\partial y_\beta} + \frac{\partial u_\beta}{\partial y_\alpha}\right) - \Gamma^k_{\alpha\beta}(\varepsilon) u_k, \\
e_{\alpha 3}(\varepsilon,\boldsymbol{u}) = e_{3\alpha}(\varepsilon,\boldsymbol{u}) &:= \frac{1}{2}\left(\frac{\partial u_3}{\partial y_\alpha} + \frac{1}{\varepsilon}\frac{\partial u_\alpha}{\partial x_3}\right) - \Gamma^\beta_{\alpha 3}(\varepsilon) u_\beta, \\
e_{33}(\varepsilon,\boldsymbol{u}) &:= \frac{1}{\varepsilon}\frac{\partial u_3}{\partial x_3},
\end{aligned} \tag{4.3}$$

and

$$\mathrm{Rig}(\Omega^1) := \left\{(r_i) \in H^1(\Omega^1;\mathbb{R}^3);\ e_{ij}(\varepsilon,\boldsymbol{r}) = 0 \text{ a.e. in } \Omega^1\right\}.$$

(b) Given any domain $\omega \subset \mathbb{R}^2$, any embedding $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega};\mathbb{R}^3)$ and any non-empty relatively open subset γ_0 of the boundary of ω such that $\boldsymbol{\theta}(\gamma_0)$ is not contained in a straight line, there exist two constants $\varepsilon_0 = \varepsilon_0(\omega,\boldsymbol{\theta})$ and $C_0 = C_0(\omega,\boldsymbol{\theta},\gamma_0)$ independent of ε such that, for all $0 < \varepsilon \le \varepsilon_0$ and for all $(u_i) \in H^1(\Omega^1;\mathbb{R}^3)$ that vanish on $\Gamma^1 := \gamma_0 \times (-1,1)$,

$$\|(u_i)\|_{H^1(\Omega^1)} + \frac{1}{\varepsilon}\left\|\frac{\partial (u_i)}{\partial x_3}\right\|_{L^2(\Omega^1)} \le \frac{C_0}{\varepsilon}\left\|\left(e_{ij}(\varepsilon,\boldsymbol{u})\right)\right\|_{L^2(\Omega^1)}. \tag{4.4}$$

Proof. Given any vector field $(u_i)\in H^1(\Omega^1;\mathbb{R}^3)$, let $u_i^\varepsilon := u_i\circ(\boldsymbol{\pi}^\varepsilon)^{-1}\in H^1(\Omega^\varepsilon)$ and

$$e_{ij}(\boldsymbol{u}^\varepsilon) := \frac{1}{2}\left(u_i^\varepsilon|_j + u_j^\varepsilon|_i\right) = \frac{1}{2}\left(\frac{\partial u_i^\varepsilon}{\partial x_j^\varepsilon} + \frac{\partial u_j^\varepsilon}{\partial x_i^\varepsilon}\right) - \Gamma_{ij}^k u_k^\varepsilon \in L^2(\Omega^\varepsilon), \tag{4.5}$$

where $\Gamma_{ij}^k := (\partial \boldsymbol{g}_i/\partial x_j^\varepsilon)\cdot \boldsymbol{g}^k \in \mathcal{C}^0(\overline{\Omega^\varepsilon})$. Then Theorem 3.1 shows that, for some constant C_0 independent of ε,

$$\begin{aligned}
&\|(u_i)\|_{H^1(\Omega^\varepsilon)} \le C_0\left(\left\|(u_i^\varepsilon)\right\|_{L^2(\Omega^\varepsilon)} + \frac{1}{\varepsilon}\left\|(e_{ij}(\boldsymbol{u}^\varepsilon))\right\|_{L^2(\Omega^\varepsilon)}\right),\\
&\inf_{(r_i^\varepsilon)\in \mathrm{Rig}(\Omega^\varepsilon)} \left\|(u_i^\varepsilon)-(r_i^\varepsilon)\right\|_{H^1(\Omega^\varepsilon)} \le \frac{C_0}{\varepsilon}\left\|(e_{ij}(\boldsymbol{u}^\varepsilon))\right\|_{L^2(\Omega^\varepsilon)},\\
&\left\|(u_i^\varepsilon)\right\|_{H^1(\Omega^\varepsilon)} \le \frac{C_0}{\varepsilon}\left\|(e_{ij}(\boldsymbol{u}^\varepsilon))\right\|_{L^2(\Omega^\varepsilon)} \quad \text{if} \quad u_i^\varepsilon = 0 \quad \text{on } \gamma_0\times(-\varepsilon,\varepsilon).
\end{aligned} \tag{4.6}$$

Since $u_i = u_i^\varepsilon\circ\boldsymbol{\pi}^\varepsilon$ in Ω^1, we have (remember that $x_\alpha^\varepsilon = x_\alpha = y_\alpha$ and $x_3^\varepsilon = \varepsilon x_3$)

$$\frac{\partial u_i}{\partial y_\alpha} = \frac{\partial u_i^\varepsilon}{\partial y_\alpha}\circ\boldsymbol{\pi}^\varepsilon, \quad \frac{\partial u_i}{\partial x_3} = \varepsilon\frac{\partial u_i^\varepsilon}{\partial x_3^\varepsilon}\circ\boldsymbol{\pi}^\varepsilon \quad \text{in } \Omega^1, \tag{4.7}$$

so that, by using the change of variable $x^\varepsilon = \boldsymbol{\pi}^\varepsilon(x)$ in the integrals below,

