NUMERICAL ERGODICITY AND UNIFORM ESTIMATE OF MONOTONE SPDES DRIVEN BY MULTIPLICATIVE NOISE*

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Abstract

We analyze the long-time behavior of numerical schemes for a class of monotone stochastic partial differential equations (SPDEs) driven by multiplicative noise. By deriving several time-independent a priori estimates for the numerical solutions, combined with the ergodic theory of Markov processes, we establish the exponential ergodicity of these schemes with a unique invariant measure, respectively. Applying these results to the stochastic Allen-Cahn equation indicates that these schemes always have at least one invariant measure, respectively, and converge strongly to the exact solution with sharp time-independent rates. We also show that these numerical invariant measures are exponentially ergodic and thus give an affirmative answer to a question proposed in [J. Cui *et al.*, Stochastic Process. