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Modified Newton Preconditioned Block GADI Method for Complex Nonlinear Systems

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Abstract. An efficient modified Newton-PBGADI method for nonlinear systems with large sparse non-Hermitian positive definite Jacobian matrices is proposed. The method integrates a modified Newton framework that achieves R-order three convergence while requiring only a single inversion of the Jacobian matrix per iteration. The resulting linear subproblems are solved by a preconditioned block GADI method. The convergence of this iterative scheme is studied and the local convergence of the overall modified Newton–PBGADI method is rigorously analyzed. Two numerical examples involving two-dimensional nonlinear PDEs validate effectiveness and computational efficiency of the method.

AMS subject classifications: 65F08, 65F10, 65F45

Key words: Complex system of nonlinear equations, modified Newton-PBGADI method, splitting iteration, convergence analysis.

1. Introduction

Let $\mathscr{F}:\mathbb{D}\subset\mathbb{C}^n\to\mathbb{C}^n$ be a nonlinear continuously differentiable function such that its Jacobian $\mathscr{F}'(x)$ is a large sparse matrix. Representing the Jacobian in the form $\mathscr{F}'(x)=U(x)+iQ(x)$, $i^2=-1$, we assume that $U(x)$ is a real symmetric positive definite matrix, $Q(x)$ a real symmetric positive semi-definite matrix, and consider the non-linear system

$$\mathscr{F}(x)=0. \tag{1.1}$$

Note that nonlinear systems of algebraic equations appear in various applications, including engineering, science, and economics — cf. Refs. [15,28,34]. These systems play a vital

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role in modeling and solving numerous real-world problems, making numerical methods for their solution indispensable tools in these fields. There are numerous well-established numerical methods for solving nonlinear equations, with the Newton method [25] being the most fundamental and extensively utilized iterative method. The primary computational expense of implementing the Newton method lies in the inversion of Jacobian matrices, and its convergence is limited to a local quadratic rate. Additionally, several extensions of the Newton method have been developed, including the Müller method and the Halley method — cf. [1, 18, 30, 31]. Among these modifications, high-order iteration methods either fall short of attaining the desired higher-order convergence or achieve it with significantly increased computational complexity. In this work, we employ the modified Newton (MN) method [17] as the solver for the nonlinear systems. Although the MN method requires evaluating function one additional time compared to the Newton method, it achieves at least third-order R-convergence.

In order to solve linear equations arising in Newton-like methods, it is essential to consider efficient strategies. Two approaches can be used — viz. direct and iterative methods. Direct methods are important techniques for solving linear systems of equations. However, they are not suitable in the case of large-scale problems because of high storage demands. Moreover, for ill-conditioned coefficient matrices, direct methods may become numerically unstable. Therefore, for large-scale sparse systems requiring dynamic adjustments, iterative methods are often more suitable. There are many types of iterative methods, such as Krylov subspace methods, multigrid methods, splitting iteration methods, etc. For splitting iteration methods, many results have already been published [2, 6, 9, 19, 21]. Thus, in 2002 Bai *et al.* [8] proposed Hermitian and skew-Hermitian splitting methods for linear non-Hermitian and positive definite systems and presented unconditional convergence analysis. Later on, the method has been extended and improved by Benzi [11], who generalized the HSS method and demonstrated that the new scheme outperforms the HSS method in certain situations, proving to be an effective preconditioner for specific linear systems in saddle point form. Recently, Bai [4, 5] proposed a two-step matrix splitting iterative method and analyzed its convergence. This approach encompasses several existing two-step matrix splitting iterative methods as special cases.

Furthermore, by using the MN method as an outer iteration and an iterative method for linear systems as the inner iteration, we can construct an iterative method for nonlinear equations. To address this challenge, various iterative methods have been developed in recent years [10, 20]. In 2010, Bai *et al.* [10] introduced the Newton-HSS method and gave a rigorous proof of its convergence. Consequently, different refinements and extensions to the method have been developed — e.g. the MN-HSS method — cf. [32], and numerical experiments show that the MN-HSS method is efficient. In 2017, Dai *et al.* [16] developed the MN-NSS method functioning as the solver for nonlinear equations. This method also demonstrates good numerical experimental results. On the other hand, it was noted that while solving complex nonlinear systems, the imaginary and real components of complex numbers can be treated separately. In particular, for Jacobian matrices, one have to work with complex arithmetic, which may hinder performance. Therefore, suitable iterative solving strategies have been used in such cases [12, 14, 24, 26, 27, 33]. For example,