$$\begin{aligned}
&\int_{\Omega^1} |u_i(x)|^2\,\mathrm{d}x = \varepsilon^{-1}\int_{\Omega^\varepsilon}\left|u_i^\varepsilon(x^\varepsilon)\right|^2\mathrm{d}x^\varepsilon,\\
&\int_{\Omega^1}\left|\frac{\partial u_i}{\partial y_\alpha}(x)\right|^2\mathrm{d}x = \varepsilon^{-1}\int_{\Omega^\varepsilon}\left|\frac{\partial u_i^\varepsilon}{\partial y_\alpha}(x^\varepsilon)\right|^2\mathrm{d}x^\varepsilon,\\
&\int_{\Omega^1}\left|\frac{\partial u_i}{\partial x_3}(x)\right|^2\mathrm{d}x = \varepsilon\int_{\Omega^\varepsilon}\left|\frac{\partial u_i^\varepsilon}{\partial x_3^\varepsilon}(x^\varepsilon)\right|^2\mathrm{d}x^\varepsilon,
\end{aligned}$$

on the one hand.

Definitions (4.3) and (4.5) of the functions $e_{ij}(\varepsilon,\boldsymbol{u})$ and $e_{ij}(\boldsymbol{u}^\varepsilon)$, combined with relations (4.7), imply that $e_{ij}(\varepsilon,\boldsymbol{u}) = (e_{ij}(\boldsymbol{u}^\varepsilon))\circ\boldsymbol{\pi}^\varepsilon$ in Ω^ε, so that we have

$$\int_{\Omega^1}\left|(e_{ij}(\varepsilon,\boldsymbol{u}))(x)\right|^2\mathrm{d}x = \varepsilon^{-1}\int_{\Omega^\varepsilon}\left|(e_{ij}(\boldsymbol{u}^\varepsilon))(x^\varepsilon)\right|^2\mathrm{d}x^\varepsilon,$$

on the other hand.

Consequently,

$$\begin{aligned}
\|(u_i^\varepsilon)\|^2_{L^2(\Omega^\varepsilon)} &= \sum_i \|u_i^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} = \sum_i \varepsilon \|u_i\|^2_{L^2(\Omega^1)} = \varepsilon \|(u_i)\|^2_{L^2(\Omega^1)}, \\
\|(e_{ij}(\boldsymbol{u}^\varepsilon))\|^2_{L^2(\Omega^\varepsilon)} &= \sum_{i,j} \|e_{ij}(\boldsymbol{u}^\varepsilon)\|^2_{L^2(\Omega^\varepsilon)} = \sum_{i,j} \varepsilon \|e_{ij}(\varepsilon,\boldsymbol{u})\|^2_{L^2(\Omega^1)} \\
&= \varepsilon \|(e_{ij}(\varepsilon,\boldsymbol{u}))\|^2_{L^2(\Omega^1)},
\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}
\|(u_i^\varepsilon)\|^2_{H^1(\Omega^\varepsilon)} &= \sum_i \|u_i^\varepsilon\|^2_{H^1(\Omega^\varepsilon)} \\
&= \sum_i \|u_i^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} + \sum_{i,\alpha} \left\|\frac{\partial u_i^\varepsilon}{\partial y_\alpha}\right\|^2_{L^2(\Omega^\varepsilon)} + \sum_i \left\|\frac{\partial u_i^\varepsilon}{\partial x_3^\varepsilon}\right\|^2_{L^2(\Omega^\varepsilon)} \\
&= \sum_i \varepsilon \|u_i\|^2_{L^2(\Omega^1)} + \sum_{i,\alpha} \varepsilon \left\|\frac{\partial u_i}{\partial y_\alpha}\right\|^2_{L^2(\Omega^1)} + \sum_i \varepsilon^{-1} \left\|\frac{\partial u_i}{\partial x_3}\right\|^2_{L^2(\Omega^1)} \\
&= \varepsilon \|(u_i)\|^2_{L^2(\Omega^1)} + \varepsilon \sum_\alpha \left\|\frac{\partial (u_i)}{\partial y_\alpha}\right\|^2_{L^2(\Omega^1)} + \varepsilon^{-1} \left\|\frac{\partial (u_i)}{\partial x_3}\right\|^2_{L^2(\Omega^1)} \\
&\geq \frac{\varepsilon}{3} \left(\|(u_i)\|_{L^2(\Omega^1)} + \sum_\alpha \left\|\frac{\partial (u_i)}{\partial y_\alpha}\right\|_{L^2(\Omega^1)} + \varepsilon^{-1} \left\|\frac{\partial (u_i)}{\partial x_3}\right\|_{L^2(\Omega^1)} \right)^2.
\end{aligned} \tag{4.9}$$

Besides, the last inequality also holds with (u_i^ε) and (u_i) replaced respectively by $(u_i^\varepsilon + r_i^\varepsilon)$ and $(u_i + r_i)$, where $r_i^\varepsilon := r_i \circ (\boldsymbol{\pi}^\varepsilon)^{-1} \in \mathrm{Rig}(\Omega^\varepsilon)$ for any given vector field $(r_i) \in \mathrm{Rig}(\Omega^1)$.

Then inequalities (4.1), (4.2) and (4.4) appearing in the statement of the theorem are deduced from inequalities (4.6) established above by using relations (4.8) in their right-hand sides and the lower bound (4.9) in their left-hand sides.

The proof is complete. □

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