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A TWO-LEVEL FINITE ELEMENT GALERKIN METHOD FOR THE NONSTATIONARY NAVIER-STOKES EQUATIONS II: TIME DISCRETIZATION *1)

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Abstract

In this article we consider the fully discrete two-level finite element Galerkin method for the two-dimensional nonstationary incompressible Navier-Stokes equations. This method consists in dealing with the fully discrete nonlinear Navier-Stokes problem on a coarse mesh with width H and the fully discrete linear generalized Stokes problem on a fine mesh with width $h << H$. Our results show that if we choose $H = O(h^{1/2})$ this method is as the same stability and convergence as the fully discrete standard finite element Galerkin method which needs dealing with the fully discrete nonlinear Navier-Stokes problem on a fine mesh with width h. However, our method is cheaper than the standard fully discrete finite element Galerkin method.

Key words: Navier-Stokes equations, Galerkin method, Finite element.

1. Introduction

Two-level finite element Galerkin method is an efficient numerical method for solving nonlinear partial differential equations, e.g., see Xu [25, 26] for steady semi-linear elliptic equations, Layton [15], Ervin, Layton and Maubach [5], Layton and Lenferink [16] and Layton and Tobiska [17] for the steady Navier-Stokes equations. This method is closely related to the nonlinear Galerkin method [1,11, 18-20, 23] and recently developed in [7,22] to solve the nonstationary Navier-Stokes equations. However,it is well known [1, 11, 18-20, 23] that a defect of the nonlinear Galerkin methods is needed to approximate solution u_h as the large eddy component y^H and the small eddy component z^h and solve the unknown components y^H and z^h simultaneously, that is to solve a coupled nonlinear and linear equations and increase computing price.

In the case of the nonlinear evolution problem, the basic idea of the two-level method is to find an approximation u_H by solving a nonlinear problem on a coarse grid with grid size H and find an approximation u^h by solving a linearized problem about the known approximation u_H on a fine grid with grid size h. The semi-discretization in space of the 3D time-dependent Navier-Stokes problem by the two-level method is considered in [7]. Furthermore, the fully discretization in space-time of the 2D and 3D time-dependent Navier-Stokes problem by the two-level method is analyzed in [22], where the local error estimates, stability and convergence are proved, but the global error estimates do not provided. In fact, this scheme is of the global first-order accurate with respect to the time step size τ.

In the recent work [10] we considered this two-level method used in [22] for solving the nonstationary, incompressible Navier-Stokes equations. If the equations is discreted by the standard finite element Galerkin method, there will be a large system of nonlinear algebraic

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equations to be solved. To overcome this difficult, we applied a two-level finite element Galerkin method for solving the nonstationary Navier-Stokes equations in the framework of mixed finite elements. This will yield a small system of nonlinear algebraic equations and a large system of linear algebraic to be solved, i.e., this method can save a large amount of computational work. For the standard finite element Galerkin method, the discrete velocity $u_h(\cdot,t)$ and pressure $p_h(\cdot,t)$ are determined in finite element spaces denoted respectively by X_h and M_h which satisfy the so-called inf-sup condition (see [3, 8]). Our two-level finite element Galerkin method consists in

- Finding $(u_H, p_H) \in (X_H, M_H)$ by solving the nonlinear Navier-Stokes problem on the coarse mesh with width H;
- Finding $(u^h, p^h) \in (X_h, M_h)$ by solving the linear generalized Stokes problem based on (u_H, p_H) on the fine mesh with width $h << H$.

In recent work [10], our main results are the following results:

$$\|u^h(t) - u_h(t)\|_{H^1} \le \kappa(t)(h + H^2) \quad \forall t \ge 0\,, \tag{1.1}$$

$$\|p^h(t) - p_h(t)\|_{L^2} \le \sigma(t)^{-1/2}\kappa(t)(h + H^2) \quad \forall t > 0\,, \tag{1.2}$$

where $\sigma(t) = \min\{1, t\}$ and (u_h, p_h) is the standard FE Galerkin approximation based on (X_h, M_h) which satisfies the following error estimates:

$$\|u(t) - u_h(t)\|_{H^1} \le \kappa(t)h, \forall t \ge 0, \tag{1.3}$$

$$\|p(t) - p_h(t)\|_{L^2} \le \sigma(t)^{-1/2}\kappa(t)h, \forall t > 0. \tag{1.4}$$

These estimates indicate that the two-level finite element Galerkin method gives the same order of approximation as the standard finite element Galerkin method if we choose $H = O(h^{1/2})$. However, in our method, the nonlinearity is only treated on the coarse grid and only the linear problem needs to be solved on the fine grid.

This paper continues our analysis of the two-level finite element Galerkin method for the Navier-Stokes equations. Here we study the time discretizations of the two-level finite element Galerkin method and the standard finite element Galerkin method in which time is discreted by the Euler implicit difference scheme. By using several discrete analogs of the Gronwall lemma, we are able to show that if we choose $H = O(h^{1/2})$, the two-level finite element Galerkin approximate solution $(u^h_\Delta(t), p^h_\Delta(t))$ is as stable and convergence as what should be verified by the standard finite element Galerkin approximate solution $(u^\Delta_h(t), p^\Delta_h(t))$, namely the numerical solutions $(u^h_\Delta(t), p^h_\Delta(t))$ and $(u^\Delta_h(t), p^\Delta_h(t))$ satisfy

$$\|u^h_\Delta(t)\|_{H^1}, \|u^\Delta_h(t)\|_{H^1} \le c(\|\bar{u}_0\|_{H^1} + \sup_{t\ge 0}\|f(t)\|_{L^2}), \tag{1.5}$$

$$\|u^h_\Delta(t) - u(t)\|_{H^1} \le \kappa(t)(h + H^2 + \Delta t), \forall t \ge 0, \tag{1.6}$$

$$(\int_0^t \|p^h_\Delta(s) - p(s)\|^2_{L^2} ds)^{1/2} \le \kappa(t)(h + H^2 + \Delta t), \forall t \ge 0, \tag{1.7}$$

$$\|u^\Delta_h(t) - u(t)\|_{H^1} \le \kappa(t)(h + \Delta t), \forall t \ge 0, \tag{1.8}$$

$$(\int_0^t \|p^\Delta_h(s) - p(s)\|^2_{L^2} ds)^{1/2} \le \kappa(t)(h + \Delta t), \forall t \ge 0, \tag{1.9}$$

where (1.6)-(1.9) hold if Δt being small. Here $\kappa(t)$ denotes a generic constant depending on the data $(\Omega, \nu, u_0, f_\infty, t)$ and is continuous with respect to time,

$$f_\infty = \sup_{t\ge 0}\{|f(t)| + |f_t(t)|\}\,,\; f_t = \frac{df}{dt};$$

in the later case, such a constant which may stand for different values at its different occurrences. However, compared with the standard finite element Galerkin method, the two-level finite element Galerkin method should be more simple since this method only needs to solve a small, nonlinear Navier-Stokes problem on the coarse mesh with width H and a large linear generalized Stokes problem on the fine mesh with width $h << H$; while the standard finite element Galerkin method needs to solve a large, nonlinear Navier-Stokes problem on the fine mesh with width h.

2. The Navier-Stokes Equations

Let Ω be a bounded domain in R^2 assumed to have a Lipschitz-continuous boundary Γ and to satisfy a further condition stated in (2.5) below. We consider the time dependent Navier-Stokes equations describing the flow of a viscous incompressible fluid confined in Ω:

$$\frac{\partial u}{\partial t} - \nu\Delta u + (u \cdot \nabla)u + \nabla p = f \quad \text{in } \Omega\,,\, t > 0\,, \tag{2.1}$$

$$\text{div} u = 0 \quad \text{in } \Omega\,,\, t > 0\,, \tag{2.2}$$

$$u = 0 \quad \text{on } \Gamma\,,\, t > 0\,, \tag{2.3}$$

$$u(0) = \bar{u}_0 \quad \text{in } \Omega\,, \tag{2.4}$$

where $u = (u_1, u_2)$ is the velocity, p is the pressure, f represents the density of body forces, $\nu > 0$ is the viscosity and $\bar{u}_0$ is the initial velocity.

In order to introduce a variational formulation, we set

$$X = H_0^1(\Omega)^2\,,\, Y = L^2(\Omega)^2\,,$$

and

$$M = L_0^2(\Omega) = \{q \in L^2(\Omega)\,;\, \int_\Omega q(x)dx = 0\}\,.$$

We denote by $(\cdot,\cdot)$, $|.|$ the inner product and norm on $L^2(\Omega)$ or $L^2(\Omega)^2$. The space $H_0^1(\Omega)$ and X are equipped with their usual scalar product and norm

$$((u,v)) = (\nabla u, \nabla v)\,,\, \|u\| = ((u,u))^{1/2}\,.$$

We define the continuous bilinear forms $a(.,.)$ and $d(.,.)$ on $X \times X$ and $X \times M$ respectively by

$$a(u,v) = \nu((u,v)), \quad \forall u, v \in X\,,$$

and

$$d(v,q) = (q, \text{div}\, v), \quad \forall v \in X, q \in M\,.$$

Next, we introduce the closed subset V of X given by

$$V = \{v \in X\,;\, d(v,q) = 0, \quad \forall q \in M\} = \{v \in X\,;\, \text{div}\, v = 0 \quad \text{in } \Omega\}\,,$$

and we denote by H the closure of V in Y. One can show (see [11, 13-15]) that

$$H = \{v \in Y\,;\, \text{div}\, v = 0 \text{ in } \Omega \text{ and } v \cdot n = 0 \text{ on } \Gamma\}\,,$$

where n denotes the unit outward normal to Γ. Also, we denote by A the unbounded linear operator on Y given by $Au = -\Delta u$. We assume that Ω is such that the domain of A is given by

$$D(A) = H^2(\Omega)^2 \cap X\,. \tag{2.5}$$

For instance, (2.5) holds if Γ is of class C^2 or if Ω is a convex plane polygonal domain, see [9].

