ASYMPTOTIC EXPANSION FOR THE DERIVATIVE OF FINITE ELEMENTS*

LIN QUN (林 群)

(*Institute of Systems Science, Academia Sinica, Beijing, China*)

ZHU QI-DING (朱起定)

(*Xiangtan University, Hunan, China*)

Abstract

It is proved in this paper that there exists an expansion for the derivative of the linear finite element approximation to a model Dirichlet problem in a polygonal domain with a piecewise uniform triangulation.

Superconvergence of the derivative of finite elements has been studied in many works, see [1—24]. To the authors' knowledge, no expansion theorem has been proved for the finite element derivative. The aim of the paper is to show that such an expansion theorem can be derived by the expansion method in [11—14] for the finite element solution and some estimates in [6, 16, 17, 22—24] for the Green function.

Consider the model problem: Find $u\in H_0^1(\Omega)$ such that

$$-\Delta u=f \text{ in } \Omega, \tag{1}$$

where Ω is a convex polygonal domain. We approximate (1) by the linear elements. First we divide Ω into several triangular subdomains $\Omega_i(i=1, 2, \cdots, M)$ meeting at a point A (see Fig. 1) and then divide each Ω_i into a uniform triangulation:

$$\Omega_i=\bigcup_j \Omega_{ij}$$

with triangles Ω_{ij} $(j=1, 2, \cdots, 4^N)$. So we have a piecewise uniform partition

$$\Omega=\bigcup_i \Omega_i=\bigcup_i\bigcup_j \Omega_{ij}. \tag{2}$$

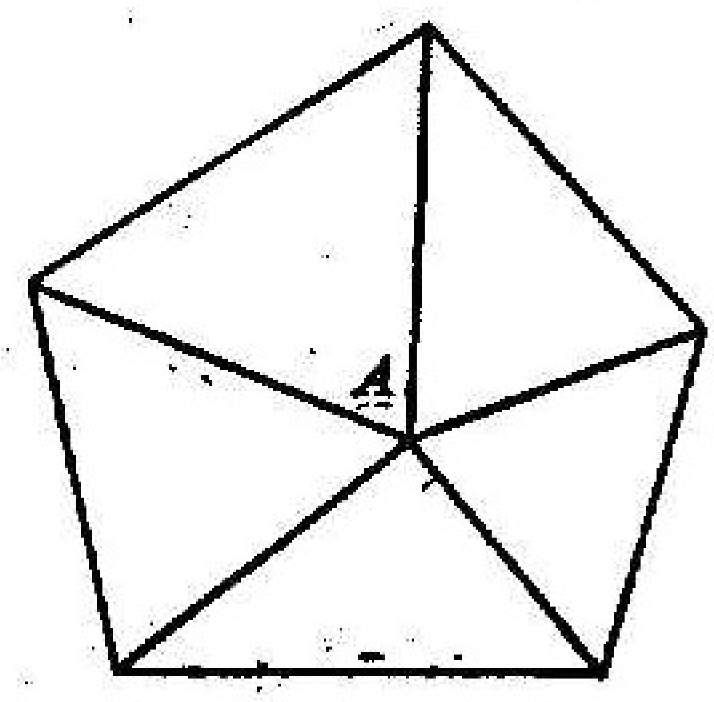

Fig. 1 (Ω)

Let $h=1/2^N$, and let u^h and u^I be the linear finite element approximation and the linear interpolation of u respect to the partition (2) respectively.

Consider an interior domain

$$\Omega_0=\{z\in\Omega: \text{dist } (z, \partial\Omega_i)\geqslant\delta>0\},$$

where δ is an arbitrary constant and $\partial\Omega_i$ the edges of $\Omega_i (1\leqslant i\leqslant M)$.

Theorem. *If $u\in H^{4,q}(\Omega)$, then there exists a function $w\in H^{2,q}(\Omega_0)$ independent of h such that*

$$\nabla(u^h-u^I)(z_0)=h^2\nabla w(z_0)+O(h^{3-2/q})\|u\|_{4,q} \tag{3}$$

for $z_0\in\Omega_0$, where $q\in(2, q_0)$ with $q_0\in(2, \infty)$ dependent on Ω (see (5)).

