East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.291211.080212a

Vol. **2**, No. 1, pp. 59-82
February 2012

A New Fourth-Order Compact Off-Step Discretization for the System of 2D Nonlinear Elliptic Partial Differential Equations

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Received 29 December 2011; Accepted (in revised version) 8 February 2012

Available online 10 February 2012

Abstract. This paper discusses a new fourth-order compact off-step discretization for the solution of a system of two-dimensional nonlinear elliptic partial differential equations subject to Dirichlet boundary conditions. New methods to obtain the fourth-order accurate numerical solution of the first order normal derivatives of the solution are also derived. In all cases, we use only nine grid points to compute the solution. The proposed methods are directly applicable to singular problems and problems in polar coordinates, which is a main attraction. The convergence analysis of the derived method is discussed in detail. Several physical problems are solved to demonstrate the usefulness of the proposed methods.

AMS subject classifications: 65N06

PACS: 02.60.Lj; 02.70.Bf

Key words: Two-dimensional nonlinear elliptic equations, off-step discretization, fourth-order finite difference methods, normal derivatives, convection-diffusion equation, Poisson equation in polar coordinates, Navier-Stokes equations of motion.

1. Introduction

We consider the system of two-dimensional (2D) nonlinear elliptic partial differential equations (PDE)

$$L\mathbf{u} \equiv \mathbf{A}\frac{\partial^2 \mathbf{u}}{\partial x^2} + \mathbf{B}\frac{\partial^2 \mathbf{u}}{\partial y^2} = \mathbf{f} \tag{1.1}$$

defined in the domain $\Omega = \{(x,y)|0 < x, y < 1\}$ with boundary $\partial\Omega$, where

$$\mathbf{A} = \text{diag}\left(A^{(i)}(x,y)\right) \in \mathbb{R}^{nXn}, \qquad \mathbf{B} = \text{diag}\left(B^{(i)}(x,y)\right) \in \mathbb{R}^{nXn},$$
$$\mathbf{u} = \left[u^{(1)}, u^{(2)}, \cdots, u^{(n)}\right]^t, \qquad \mathbf{f} = \left[f^{(1)}, f^{(2)}, \cdots, f^{(n)}\right]^t,$$

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and t denotes the transpose of the matrix. We consider $i = 1(1)n$ throughout this paper. Each $f^{(i)}$ is a function of x, y, $u^{(1)}$, $u^{(2)}$, $\cdots$, $u^{(n)}$, $u_x^{(1)}$, $u_x^{(2)}$, $\cdots$, $u_x^{(n)}$ $u_y^{(1)}$, $u_y^{(2)}$, $\cdots$, $u_y^{(n)}$; and the system (1.1) is subject to the Dirichlet boundary conditions given by

$$u^{(i)}(x,y) = u_0^{(i)}(x,y), \qquad (x,y) \in \partial\Omega \tag{1.2}$$

where $u_0^{(i)}$ are continuous functions in $\partial\Omega$. In addition, Eqs. (1.1) are assumed to satisfy the ellipticity conditions $A^{(i)}B^{(i)} > 0$ in Ω. Further, $\forall\ (x,y) \in \Omega$ we assume that

(i) $u^{(i)}(x,y) \in C^6$;

(ii) $A^{(i)}(x,y)$, $B^{(i)}(x,y) \in C^4$;

(iii) $f^{(i)}(x,y,u^{(1)},u^{(2)},\cdots,u^{(n)},u_x^{(1)},u_x^{(2)},\cdots,u_x^{(n)},u_y^{(1)},u_y^{(2)},\cdots,u_y^{(n)})$ is differentiable, and for $j = 1,2,\cdots,n$

(iv) $\partial f^{(i)}/\partial u^{(j)} \geqslant 0$;

(v) $|\partial f^{(i)}/\partial u_x^{(j)}| \leqslant C$ and $|\partial f^{(i)}/\partial u_y^{(j)}| \leqslant D$,

where C and D are positive constants and C^m denotes the set of all functions of x and y with partial derivatives up to order m continuous in Ω [3]. Conditions (iii), (iv) and (v) guarantee the existence and uniqueness of the solution of the given boundary value problem.

The present paper is concerned with solving the system (1.1) of 2D nonlinear elliptic PDE with variable coefficients by a new compact 9-point fourth-order off-step finite difference discretization. Such systems of equations arise in various important mathematical models in science and engineering. For linear elliptic problems, there has been considerable work done on the development of high order compact schemes and the convergence of relevant iterative solution methods — e.g. [5–10]. Ananthakrishnaiah & Saldanha [4] framed a 13-point fourth-order compact scheme for the solution of a scalar nonlinear elliptic PDE, which was later extended to a system of equations [18]. A variety of high order compact schemes have been developed for the solution of 2D steady state Navier-Stokes (N-S) equations in stream function vorticity form in Cartesian coordinates — e.g. [1,2,19–21].

One of the present authors previously proposed fourth-order difference methods for 2D nonlinear elliptic boundary value problems with variable coefficients using only 9 grid points of a single compact cell, with application to the singular problems involving Poisson equation and the Navier-Stokes equations in polar coordinates [12]. Subsequently, fourth-order accurate estimates were developed for the first order normal derivatives $(\partial u/\partial n)$ [13]. However, these methods could not be applied to singular elliptic problems directly, due to terms such as $1/r_{l-1}$ in polar coordinates that create difficulties at $l = 1$ where $r_0 = 0$. In such cases, a suitable difference approximation valid at $r = 0$ or a suitable modification at the singular point is required. Consequently, Mohanty and Singh [14] derived a new compact fourth-order discretization for the solution of singularly perturbed 2D nonlinear elliptic problems and the estimates of $(\partial u/\partial n)$, in an approach referred to as *off-step* discretization.