Moreover, we define the trilinear form

$$b(u,v,w) = ((u\cdot\nabla)v,w) + \frac{1}{2}((\operatorname{div} u)v,w)$$
$$= \frac{1}{2}((u\cdot\nabla)v,w) - \frac{1}{2}((u\cdot\nabla)w,v), \quad \forall u,v,w \in X\,.$$

For a given $f \in L^\infty(R^+\,;Y)$ and a given $\bar{u}_0 \in H$, the variational formulation of (2.1)-(2.4) reads: find a pair (u,p) with

$$u \in L^\infty(R^+\,;\,H) \cap L^2(0,T\,;\,V)\,, u_t \in L^2(0,T,V')\,, p \in D'(\Omega\times(0,T)), \quad \forall T>0\,,$$

such that

$$(u_t,v) + a(u,v) + b(u,u,v) - d(v,p) + d(u,q) = (f,v), \quad \forall (v,q) \in (X,M)\,, \tag{2.6}$$

$$u(0) = \bar{u}_0\,. \tag{2.7}$$

It is classical [8, 12, 14, 24] that (2.6)-(2.7) possesses a unique solution (u,p) which satisfies the following regularity results Lemma 2.1 below.

Lemma 2.1. *Let $f \in L^\infty(R^+\,;Y)$, $f_t \in L^\infty(R^+\,;Y)$ and $\bar{u}_0 \in D(A)\cap V$ be given. Then, the solution (u,p) of (2.6)-(2.7) satisfies*

$$\|u(t)\| + |Au(t)| + \|u_t(t)\| + |Au_t(t)| + |u_{tt}(t)| \le \kappa(t), \quad \forall t>0\,. \tag{2.8}$$

Here $\kappa(t)$ denotes a generic constant depending on the data $(\Omega,\nu,u_0,f_\infty,f_{1,\infty},t)$ and is continuous with respect to time,

$$f_\infty = \sup_{t\ge 0}|f(t)|, f_{1,\infty} = \sup_{t\ge 0}|f_t(t)|,\; f_t = \frac{df}{dt}\,.$$

Hereafter, we will denote by c a generic constant depending on the data (Ω,ν,f_∞) and $c_0, c_1, \cdots$, denote some positive constants depending only on (Ω,ν,f_∞). Finally, we also will use the following Poincare inequality:

$$\lambda_1|v|^2 \le \|v\|^2, \forall v \in X, \tag{2.9}$$

where λ_1 is the least eigenvalue of the operator A.

3. Finite Element Galerkin Approximation

From now on, h will be a real positive parameter tending to 0. We let $\tau_h = \tau_h(\Omega)$ be a uniformly regular mesh of Ω made of n-simplices K with mesh size h. We construct velocity-pressure finite element spaces $(X_h,M_h) \subset (X,M)$ based upon the mesh $\tau_h(\Omega)$ and define the subspace V_h of X_h given by

$$V_h = \{v_h \in X_h\,;\, d(v_h,q_h) = 0, \quad \forall q_h \in M_h\}\,. \tag{3.1}$$

Let $P_h : Y \to X_h$ and $\rho_h : M \to M_h$ denote the L^2-orthogonal projections defined respectively by

$$(P_h v, v_h) = (v, v_h,), \quad \forall v \in Y\,,\, v_h \in X_h\,,$$

and

$$(\rho_h q, q_h) = (q, q_h), \quad \forall q \in M\,,\, q_h \in M_h\,.$$

We assume that the couple (X_h,M_h) satisfies the following approximation properties:

for each $v \in D(A)$ and $q \in H^1(\Omega) \cap M$, there exist approximations $I_h v \in X_h$ and $J_h q \in M_h$ such that

$$\|v - I_h v\| \le ch|Av|\,,\ |q - J_h q| \le ch\|q\|_1\,, \tag{3.2}$$

together with the inverse inequality

$$\|v_h\|_{L^\infty} + \|v_h\| \le ch^{-1}|v_h|, \forall v_h \in X_h\,, \tag{3.3}$$

and the so-called inf-sup inequality: for each $q_h \in M_h$, there exists $v_h \in X_h, v_h \neq 0$ such that

$$d(v_h, q_h) \ge \bar{\beta}|q_h|\|v_h\|\,, \tag{3.4}$$

where $\bar{\beta} > 0$ is a constant independent of h, where $\|.\|_1$ denotes the usual norm of the Sobolev space $H^1(\Omega)$.

The following properties which are classical consequences of (3.2)-(3.4) (see [1, 3, 8]) will be very useful

$$\|P_h v\| \le c\|v\|, \quad \forall v \in X\,, \tag{3.5}$$

$$|v - P_h v| \le ch\|v\|, \quad \forall v \in X\,, \tag{3.6}$$

$$|v - P_h v| + h\|v - P_h v\| \le ch^2|Av,| \quad \forall v \in D(A). \tag{3.7}$$

It is well known that $b(\cdot,\cdot,\cdot)$ satisfies the following properties (see [1,8, 12, 17, 19]):

$$b(u, v, w) = -b(u, w, v)\,, \tag{3.8}$$

$$\begin{aligned} |b(u, v, w)| \le\ & \frac{1}{2}c_0|u|^{1/2}\|u\|^{1/2}\|v\||w|^{1/2}\|w\|^{1/2} \\ & + \frac{1}{2}c_0\|u\||v|^{1/2}\|v\|^{1/2}|w|^{1/2}\|w\|^{1/2}, \forall u, v, w \in X, \end{aligned} \tag{3.9}$$

$$|b(u, v, w)| \le c_0\|u\|\|v\|\|w\|, \forall u, v, w \in X. \tag{3.10}$$

The Galerkin approximation of (2.6)-(2.7) based on (X_h, M_h) reads:
Find $(u_h, p_h) \in H^1(0, T\,; X_h) \times L^2(0, T\,; M_h)\,, \forall T > 0\,$, such that

$$\begin{aligned} &(u_{h,t}, v_h) + a(u_h, v_h) + b(u_h, u_h, v_h) - d(v_h, p_h) + d(u_h, q_h) \\ &= (f, v_h), \quad \forall (v_h, q_h) \in (X_h, M_h)\,, \end{aligned} \tag{3.11}$$

$$u_h(0) = P_h\bar{u}_0\,. \tag{3.12}$$

The following error estimates are classical (see [1, 3,12,21]).
Theorem 3.1. *Under the assumptions (3.2)-(3.4), let* $f \in L^\infty(R^+\,; Y)$, $f_t \in L^\infty(R^+\,; Y)$ *and* $\bar{u}_0 \in D(A) \cap V$ *be given. Then, (3.11)-(3.12) possesses a unique solution* (u_h, p_h) *and the following error estimates hold:*

$$|u(t) - u_h(t)| + |u_t(t) - u_{h,t}(t)| + h\|u(t) - u_h(t)\| \le \kappa(t)h^2. \quad \forall t \ge 0\,, \tag{3.13}$$

$$|p(t) - p_h(t)| \le \sigma(t)^{-1/2}\kappa(t)h. \quad \forall t > 0, \tag{3.14}$$

4. Fully Discrete Two-Level Finite Element Galerkin Method

In this section, we first recall the spatial discrete two-level finite element Galerkin method. Then we consider the fully discrete two-level finite element Galerkin method by applying the Euler implicit difference scheme of time discretization to the spatial discrete two-level finite element Galerkin approximation. In order to justify the efficiency of this method, we also consider the fully discrete standard finite element Galerkin method.

Here we choose a coarse mesh width H and a fine mesh width $h << H$, and construct associated conforming finite element spaces (X_H, M_H) and (X_h, M_h), where $(X_H, M_H) \subset (X_h, M_h)$. Now we consider the following spatial discrete two-level finite element Galerkin method: Find $(u^h, p^h) \in H^1(0,T;X_h) \times L^2(0,T;M_h), \forall T > 0$ as follows:

- **Step I: Solve nonlinear problem on coarse mesh**

  Find $(u_H, p_H) \in (X_H, M_H)$ such that, for all $(v,q) \in (X_H, M_H)$

$$(u_{H_t}, v) + a(u_H, v) + b(u_H, u_H, v) - d(v, p_H) + d(u_H, q) = (f, v), \tag{4.1}$$

$$u_H(0) = P_H \bar{u}_0 \,. \tag{4.2}$$

- **Step II: Update on fine mesh with linear Stokes problem**

  Find $(u^h, p^h) \in (X_h, M_h)$ such that, for all $(v,q) \in (X_h, M_h)$

$$(u_t^h, v) + a(u^h, v) + b(u_H, u_H, v) - d(v, p^h) + d(u^h, q) = (f, v), \tag{4.3}$$

$$u^h(0) = P_h \bar{u}_0. \tag{4.4}$$

The existence and uniqueness of a solution (u^h, p^h) of the two-level finite element Galerkin problem (4.1)-(4.4) are known (see [10]), i.e.,

Theorem 4.1. *Under the assumptions (3.2)-(3.4), let $f \in L^\infty(R^+;Y)$, $f_t \in L^\infty(R^+;Y)$ and $\bar{u}_0 \in D(A) \cap V$ be given. Then, for $h < H$, the problem (4.1)-(4.4) possesses a unique solution (u^h, p^h) defined for $t \geq 0$ with*

$$u^h \in C^\infty(0,T;X_h)\,,\ p^h \in C^\infty(0,T;M_h) \ \textit{for all}\ T > 0.$$

Moreover, we also have the following error estimates for the two-level finite element Galerkin method (see [10]):

Theorem 4.2. *Under the assumptions of Theorem 4.1, the solution (u^h, p^h) of problem (4.1)-(4.4) satisfies*

$$\|u(t) - u^h(t)\| \leq \kappa(t)(h + H^2), \forall t \geq 0\,, \tag{4.5}$$

$$|p(t) - p^h(t)| \leq \sigma(t)^{-1/2}\kappa(t)(h + H^2), \forall t > 0, \tag{4.6}$$

where $\sigma(t) = \min\{1, t\}$.

Now, we will consider some time discretization schemes related to the spatial discrete standard finite element Galerkin method and two-level finite element Galerkin method.

Let $0 = t_0 < t_1 < t_2 < \cdots$ be a division of time interval $[0, \infty), \Delta t$ is the time step size, $t_k = k\Delta t$. By combining problem (3.11)-(3.12) with the Euler implicit difference scheme, we construct the following numerical methods related to the spatial discrete standard finite element Galerkin method and two-level finite element Galerkin method:

Standard finite element Galerkin method: Find $(u_h^{k+1}, p_h^{k+1}) \in (X_h, M_h)$ such that

$$\begin{aligned}\frac{1}{\Delta t}(u_h^{k+1} - u_h^k, v) &+ a(u_h^{k+1}, v) - d(v, p_h^{k+1}) \\ &+ d(u_h^{k+1}, q) + b(u_h^{k+1}, u_h^{k+1}, v) = (f^{k+1}, v) \quad \forall (v,q) \in (X_h, M_h)\,,\end{aligned} \tag{4.7}$$

$$u_h^0 = P_h \bar{u}_0\,, \tag{4.8}$$

where

$$f^{k+1}=\frac{1}{\Delta t}\int_{t_k}^{t_{k+1}}f(t)dt,|f^{k+1}|\le f_\infty.$$

Two-level finite element Galerkin method:

- Find $(u_H^{k+1},p_H^{k+1})\in(X_H,M_H)$ such that

$$\frac{1}{\Delta t}(u_H^{k+1}-u_H^k,v)+a(u_H^{k+1},v)+b(u_H^{k+1},u_H^{k+1},v)$$
$$-\ d(v,p_H^{k+1})+d(u_H^{k+1},q)=(f^{k+1},v),\quad \forall(v,q)\in(X_H,M_H). \tag{4.9}$$

- Find $(u_{k+1}^h,p_{k+1}^h)\in(X_h,M_h)$ such that

$$\frac{1}{\Delta t}(u_{k+1}^h-u_k^h,v)+a(u_{k+1}^h,v)+b(u_H^{k+1},u_H^{k+1},v)$$
$$-\ d(v,p_{k+1}^h)+d(u_{k+1}^h,q)=(f^{k+1},v),\quad \forall(v,q)\in(X_h,M_h), \tag{4.10}$$

$$u_H^0=P_H\bar u_0,u_0^h=P_h\bar u_0. \tag{4.11}$$

Remark 4.1. Here the notations are such that a lower mark with h refers to the standard finite element Galerkin method, while an upper mark with h refers to the two-level finite element Galerkin method, where (u_h^k,p_h^k) is expected to be the approximation of $(u_h(t_k),p_h(t_k))$ and (u_k^h,p_k^h) is expected to be the approximation of $(u^h(t_k),p^h(t_k)),\forall k\ge 0$.

Next, we will give some further properties of the trilinear form b which will be very useful in sequel numerical analysis. First, we need to introduce the analogue $A_h:X_h\to X_h,A_h:M_h\to M_h$ of the Laplace operator $A=-\Delta$ given by

$$(A_hu_h,v_h)=((u_h,v_h)),\forall u_h,v_h\in X_h,$$

$$(v_h,\nabla_hq_h)=-d(v_h,q_h),\forall(v_h,q_h)\in(X_h,M_h),$$

$$d(v_h,A_hq_h)=d(A_hv_h,q_h)=-(A_hv_h,\nabla_hq_h),\forall(v_h,q_h)\in(X_h,M_h).$$

Lemma 4.3. *The trilinear form b satisfies the following estimates:*

$$|b(u_{h_1},v_{h_2},w_{h_3})|+|b(v_{h_2},u_{h_1},w_{h_3})|\le c_1\|u_{h_1}\|^{1/2}|A_{h_1}u_{h_1}|^{1/2}\|v_{h_2}\||w_{h_3}|, \tag{4.12}$$

$$|b(u_{h_1},v_{h_2},w_{h_3})|+|b(u_{h_1},w_{h_3},v_{h_2})|\le c_1|u_{h_1}|\|v_{h_2}\|\|w_{h_3}\|^{1/2}|A_{h_3}w_{h_3}|^{1/2}, \tag{4.13}$$

$$|b(u_{h_1},u_{h_1},w_{h_3})|\le c_1|u_{h_1}|^{1/2}\|u_{h_1}\||A_{h_1}u_{h_1}|^{1/2}|w_{h_3}|, \tag{4.14}$$

for any $u_{h_1}\in X_{h_1},v_{h_2}\in X_{h_2},w_{h_3}\in X_{h_3}$, where X_{h_1},X_{h_2} and X_{h_3} are three finite element spaces corresponding to mesh parameters h_1,h_2 and h_3, respectively.

Proof. To prove (4.12)-(4.13), we will need the discrete analogues of several Sobolev inequalities borrowed from Heywood-Rannacher [12], namely for any $h>0$,

$$\|\phi_h\|_{L^6}\le c\|\phi_h\|,\forall\phi_h\in X_h, \tag{4.15}$$

$$\|\phi_h\|_{L^\infty}+\|\nabla\phi_h\|_{L^3}\le c\|\phi_h\|^{1/2}|A_h\phi_h|^{1/2},\forall\phi_h\in X_h. \tag{4.16}$$

Moreover, we note that for any $u_{h_1}\in X_{h_1},v_{h_2}\in X_{h_2}$ and $w_{h_3}\in X_{h_3}$,

$$|b(u_{h_1},v_{h_2},w_{h_3})|\le c\|u_{h_1}\|_{L^\infty}\|v_{h_2}\||w_{h_3}|+c\|\nabla u_{h_1}\|_{L^3}\|v_{h_2}\||w_{h_3}|,$$

$$|b(v_{h_2},u_{h_1},w_{h_3})|\le c\|v_{h_2}\|_{L^6}\|\nabla u_{h_1}\|_{L^3}|w_{h_3}|+c\|v_{h_2}\|\|u_{h_1}\|_{L^\infty}|w_{h_3}|,$$

which and (4.15)-(4.16) imply (4.12). Next, (4.13) follows from (4.15)-(4.16) and the following estimates:

$$|b(u_{h_1},v_{h_2},w_{h_3})| \le c|u_{h_1}|\|v_{h_2}\|\|w_{h_3}\|_{L^\infty} + c|u_{h_1}|\|\nabla w_{h_3}\|_{L^3}\|v_{h_2}\|_{L^6},$$

$$|b(u_{h_1},w_{h_3},v_{h_2})| \le c|u_{h_1}|\|\nabla w_{h_3}\|_{L^3}\|v_{h_2}\|_{L^6} + c|u_{h_1}|\|v_{h_2}\|\|w_{h_3}\|_{L^\infty}.$$

To Consider (4.14), we will prove the following estimate

$$\|\phi_h\|_{L^\infty} \le c|\phi_h|^{1/2}|A_h\phi_h|^{1/2}, \forall h>0, \phi_h \in X_h. \tag{4.17}$$

In fact, for any $h>0$ and $\phi_h \in X_h$, let $w \in H_0^1(\Omega)^2 \cap H^2(\Omega)^2$ be such that $Aw = A_h\phi_h$ in Ω. Clearly $|Aw| = |A_h\phi_h|$, and by a standard argument using in [12], one obtains the error estimate

$$|w-\phi_h| + h\|w-\phi_h\| \le ch^2|A_h\phi_h|. \tag{4.18}$$

Letting $\hat{w}$ denote the piecewise linear approximation of w. Then notice, by (3.3), (4.18) and a dimensional argument (see [4, 6]), that

$$\|\phi_h\|_{L^\infty} \le \|\phi_h - \hat{w}\|_{L^\infty} + \|\hat{w}\|_{L^\infty} \le ch^{-1}|\phi_h - \hat{w}| + \|\hat{w}\|_{L^\infty}$$
$$\le ch^{-1}(|\phi_h - w| + |w-\hat{w}|) + \|w-\hat{w}\|_{L^\infty} + \|w\|_{L^\infty},$$

and further,

$$\|\phi_h\| \le ch^{-1}|\phi_h|, |A_h\phi_h| \le ch^{-1}\|\phi_h\|,$$

$$|w-\phi_h| \le ch^2|A_h\phi_h| \le ch|\phi_h|^{1/2}|A_h\phi_h|^{1/2},$$

$$|w-\hat{w}| \le ch^2|Aw| = ch^2|A_h\phi_h| \le ch|\phi_h|^{1/2}|A_h\phi_h|^{1/2},$$

$$\|w-\hat{w}\|_{L^\infty} \le ch|Aw| = ch|A_h\phi_h| \le c|\phi_h|^{1/2}|A_h\phi_h|^{1/2},$$

$$\|w\|_{L^\infty} \le c|w|^{1/2}|Aw|^{1/2} \le c(|w-\phi_h| + |\phi_h|)^{1/2} \le c|\phi_h|^{1/2}|A_h\phi_h|^{1/2}.$$

Hence, we have obtain (4.14).

Moreover, we also need the following estimates on $(u_h(t), p_h(t))$ which are borrowed from [1] and [12].

Lemma 4.4. *Under the assumptions of Theorem 4.1, the solution (u_h, p_h) of (3.11)-(3.12) satisfies:*

$$|u_h(t)| + \|u_h(t)\| + \|u_{h,t}(t)\| + |A_hu_h(t)| + |u_{h,tt}(t)| \le \kappa(t), \forall t \ge 0. \tag{4.19}$$

5. Stability Analysis

Our aim is now to derive the stability of the solution sequence $\{u_h^k\}$ corresponding to the standard finite element Galerkin method and the solution sequence $\{u_k^h\}$ corresponding to the two-level finite element Galerkin method. First, we need the following Gronwall lemma.

Lemma 5.1. *Assume that $\alpha, \gamma > 0, a_k \ge 0, b_k \ge 0, \forall k \ge 0$ such that*

$$(1+\alpha\Delta t)a_{k+1} - a_k + b_{k+1} \le \gamma\Delta t. \tag{5.1}$$

Then,

$$a_J + \sum_{k=1}^{J}(1+\alpha\Delta t)^{-(J+1-k)}b_k \le (1+\alpha\Delta t)^{-J}a_0 + \alpha^{-1}\gamma, \forall J \ge 0, \tag{5.2}$$

where we have used the following notation:

$$\sum_{k=1}^{0} b_k = 0.$$

Proof. From (5.1), we have

$$(1+\alpha\Delta t)^{k+1}a_{k+1} - (1+\alpha\Delta t)^k a_k + (1+\alpha\Delta t)^k b_{k+1} \le \gamma\Delta t(1+\alpha\Delta t)^k.$$

Summing above inequality for $k = 0, \cdots, J-1$, we obtain

$$(1+\alpha\Delta t)^J a_J + \sum_{k=1}^{J}(1+\alpha\Delta t)^{k-1}b_k$$

$$\le a_0 + \gamma\Delta t\sum_{k=0}^{J-1}(1+\alpha\Delta t)^k \le a_0 + \alpha^{-1}\gamma(1+\alpha\Delta t)^J,$$

which yields (5.2).

Now, we will consider the stability in $L^\infty(R^+;Y)$ of the solution sequences $\{u_\mu^k, p_\mu^k\}, \mu = h, H$ and $\{u_k^h, p_k^h\}$.

Theorem 5.2. *Under the assumptions of Theorem 4.1, the solution sequence* $\{u_\mu^k\}, \mu = h, H$ *generated by the standard finite element Galerkin method satisfies the following absolute stability in* $L^\infty(R^+;Y)$*:*

$$|u_\mu^J|^2 + \sum_{k=1}^{J}(1+\nu\lambda_1\Delta t)^{-(J+1-k)}\left(\frac{\nu}{2}\|u_\mu^k\|^2\Delta t + |u_\mu^k - u_\mu^{k-1}|^2\right) \le M_0^2, \quad \forall J \ge 0, \tag{5.3}$$

where

$$M_0^2 = 2|\bar{u}_0|^2 + \frac{2}{\nu^2\lambda_1^2}f_\infty^2.$$

Proof. Taking $(v,q) = (u_h^{k+1}, p_h^{k+1})$ in (4.7), $(v,q) = (u_H^{k+1}, p_H^{k+1})$ in (4.9), respectively, using (3.8) and the relation:

$$2(u-v,u) = |u|^2 - |v|^2 + |u-v|^2, \quad \forall u,v \in X_\mu, \tag{5.4}$$

we obtain

$$\frac{1}{2\Delta t}(|u_\mu^{k+1}|^2 - |u_\mu^k|^2 + |u_\mu^{k+1} - u_\mu^k|^2) + \nu\|u_\mu^{k+1}\|^2 = (f^k, u_\mu^{k+1}). \tag{5.5}$$

From (2.9), $\lambda_1|u_\mu^k|^2 \le \|u_\mu^k\|^2$. Hence we derive from (5.5) that

$$(f^{k+1}, u_\mu^{k+1}) \le |f^{k+1}||u_\mu^{k+1}| \le \frac{\nu}{4}\|u_\mu^{k+1}\|^2 + \frac{1}{\nu\lambda_1}|f^{k+1}|^2, \tag{5.6}$$

$$(1+\nu\lambda_1\Delta t)|u_\mu^{k+1}|^2 - |u_\mu^k|^2 + |u_\mu^{k+1} - u_\mu^k|^2 + \frac{\nu}{2}\|u_\mu^{k+1}\|^2\Delta t \le \frac{2}{\nu\lambda_1}|f^{k+1}|^2\Delta t. \tag{5.7}$$

We set

$$a_k = |u_\mu^k|^2, \gamma = \frac{2}{\nu\lambda_1}f_\infty^2, \alpha = \nu\lambda_1,$$

$$b_{k+1} = \frac{\nu}{2}\|u_\mu^{k+1}\|^2\Delta t + |u_\mu^{k+1} - u_\mu^k|^2, b_0 = 0.$$

Applying Lemma 5.1 to (5.7) with

$$a_0 = |u_\mu^0|^2 = |P_\mu \bar{u}_0|^2 \le |\bar{u}_0|^2,$$

we obtain (5.3).

Theorem 5.3. *Under the assumptions of Theorem 4.1, the solution sequence* $\{u_k^h\}$ *satisfies*

$$|u_J^h|^2 + \sum_{k=1}^{J} (1+\nu\lambda_1\Delta t)^{-(J+1-k)} \Big(\frac{\nu}{4}\|u_k^h\|^2\Delta t + |u_k^h - u_{k-1}^h|^2\Big) \le c_2(1+M_0^2)M_0^2. \tag{5.8}$$

Proof. A similar argument to the proof of Theorem 5.2 can yields

$$\begin{aligned}(1+\nu\lambda_1\Delta t)|u_{k+1}^h|^2 - |u_k^h|^2 + |u_{k+1}^h - u_k^h|^2 + \frac{\nu}{2}\|u_{k+1}^h\|^2\Delta t \\ + 2b(u_H^{k+1}, u_H^{k+1}, u_{k+1}^h)\Delta t \le \frac{2}{\nu\lambda_1} f_\infty^2 \Delta t.\end{aligned} \tag{5.9}$$

From (3.9), we derive

$$\begin{aligned}|b(u_H^{k+1}, u_H^{k+1}, u_{k+1}^h)| &\le c_0 |u_H^{k+1}|^{1/2}\|u_H^{k+1}\|^{3/2}|u_{k+1}^h|^{1/2}\|u_{k+1}^h\|^{1/2} \\ &\le \frac{\nu}{4}\|u_H^{k+1}\|^2|u_{k+1}^h| + \frac{c_0^2}{\nu}|u_H^{k+1}|\|u_H^{k+1}\|\|u_{k+1}^h\| \\ &\le \frac{\nu}{8}\|u_{k+1}^h\|^2 + \frac{\nu}{4}\|u_H^{k+1}\|^2|u_{k+1}^h| + \frac{2c_0^4}{\nu^3}|u_H^{k+1}|^2\|u_H^{k+1}\|^2.\end{aligned} \tag{5.10}$$

Combining (5.9) with (5.10) and using (5.3) yield

$$\begin{aligned}&(1+\nu\lambda_1\Delta t)^{k+1}|u_{k+1}^h|^2 - (1+\nu\lambda_1\Delta t)^k|u_k^h|^2 + b_{k+1}(1+\nu\lambda_1\Delta t)^k \\ \le &\frac{\nu}{2}(1+\nu\lambda_1\Delta t)^k\|u_H^{k+1}\|^2(|u_{k+1}^h| + 8c_0^4\nu^{-4}M_0^2)\Delta t + \frac{2}{\nu\lambda_1}f_\infty^2\Delta t,\end{aligned} \tag{5.11}$$

where

$$b_{k+1} = |u_{k+1}^h - u_k^h|^2 + \frac{\nu}{4}\|u_{k+1}^h\|^2\Delta t.$$

Setting $\sigma = \sup_{k\ge 0}|u_k^h|$ and summing (5.11) for $k = 0, \cdots, J-1$ and using (5.3),

$$\begin{aligned}&(1+\nu\lambda_1\Delta t)^J|u_J^h|^2 + \sum_{k=1}^{J}(1+\nu\lambda_1\Delta t)^{k-1}b_k \\ \le &|u_0^h|^2 + (\sigma + 8c_0^4\nu^{-4}M_0^2)M_0^2(1+\nu\lambda_1\Delta t)^J + \frac{2}{\nu^2\lambda_1^2}f_\infty^2(1+\nu\lambda_1\Delta t)^J,\end{aligned} \tag{5.12}$$

or

$$|u_J^h|^2 \le (1+\nu\lambda_1\Delta t)^{-J}|\bar{u}_0|^2 + (\sigma + 8c_0^4\nu^4 M_0^2)M_0^2 + \frac{2}{\nu^2\lambda_1^2}f_\infty^2. \tag{5.13}$$

Hence, we derive from (5.13) that

$$\frac{1}{2}\sigma^2 \le |\bar{u}_0|^2 + \frac{1}{2}M_0^4 + 8c_0^4\nu^{-4}M_0^4 + \frac{2}{\nu^2\lambda_1^2}f_\infty^2,$$

which yields

$$\sigma \le c(1+M_0)M_0. \tag{5.14}$$

Combining (5.12) and (5.14) yields (5.8).

Consider now the stability of the sequences $\{u_h^k\}$ and $\{u_k^h\}$ in $L^\infty(R^+;X)$. Here we need the discrete uniform Gronwall lemma:

Lemma 5.4. *Let $a_k, d_k, \gamma_k, k \geq 0$, be nonnegative real numbers such that*

$$a_{k+1} - a_k \leq d_k a_k \Delta t + \gamma_{k+1}\Delta t, \tag{5.15}$$

then for all $J \geq 0$

$$a_J \leq \exp\{\sum_{k=0}^{J-1} d_k \Delta t\}(a_0 + \sum_{k=0}^{J-1}\gamma_k \Delta t). \tag{5.16}$$

If for some fixed $r > 0$ and arbitrary $k_0 \geq 0$, a_k, d_k, γ_k, for all $k \geq k_0$, satisfy

$$\sum_{i=k+1}^{r+k} d_i \Delta t \leq \sigma_1, \sum_{i=k+1}^{r+k}\gamma_i \Delta t \leq \sigma_2, \sum_{i=k+1}^{r+k} a_i \Delta t \leq \sigma_3, \tag{5.17}$$

then

$$a_k \leq (\sigma_2 + \frac{\sigma_3}{t_r})\exp\{\sigma_1\}, \forall k \geq r+1. \tag{5.18}$$

This proof is classical (see Shen [23]).