* Received March 31, 1984.

Proof. Let $G_{z_0}(z)$ be the Green function with singularity at z_0 and $G^h_{z_0}(z)$ the linear finite element approximation to $G_{z_0}(z)$, Then, from [11, 13, 14], we have

$$(u^h-u^I)(z_0)=\int_\Omega \nabla G^h_{z_0}\nabla(u-u^I)dz$$
$$=h^2P_h(z_0)+h^2Q_h(z_0)+O(h^3)\|u\|_{4,q}\|G^h_{z_0}\|_{1,p}$$

with the piecewise linear functions P_h and Q_h defined by

$$P_h(z_0)=\sum_{i=1}^M\int_{\Omega_i} G^h_{z_0}D_i^4u\,dz,$$
$$Q_h(z_0)=\sum_{i=1}^M\left(\int_{\partial\Omega_i} G^h_{z_0}D_i^3u\,dz+G^h_{z_0}(A)D_i^2u(A)\right), \quad (4)$$

where D_i^j denotes some linear combinations of some order ($\leqslant j$) derivatives of u, $\frac{1}{p}+\frac{1}{q}=1$.

Let z' and z'' be any two vertices at a typical triangle element $\Omega_{ij}\subset\Omega_0$. Then

$$(u^h-u^I)(z')-(u^h-u^I)(z'')=\int_\Omega \nabla(G^h_{z'}-G^h_{z''})\nabla(u-u^I)dz$$
$$=h^2(P_h(z')-P_h(z''))+h^2(Q_h(z')-Q_h(z''))$$
$$+O(h^3)\|u\|_{4,q}\|G^h_{z'}-G^h_{z''}\|_{1,p}.$$

Setting

$$Q(z_0)=\sum_{i=1}^M\left(\int_{\partial\Omega_i} G_{z_0}D_i^3u\,dz+G_{z_0}(A)D_i^2u(A)\right)\in H^{2,q}(\Omega_0)$$

and using the inequalities due to Frehse, Rannacher and Scott[5,6,16] and Schatz and Wahlbin[17]

$$\|G^h_{z'}-G^h_{z''}\|_{1,p}\leqslant Kh^{1-2/q},$$
$$|G^h_{z'}(z)-G_{z'}(z)|\leqslant Kh^{2-2/q} \quad \text{for } z\in\partial\Omega_i$$

we obtain

$$(u^h-u^I)(z')-(u^h-u^I)(z'')=h^2(P_h(z')-P_h(z''))$$
$$+h^2(Q(z')-Q(z''))+O(h^{4-2/q})\|u\|_{4,q},$$

hence, for $z_0\in\Omega_{ij}$,

$$\nabla(u^h-u^I)(z_0)=h^2\nabla P_h(z_0)+h^2\nabla Q^I(z_0)+O(h^{3-2/q})\|u\|_{4,q}.$$

Let

$$F(z)=D_i^4u(z) \quad \text{if } z\in\Omega_i$$

and find $P\in H_0^1(\Omega)\cap H^{2,q}(\Omega)$ such that

$$-\Delta P=F \quad \text{in } \Omega.$$

Then

$$P(z_0)=\sum_{i=1}^M\int_{\Omega_i} G_{z_0}D_i^4u\,dz\in H^{2,q}(\Omega). \quad (5)$$

Note from (4) that P_h is just the linear finite element approximation to P, we have by [16]

$$|\nabla(P_h-P)|\leqslant Kh^{1-2/q}\|P\|_{2,q,\Omega},$$
$$|\nabla(Q^I-Q)|\leqslant Kh^{1-2/q}\|Q\|_{2,q,\Omega_0}$$

and (3) follows with $w=P+Q$.

Let $\overline{\nabla}$ be the nodal point averaged gradient defined by Křížek and Neittaanmäki[18]. Then it is easy to prove that