Theorem 5.5. *Under the assumptions of Theorem 4.1, the sequences $\{u_\mu^k\}, \mu = h, H$ and $\{u_k^h\}$ satisfy the following stability in $L^\infty(R^+ : X)$:*

$$\|u_\mu^J\|^2 \leq M_1^2, \frac{\nu}{2}\sum_{k=1}^{J}(1+\nu\lambda_1\Delta t)^{-(J+1-k)}|A_\mu u_\mu^k|^2\Delta t \leq (1+M_0^8)(1+M_0^2M_1^2)M_1^2, \forall J \geq 0, \tag{5.19}$$

$$\|u_J^h\|^2 + \frac{\nu}{2}\sum_{k=1}^{J}(1+\nu\lambda_1\Delta t)^{-(J+1-k)}|A_h u_k^h|^2\Delta t \leq (1+M_0^8)(1+M_0^2M_1^2)M_1^2, \forall J \geq 0, \tag{5.20}$$

where r is a fixed integer,

$$M_1^2 = ce^{c(1+M_0^8)M_0^4 t_r}(1+t_r+t_r^{-1})(\|\bar{u}_0\|^2 + M_0^2)(1+M_0^4).$$

Proof. Taking $(v,q) = (A_h u_h^{k+1}, A_h p_h^{k+1})$ in (4.7) and $(v,q) = (A_H u_H^{k+1}, A_H p_H^{k+1})$ in (4.9) and using the relation: for $\mu = h, H$

$$(a_\mu - b_\mu, 2A_\mu a_\mu) = ((a_\mu - b_\mu, a_\mu)) = \|a_\mu\|^2 - \|b_\mu\|^2 + \|a_\mu - b_\mu\|^2, \forall a_\mu, b_\mu \in X_\mu, \tag{5.21}$$

we derive

$$\begin{aligned}&\frac{1}{2\Delta t}(\|u_\mu^{k+1}\|^2 - \|u_\mu^k\|^2 + \|u_\mu^{k+1} - u_\mu^k\|^2) + \nu|A_\mu u_\mu^{k+1}|^2 + b(u_\mu^{k+1}, u_\mu^{k+1}, A_\mu u_\mu^{k+1})\\ &= (f^{k+1}, A_\mu u_\mu^{k+1}).\end{aligned} \tag{5.22}$$

Moreover, we can obtain from (3.8), (4.7) and (4.9) that

$$\begin{aligned}&\frac{1}{\Delta t}|u_\mu^{k+1} - u_\mu^k|^2 + \frac{\nu}{2}(\|u_\mu^{k+1}\|^2 - \|u_\mu^k\|^2 + \|u_\mu^{k+1} - u\mu^k\|^2)\\ &+ b(u_\mu^{k+1} - u_\mu^k, u_\mu^k, u_\mu^{k+1} - u_\mu^k) + b(u_\mu^k, u_\mu^k, u_\mu^{k+1} - u_\mu^k)\\ &= (f^{k+1}, u_\mu^{k+1} - u_\mu^k).\end{aligned} \tag{5.23}$$

From (3.8) and (5.3), we have

$$|b(u_\mu^{k+1}-u_\mu^k,u_\mu^k,u_\mu^{k+1}-u_\mu^k)| \le c_0|u_\mu^k|^{1/2}\|u_\mu^k\|^{1/2}|u_\mu^{k+1}-u_\mu^k|^{1/2}\|u_\mu^{k+1}-u_\mu^k\|^{3/2}$$
$$\le \frac{\nu}{8}\|u_\mu^{k+1}-u_\mu^k\|^2+cM_0^4\|u_\mu^k\|^2,$$
$$|b(u_\mu^k,u_\mu^k,u_\mu^{k+1}-u_\mu^k)| \le \frac{\nu}{8}\|u_\mu^{k+1}-u_\mu^k\|^2+c(1+M_0^2)\|u_\mu^k\|^2,$$
$$|(f^{k+1},u_\mu^{k+1}-u_\mu^k)| \le \frac{\nu}{8}\|u_\mu^{k+1}-u_\mu^k\|^2+\frac{2}{\nu\lambda_1}|f^{k+1}|^2.$$

Combining (5.20) with above estimates and using Theorem 5.2 yield

$$\frac{1}{\Delta t}|u_\mu^{k+1}-u_\mu^k|^2+\nu\|u_\mu^{k+1}\|^2+\nu\|u_\mu^{k+1}-u_\mu^k\|^2 \le c(1+M_0^4)\|u_\mu^k\|^2+\frac{4}{\nu\lambda_1}f_\infty^2. \tag{5.24}$$

On the other hand , we derive from (4.14) and (5.24) that

$$|b(u_\mu^{k+1},u_\mu^{k+1},A_\mu u_\mu^{k+1})| \le c_1|u_\mu^{k+1}|^{1/2}\|u_\mu^{k+1}\||A_\mu u^{k+1}|^{3/2}$$
$$\le \frac{\nu}{4}|A_\mu u_\mu^{k+1}|^2+(\frac{2}{\nu})^3c_1^4|u_\mu^{k+1}|^2\|u_\mu^{k+1}\|^4$$
$$\le \frac{\nu}{8}|A_\mu u_\mu^{k+1}|^2+c(1+M_0^8)M_0^2\|u_\mu^k\|^4+cM_0^6, \tag{5.25}$$
$$|(f^{k+1},A_\mu u_\mu^{k+1})| \le \frac{\nu}{8}|A_\mu u_\mu^{k+1}|^2+2\nu^{-1}f_\infty^2\,. \tag{5.26}$$

Combining (5.22) with (5.25)-(5.26) yields

$$\|u_\mu^{k+1}\|^2-\|u_\mu^k\|^2+\frac{3}{2}\nu\Delta t|A_\mu u_\mu^{k+1}|^2 \le c(1+M_0^8)M_0^2\Delta t\|u_\mu^k\|^4+c(1+M_0^4)M_0^2\Delta t. \tag{5.27}$$

In particularly, (5.27) yields

$$a_{k+1}-a_k \le d_k a_k\Delta t+\gamma_k\Delta t, \tag{5.28}$$

where

$$a_k=\|u^k\|^2, d_k=c(1+M_0^8)M_0^2\|u_\mu^k\|^2, \gamma_k=c(1+M_0^4)M_0^2.$$

Recalling (5.7) and (5.3), we have easily checked that a_k, d_k, γ_k satisfy the assumptions of Lemma 5.3, namely for some fixed $r>0$ and any $k_0\ge 0$,

$$\sum_{i=k+1}^{r+k} a_i\Delta t \le \sigma_3 = c(1+t_r)M_0^2,$$
$$\sum_{i=k+1}^{r+k} \gamma_i\Delta t \le \sigma_2 = c(1+M_0^4)M_0^2 t_r,$$
$$\sum_{i=k+1}^{r+k} d_i\Delta t \le \sigma_1 = c(1+M_0^8)M_0^4 t_r, \forall k\ge k_0.$$

Applying Lemma 5.3 to (5.28), we obtain

$$\|u^k\|^2 \le \exp\{\sum_{i=0}^{k-1} d_i\Delta t\}(\|\bar u_0\|^2+\sum_{i=0}^{k-1}\gamma_i\Delta t) \le e^{\sigma_1}(\|\bar u_0\|^2+\sigma_2), \forall 0\le k\le r,$$

$$\|u_\mu^k\|^2 \le e^{\sigma_1}(\sigma_2 + \frac{\sigma_3}{t_r}), \forall k \ge r+1.$$

Hence, we obtain

$$\|u_\mu^k\|^2 \le M_1^2, \forall k \ge 0. \tag{5.29}$$

Using again (2.9), (5.29) and (5.27), we obtain

$$(1+\nu\lambda_1\Delta t)\|u_\mu^{k+1}\|^2 - \|u_\mu^k\|^2 + \frac{\nu}{2}|A_\mu u_\mu^{k+1}|^2\Delta t$$

$$\le (1+M_0^8)M_0^2M_1^4 + c(1+M_0^4)M_0^2\Delta t, \tag{5.30}$$

which and (5.29) yields (5.19).

Moreover, we take $(v,q) = (A_h u_{k+1}^h, A_h p_{k+1}^h)$ in (4.10) and use (5.21). Then,

$$\frac{1}{2\Delta t}(\|u_{k+1}^h\|^2 - \|u_k^h\|^2 + \|u_{k+1}^h - u_k^h\|^2) + \nu|A_h u_{k+1}^h|^2$$

$$+ b(u_H^{k+1}, u_H^{k+1}, A_h u_{k+1}^h) = (f^{k+1}, A_h u_{k+1}^h). \tag{5.31}$$

From (4.14), (5.3) and the Young inequality, we have

$$(f^{k+1}, A_h u_{k+1}^h) \le \frac{\nu}{8}|A_h u_{k+1}^h|^2 + 2\nu^{-1}|f^{k+1}|^2, \tag{5.32}$$

$$\begin{aligned}|b(u_H^{k+1}, u_H^{k+1}, A_h u_{k+1}^h)| &\le c_1|u_H^{k+1}|^{1/2}|A_H u_H^{k+1}|^{1/2}\|u_H^{k+1}\||A_h u_{k+1}^h| \\ &\le \frac{\nu}{8}|A_h u_{k+1}^h|^2 + 2\nu^{-1}c_1^2|u_H^{k+1}|\|u_H^{k+1}\|^2|A_H u_H^{k+1}| \\ &\le \frac{\nu}{8}(|A_h u_{k+1}^h|^2 + 2|A_H u_H^{k+1}|^2) + cM_0^2\|u_H^{k+1}\|^4. \end{aligned} \tag{5.33}$$

Combining (5.31) with (5.32)-(5.33) and using (5.29), we obtain

$$\|u_{k+1}^h\|^2 - \|u_k^h\|^2 + \frac{3}{2}\nu|A_h u_{k+1}^h|^2\Delta t \le \frac{\nu}{2}|A_H u_H^{k+1}|^2\Delta t + cM_0^2M_1^4\Delta t, \tag{5.34}$$

which and (2.9) yield

$$(1+\nu\lambda_1\Delta t)^{k+1}\|u_{k+1}^h\|^2 - (1+\nu\lambda_1\Delta t)^k\|u_k^h\|^2 + \frac{\nu}{2}(1+\nu\lambda_1\Delta t)^k|A_h u_{k+1}^h|^2\Delta t$$

$$\le (\frac{\nu}{2}|A_H u_H^{k+1}|^2 + c(1+M_1^4)M_0^2)(1+\nu\lambda_1\Delta t)^k\Delta t. \tag{5.35}$$

Summing (5.35) for $k = 0, 1, \cdots, J-1$, we obtain

$$(1+\nu\lambda_1\Delta t)^J\|u_J^h|^2 + \frac{\nu}{2}\sum_{k=1}^{J}(1+\nu\lambda_1\Delta t)^{k-1}|A_h u_k^h|^2\Delta t$$

$$\le \|\bar{u}_0\|^2 + \frac{\nu}{2}\sum_{k=1}^{J}(1+\nu\lambda_1\Delta t)^{k-1}|A_H u_H^k|^2\Delta t$$

$$+ c(1+M_1^4)M_0^2(1+\nu\lambda_1\Delta t)^J, \forall J \ge 0, \tag{5.36}$$

which and (5.19) yields

$$\|u_J^h\|^2 + \frac{\nu}{2}\sum_{k=1}^{J}(1+\nu\lambda_1\Delta t)^{-(J+1-k)}|A_h u_k^h|^2\Delta t$$

$$\leq (1+M_0^8)(1+M_0^2M_1^2) + M_0^2(1+M_0^2M_1^2)M_1^2, \forall J \geq 0, \tag{5.37}$$

namely, (5.20) holds.

Remark. Similarly, we can prove that the semi-discrete numerical solution $u_h(t)$ and $u_H(t)$ are stability in $L^\infty(R^+;X)$:

$$\|u_h(t)\|^2 + \|u_H(t)\|^2 \leq M_1^2, \forall t \geq 0. \tag{5.38}$$

6. Convergence Analysis

In this section, we shall analyze the rates of convergence on the numerical solutions corresponding to the standard finite element Galerkin method and the two-level finite element Galerkin method. For the standard finite element Galerkin method, we introduce the numerical solution $(u_\mu^\Delta, p_\mu^\Delta), \mu = h, H$ defined by

$$u_\mu^\Delta(t) = \alpha(t)u_\mu^k + \beta(t)u_\mu^{k+1}\,,\ p_\mu^\Delta(t) = p_\mu^{k+1}, \quad \forall t \in [t_k, t_{k+1}]\,,$$

and for the two-level finite element Galerkin method we define the numerical solution (u_Δ^h, p_Δ^h) as follows :

$$u_\Delta^h(t) = \alpha(t)u_k^h + \beta(t)u_{k+1}^h\,,\ p_\Delta^h(t) = p_{k+1}^h, \quad \forall t \in [t_k, t_{k+1}]\,,$$

where

$$\alpha(t) = \frac{t_{k+1}-t}{\Delta t}, \beta(t) = \frac{t-t_k}{\Delta t}\,.$$

In this section, we will frequently use a discrete version of the Gronwall inequality in a slightly more general form (see [13]):

Lemma 6.1. *Let $\Delta t, \gamma$ and a_k, b_k, d_k, γ_k, for integers $k \geq 0$, be nonnegative numbers such that*

$$a_J + \sum_{k=0}^{J} b_k\Delta t \leq \sum_{k=0}^{J} d_k a_k \Delta t + \sum_{k=0}^{J}\gamma_k\Delta t + \gamma, \forall J \geq 0. \tag{6.1}$$

Suppose that

$$d_k\Delta t < 1, \text{ and set } \sigma_k = (1-d_k\Delta t)^{-1}, \forall k \geq 0. \tag{6.2}$$

Then,

$$a_J + \sum_{k=0}^{J} b_k\Delta t \leq \exp\Big(\sum_{k=0}^{J}\sigma_k d_k\Delta t\Big)\Big\{\sum_{k=0}^{J}\gamma_k\Delta t + \gamma\Big\}, \forall J \geq 0. \tag{6.3}$$

Now, we first consider some convergence results on the solution sequences $\{u_h^k\}$ and $\{u_k^h\}$ in $L^\infty(R^+;X)$.

Theorem 6.2. *Assume that the assumptions of Theorem 4.1 are valid and Δt satisfies the following convergence condition:*

$$\Delta t \leq \frac{\nu^3}{64}c_1^{-4}M_1^{-4}. \tag{6.4}$$

Then, for $\mu = h, H$

$$\|u_\mu(t_J) - u_\mu^J\|^2 + \frac{\nu}{2}\sum_{k=0}^{J}|A_\mu(u_\mu(t_k) - u_\mu^h)|^2\Delta t \leq \kappa(t_J)(\Delta t)^2, \quad \forall J \geq 0\,. \tag{6.5}$$

Proof. For simplicity, we only prove (6.5) for $\mu = h$. By virtue of the Taylor expansion with the integral remainder (see Girault-Raviart [11]):

$$\frac{1}{\Delta t}(u(t_{k+1}) - u(t_k), v) = (\frac{du}{dt}(t_{k+1}), v) + \frac{1}{\Delta t}\int_{t_k}^{t_{k+1}} (t - t_{k+1})(\frac{d^2u}{dt^2}(t), v)dt, \quad \forall v \in X_h \tag{6.6}$$

we derive from (3.11)-(3.12) that

$$\frac{1}{\Delta t}(u_h(t_{k+1}) - u_h(t_k), v) + a(u_h(t_{k+1}), v) + b(u_h(t_{k+1}), u_h(t_{k+1}), v) - d(v, p_h(t_{k+1})) + d(u_h(t_{k+1}), q) = (f(t_{k+1}), v) + \frac{1}{\Delta t}\int_{t_k}^{t_{k+1}} (t - t_{k+1})(u_{h,tt}(t), v)dt, \quad \forall (v, q) \in (X_h, M_h)\,. \tag{6.7}$$

By setting $E_\mu^k = u_\mu(t_k) - u_\mu^k$ and $\delta_\mu^k = p_\mu(t_k) - p_\mu^k, \mu = h, H$, we derive from (6.7) and (4.3) that

$$\frac{1}{\Delta t}(E_h^{k+1} - E_h^k, v) + a(E_h^{k+1}, v) + b(E_h^{k+1}, u_h(t_{k+1}), v) + b(u_h^{k+1}, E_h^{k+1}, v) - d(v, \delta_h^{k+1}) + d(E_h^{k+1}, q) = (E_*^{k+1}, v), \forall (v, q) \in (X_h, M_h)\,, \tag{6.8}$$

where

$$(E_*^{k+1}, v) = \frac{1}{\Delta t}\int_{t_k}^{t_{k+1}} (f(t_{k+1}) - f(t), v)dt + \frac{1}{\Delta t}\int_{t_k}^{t_{k+1}} (t - t_{k+1})(u_{h,tt}(t), v)dt$$

We now aim to estimate $|E_*^{k+1}|$, where

$$|E_*^{k+1}| = \sup_{v \in X_h} \frac{(E_*^{k+1}, v)}{|v|},$$

and

$$\frac{1}{\Delta t}|\int_{t_k}^{t_{k+1}} (f(t_{k+1}) - f(t), v)dt| \leq \frac{1}{\Delta t}|\int_{t_k}^{t_{k+1}}\int_t^{t_{k+1}} f_\tau d\tau dt||v| \leq (\Delta t)^{1/2}(\int_{t_k}^{t_{k+1}} |f_t|^2 dt)^{1/2}|v|, \tag{6.9}$$

$$\frac{1}{\Delta t}|\int_{t_k}^{t_{k+1}} (t - t_{k+1})(u_{h,tt}(t), v)dt| \leq (\Delta t)^{1/2}(\int_{t_k}^{t_{k+1}} |u_{h,tt}|^2 dt)^{1/2}|v|. \tag{6.10}$$

Estimating (6.9)-(6.10), we obtain

$$|E_*^{k+1}|^2 \leq \Delta t \int_{t_k}^{t_{k+1}} (|f_t|^2 + |u_{h,tt}|^2)dt\,. \tag{6.11}$$

Next, by taking $(v, q) = (A_h E_h^{k+1}, A_h \delta_h^{k+1})$ in (6.8) and the relation (5.21), we obtain

$$\frac{1}{2\Delta t}(\|E_h^{k+1}\|^2 - \|E_h^k\|^2 + \|E_h^{k+1} - E_h^k\|^2) + \nu|A_h E_h^{k+1}|^2 + b(E_h^{k+1}, u_h(t_{k+1}), A_h E_h^{k+1}) + b(u_h^{k+1}, E_h^{k+1}, A_h E_h^{k+1}) = (E_*^{k+1}, A_h E_h^{k+1})\,. \tag{6.12}$$

Thanks to (4.13), we have

$$|b(E_h^{k+1},u_h(t_{k+1}),A_hE_h^{k+1})| \le c_0\|E_h^{k+1}\|^{1/2}|A_hE_h^{k+1}\|^{3/2}\|u_h(t_{k+1})\|$$
$$\le \frac{\nu}{4}|A_hE_h^{k+1}|^2+(\frac{2}{\nu})^3c_1^4\|u_h(t_{k+1})\|^4\|E_h^{k+1}\|^2\,, \tag{6.13}$$

$$|b(u_h^{k+1},E_h^{k+1},A_hE_h^{k+1})| \le \frac{\nu}{4}|A_hE_h^{k+1}|^2+\frac{8}{\nu^3}c_1^4\|u_h^{k+1}\|^4\|E_h^{k+1}\|^2, \tag{6.14}$$

$$|(E_*^{k+1},A_hE_h^{k+1})| \le \frac{\nu}{8}|A_hE_h^{k+1}|^2+\frac{2}{\nu}|E_*^{k+1}|^2\,. \tag{6.15}$$

By combining (6.12) with (6.13)-(6.15) and using (6.11) , (5.19) and (5.38), we obtain

$$a_{k+1}-a_k+b_{k+1}\Delta t \le d_{k+1}a_{k+1}\Delta t+\gamma_{k+1}\Delta t\,, \tag{6.16}$$

where

$$a_k=\|E_h^k\|^2, b_k=|A_hE_h^k|^2, d_{k+1}=\frac{16}{\nu^3}c_1^4M_1^2(\|u_h(t_{k+1})\|^2+\|u_h^{k+1}\|^2),$$
$$\gamma_{k+1}=\frac{4}{\nu}\Delta t\int_{t_k}^{t_{k+1}}(|f_t|^2+|u_{h,tt}|^2)dt, \gamma_0=0.$$

Summing (6.16) for k from 0 to $J-1$ and using the fact of $E_h^0=0$, we obtain

$$a_J+\frac{\nu}{2}\sum_{k=0}^J b_k\Delta t \le \sum_{k=0}^J d_ka_k\Delta t+\sum_{k=0}^J\gamma_k\Delta t. \tag{6.17}$$

From (5.19), (5.38) and the convergence condition (6.4), we have

$$d_k\Delta t \le \frac{32}{\nu^3}c_1^4M_1^4\Delta t \le \frac{1}{2}, \sigma_k=(1-d_k\Delta t)\le 2. \tag{6.18}$$

Hence, we can apply Lemma 6.1 to (6.17), then

$$a_J+\sum_{k=0}^J b_k\Delta t \le \frac{4}{\nu}\exp(2\sum_{k=0}^J d_k\Delta t)\int_0^{t_J}(|f_t|^2+|u_{h,tt}|^2)dt(\Delta t)^2\,,$$

namely (6.5) holds.

Theorem 6.3. *Assume that the assumptions of Theorem 6.2 are valid. Then*

$$\|u^h(t_J)-u_J^h\|^2+\nu\sum_{k=0}^J|A_h(u^h(t_k)-u_k^h)|^2\Delta t \le \kappa(t_J)(\Delta t)^2 \quad \forall J\ge 0. \tag{6.19}$$

Proof. An exact similar argument to the proof of (6.5) can yields

$$\frac{1}{\Delta t_{k+1}}(E_{k+1}^h-E_k^h,v)+a(E_{k+1}^h,v)-d(v,\delta_{k+1}^h)+d(E_{k+1}^h,q)+b(E_H^{k+1},u_H(t_k),v)$$
$$+b(u_H^k,E_H^{k+1},v)=(E_{k+1}^*,v), \forall(v,q)\in(X_h,M_h)\,, \tag{6.20}$$

where $E_k^h=u^h(t_k)-u_k^h, \delta_k^h=p^h(t_k)-p_k^h$ and

$$(E_{k+1}^*,v)=\frac{1}{\Delta t}\int_{t_k}^{t_{k+1}}(f(t_{k+1})-f(t),v)dt+\frac{1}{\Delta t}\int_{t_k}^{t_{k+1}}(t-t_{k+1})(u_{tt}^h(t),v)dt\,.$$

The estimate of $|E^*_{k+1}|$ is similar to ones of $|E^{k+1}_*|$, namely

$$|E^*_{k+1}|^2 \le (\Delta t)^2 \int_{t_k}^{t_{k+1}} (|f_t|^2 + |u^h_{tt}|^2) dt\,. \tag{6.21}$$

Next, by taking $(v,q) = (A_h E^h_{k+1}, A_h \delta^h_{k+1})$ in (6.20) and using (5.21) and (6.21), we find

$$\begin{aligned}
&\frac{1}{2\Delta t}(\|E^h_{k+1}\|^2 - \|E^h_k\|^2 + \|E^{k+1} - E^k\|^2) + \nu|A_h E^h_{k+1}|^2 \\
&+ b(E^{k+1}_H, u_H(t_{k+1}), A_h E^h_{k+1}) + b(u^{k+1}_H, E^{k+1}_H, A_h E^h_{k+1}) \\
=\ & (E^*_{k+1}, A_h E^h_{k+1}) \le \frac{\nu}{4}|E^{k+1}|^2 + \frac{1}{\nu}|E^*_{k+1}|^2\,.
\end{aligned} \tag{6.22}$$

Moreover, by (4.12)-(4.14), (5.19) and (5.38), we have

$$\begin{aligned}
&|b(E^{k+1}_H, u_H(t_{k+1}), A_h E^h_{k+1}) + b(u^{k+1}_H, E^{k+1}_H, A_h E^h_{k+1})| \\
\le\ & c_1(\|u_H(t_{k+1})\| + \|u^{k+1}_H\|)\|E^{k+1}_H\|^{1/2}|A_H E^{k+1}_H|^{1/2}|A_h E^h_{k+1}| \\
\le\ & \frac{\nu}{4}(|A_h E^h_{k+1}|^2 + |A_H E^{k+1}_H|^2) + \frac{4}{\nu^3}c_1^4 M_1^2(\|u_H(t_{k+1})\|^2 + \|u^{k+1}_H\|^2)\|E^{k+1}_H\|^2.
\end{aligned} \tag{6.23}$$

Combining (6.22) with (6.23) yields

$$\begin{aligned}
&\|E^{k+1}\|^2 - \|E^k\|^2 + \nu|A_h E^h_{k+1}|^2 \Delta t \\
\le\ & \frac{\nu}{2}|A_H E^{k+1}_H|^2 \Delta t + d_{k+1}\|E^h_{k+1}\|^2 \Delta t + \gamma_{k+1}\Delta t\,,
\end{aligned} \tag{6.24}$$

where

$$d_k = \frac{4}{\nu^3}c_1^4 M_1^2(\|u_H(t_k)\|^2 + \|u^k_H\|^2),$$

$$\gamma_{k+1} = \frac{\nu}{2}|A_H E^{k+1}_H|^2 + \frac{2}{\nu}\Delta t \int_{t_k}^{t_{k+1}} (|f_t|^2 + |u^h_{tt}|^2|^2) dt\,, \gamma_0 = 0.$$

Summing (6.24) for $k = 0, \cdots, J-1$ and using the fact of $E^h_0 = 0$, we obtain

$$a_J + \sum_{k=0}^{J} b_k \Delta t \le \sum_{k=0}^{J} a_k d_k \Delta t + \sum_{k=0}^{J} \gamma_k \Delta t, \tag{6.25}$$

where $a_k = \|E^h_k\|$. From (5.19), (5.38) and the convergence condition (6.4), we check that d_k and σ_k satisfy (6.18). Hence, we can apply Lemma 6.1 to (6.25) and using (6.5), we obtain (6.19).

Secondly, we will provide the convergence results of the pressure sequences $\{p^k_h\}$ and $\{p^h_k\}$.

Theorem 6.4. *Under the assumptions of Theorem 6.2, the sequences $\{p_k\}$ and $\{p_k\}$ satisfy respectively*

$$\sum_{k=1}^{J} |p_h(t_k) - p^k_h|^2 \Delta t \le \kappa(t_J)(\Delta t)^2, \forall J \ge 1\,, \tag{6.26}$$

$$\sum_{k=1}^{J} |p^h(t_k) - p^h_k|^2 \Delta t \le \kappa(t_J)(\Delta t)^2, \quad \forall J \ge 1. \tag{6.27}$$

Proof. Thanks to (4.12)-(4.14), we derive from (6.8) and (6.20) that

$$\frac{|E_h^{k+1}-E_h^k|}{\Delta t} \leq \nu|A_h E_h^{k+1}| + c_1(\|u_h(t_{k+1})\| + \|u_h^{k+1}\|)|A_h E_h^{k+1}|^{1/2}\|E_h^{k+1}\|^{1/2}|E_*^{k+1}|, \tag{6.28}$$

$$\frac{|E_{k+1}^h-E_k^h|}{\Delta t} \leq \nu|A_h E_{k+1}^h| + c_1(\|u_H(t_{k+1})\| + \|u_H^{k+1}\|)|A_H E_H^{k+1}|^{1/2}\|E_H^{k+1}\|^{1/2} + |E_{k+1}^*|. \tag{6.29}$$

Using again (2.9), (3.4), (3.10), (5.19) and (5.38), we imply

$$|p_h(t_{k+1}) - p_h^{k+1}|^2\Delta t \leq c|A_h E_h^{k+1}|^2\Delta t + c\|E_h^{k+1}\|^2\Delta t + c\|E_*^{k+1}\|^2\Delta t, \tag{6.30}$$

$$|p^h(t_k) - p_k^h|^2\Delta t \leq c(|A_h E_{k+1}^h|^2 + |A_H E_H^{k+1}|^2)\Delta t + c\|E_H^{k+1}\|^2\Delta t + c|E_{k+1}^*|^2\Delta t. \tag{6.31}$$

By summing (6.30) and (6.31) for $k = 0, 1, ..., J-1$ and using (6.5) and (6.19, we obtain (6.26) and (6.27), respectively.

By combining Theorem 3.1, Lemma 4.4 with Theorem 6.2, one finds

$$\begin{aligned}&\|u(t) - u_h^\Delta(t)\| \leq \|u(t) - u_h(t)\| \\ &+ \alpha(t)\|u_h(t) - u_h(t_k)\| + \beta(t)\|u_h(t) - u_h(t_{k+1})\| \\ &+ \alpha(t)\|u_h(t_k) - u_h^k\| + \beta(t)\|u_h(t_{k+1}) - u_h^{k+1}\| \\ &\leq \kappa(t_{k+1})h + \sup_{t\in[t_k,t_{k+1}]}\|u_{h,t}(t)\|\Delta t \leq \kappa(t_{k+1})(h+\Delta t), \forall t \in [t_k, t_{k+1}],\end{aligned} \tag{6.32}$$

which yields the following convergence result .

Theorem 6.5. *Under the assumptions of Theorem 6.1 , the numerical solution (u_Δ, p_Δ) corresponding to the finite element Galerkin method is of the following convergence rate:*

$$\|u(t) - u_h^\Delta(t)\| \leq \kappa(t)\{h + \Delta t\}, \forall t \geq 0, \tag{6.33}$$

$$(\int_0^t |p(t) - p_h^\Delta(t)|^2 dt)^{1/2} \leq \kappa(t)\{h + \Delta t\}, \forall t \geq 0. \tag{6.34}$$

Proof. Using (6.32), we prove easily that (6.33) holds. For convenience, we let $t = t_J$. Recalling again Theorem 3.1 and Lemma 4.4, we can derive from (3.4) and (3.10) that

$$(\int_0^t |p(t) - p_h(t)|^2 dt)^{1/2} \leq \kappa(t)h, \int_0^t |p_{h,s}(s)|^2 ds \leq \kappa(t), \forall t \geq 0. \tag{6.35}$$

Hence, we have

$$\begin{aligned}&\int_{t_0}^t |p(t) - p_h^\Delta(t)|^2 dt \leq 2\int_{t_0}^t |p(t) - p_h(t)|^2 dt \\ &+ 4\sum_{k=1}^J \int_{t_{k-1}}^{t_k} |p_h(t) - p_h(t_k)|^2 dt + 4\sum_{k=1}^J |p_h(t_k) - p_h^k|^2\Delta t,\end{aligned}$$

$$\sum_{k=1}^{J}\int_{t_{k-1}}^{t_k}|p_h(t)-p_h(t_k)|^2dt \le \sum_{k=1}^{J}\int_{t_{k-1}}^{t_k}|\int_{t}^{t_k}p_{h,s}(s)ds|^2dt$$

$$\le \sum_{k=1}^{J}\int_{t_{k-1}}^{t_k}|p_{h,t}|^2dt(\Delta t)^2 = \int_0^t|p_{h,s}|^2ds(\Delta t)^2 \le \kappa(t)(\Delta t)^2\,.$$

Combining (6.15) and above estimates yields (6.34). #

Similarly , we also obtain the convergence rate of the numerical solution (u^h_Δ, p^h_Δ) corresponding to the two-level finite element Galerkin method.

Theorem 6.6. *Under the assumptions of Theorem 6.2, (u^h_Δ, p^h_Δ) is of the following convergence rate:*

$$\|u(t)-u^h_\Delta(t)\| \le \kappa(t)\{h+H^2+\Delta t\}, \quad \forall t\ge 0\,, \tag{6.36}$$

$$(\int_0^t|p(t)-p^h_\Delta(t)|^2dt)^{1/2} \le \kappa(t)\{h+H^2+\Delta t\}, \quad \forall t\ge 0\,. \tag{6.37}$$

This proof can be omitted.

7. Numerical Test

In this section we present the results of numerical experiments with the one-level finite element Galerkin method (1-level method) and the two-level finite element Galerkin method (2-level method) described in section 4. Specifically, we consider a simple example which is a nonphysical example with $p=0$. The domain Ω is the unit square $\{0<x<1, 0<y<1\}$. The finite element discretization uses a quadrilaterals mesh with the Q_1 element. In the Q_1 element, piecewise bilinear functions on quadrilaterals are used to approximate the velocity u. We set $\nu=0.1$, $T=1$ and the exact solution be given by

$$u_1(x,y,t) = 10x^2(x-1)^2y(y-1)(2y-1)cos(t), \tag{7.1}$$

$$u_2(x,y,t) = -10x(x-1)(2x-1)y^2(y-1)^2cos(t). \tag{7.2}$$

If we compute the solution $u^\Delta_h(t)$, denoted as $u_{one}(t)$, by using 1-level method and $u^h_\Delta(t)$, denoted as $u_{two}(t)$ by using 2-level method, respectively, the theoretical rate of convergence for the velocity u should be:

$$\|u(t)-u_{one}(t)\| \le \kappa(t)(h+\Delta t), \quad \|u(t)-u_{two}(t)\| \le \kappa(t)(h+H^2+\Delta t). \tag{7.3}$$

From the error estimate (7.3), we should choose a time step $\Delta t = O(h)$, $H=O(h^{1/2})$ for 1-level method and 2-level method. Since we have an exact solution for this example, we can compute the L^2- and H^1-errors for 1-level method and 2-level method and verify that they agree with the predicted results.

For 1-level method, we provide the L^2- and H^1-global relative errors

$$|||u-u_{one}|||_0 = (\frac{1}{T}\int_0^T\frac{|u(t)-u_{one}(t)|^2}{|u(t)|^2}dt)^{1/2}, \tag{7.4}$$

$$|||u-u_{one}|||_1 = (\frac{1}{T}\int_0^T\frac{\|u(t)-u_{one}(t)\|^2}{\|u(t)\|^2}dt)^{1/2}, \tag{7.5}$$

and the CPU time spent for these numerical computations in TABLE 1 with some h and Δt values; and for 2-level method, we provide the L^2- and H^1-global relative errors

$$|||u-u_{two}|||_0 = (\frac{1}{T}\int_0^T\frac{|u(t)-u_{two}(t)|^2}{|u(t)|^2}dt)^{1/2}, \tag{7.6}$$

$$|||u-u_{two}|||_1=(\frac{1}{T}\int_0^T\frac{\|u(t)-u_{two}(t)\|^2}{\|u(t)\|}dt)^{1/2}, \tag{7.7}$$

and the CPU time spent for these numerical computations in TABLE 2 with some H, h and Δt values.

Table 1. Global relative errors and CPU times on $[0,T]$ for 1-level method

h	Δt	$\|\|\|u-u_{one}\|\|\|_0$	$\|\|\|u-u_{one}\|\|\|_1$	CPU time (seconds)
$\frac{1}{9}$	$\frac{1}{9}$	0.002837499359	0.01765409989	1
$\frac{1}{16}$	$\frac{1}{16}$	0.001630965379	0.01674963354	14
$\frac{1}{25}$	$\frac{1}{25}$	0.001226508960	0.01808277793	260
$\frac{1}{36}$	$\frac{1}{36}$	0.0008157798295	0.01759956684	2820
$\frac{1}{49}$	$\frac{1}{49}$	0.0006580201631	0.01815739471	21136

Table 2. Global relative errors and CPU times on $[0,T]$ for 2-level method

H	h	Δt	$\|\|\|u-u_{two}\|\|\|_0$	$\|\|\|u-u_{two}\|\|\|_1$	CPU time(seconds)
$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{9}$	0.002837499141	0.01765409948	1
$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{16}$	0.001761768664	0.01797411308	2
$\frac{1}{5}$	$\frac{1}{25}$	$\frac{1}{25}$	0.001182498178	0.01808276758	12
$\frac{1}{6}$	$\frac{1}{36}$	$\frac{1}{36}$	0.0008432068195	0.01758127961	123
$\frac{1}{7}$	$\frac{1}{49}$	$\frac{1}{49}$	0.0006296068716	0.01815738914	313

Two Tables show that the L^2- and H^1-global relative errors of two methods are almost same, but 2-level method spends less time than 1-level method.

We want to compare the L^2- and H^1-error of 1-level method with 2-level method at each time $t \in [0,T]$. To do this, for fixed values of $H=\frac{1}{7}$, $h=\frac{1}{49}$ and $\Delta t=\frac{1}{49}$ we plot the L^2-error curves as FIG.1 and the H^1-error curves as FIG.2 for 1-level method and 2-level method on $[0,T]$.

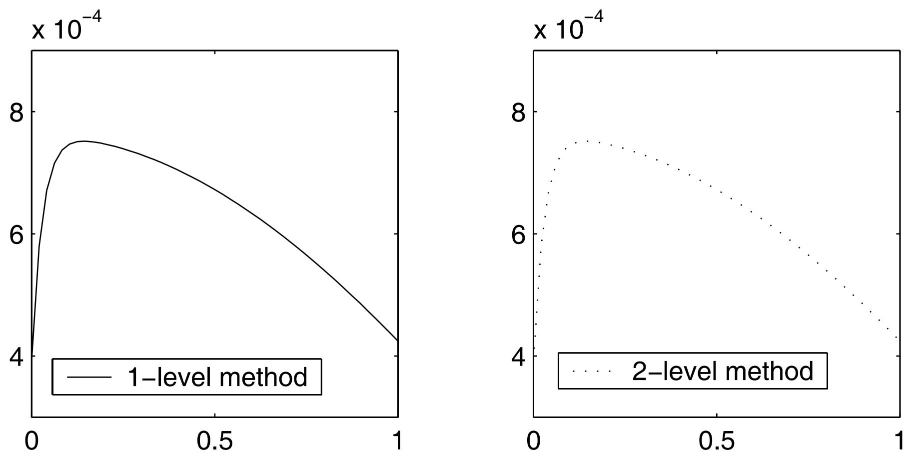

Fig.1. the L^2-error curves on $[0,T]$ using 1-level method and 2-level method.

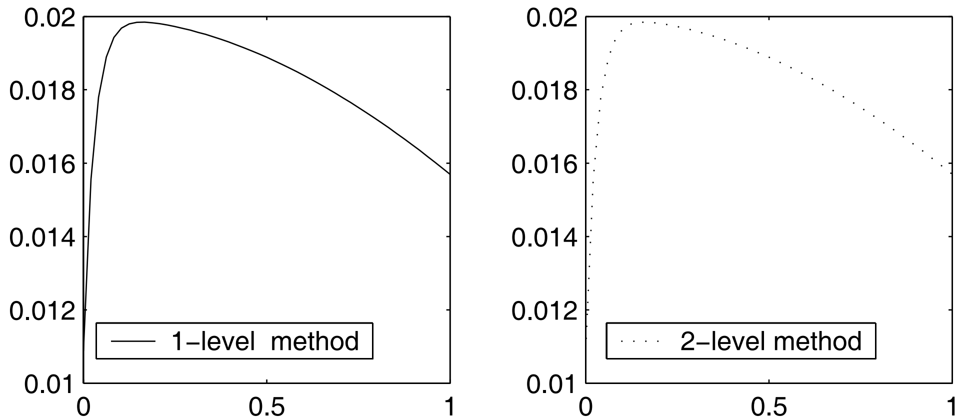

Fig.2. the H^1-error curves on $[0,T]$ using 1-level method and 2-level method.

The curves suggest that there is no significant difference between the two methods in terms of the L^2- and H^1- errors. But, the CPU time spent by 2-level method is shorter than the 1-level finite element Galerkin method, i.e., 2-level method spends 313 seconds and 1-level finite element Galerkin method spends 21136 seconds.

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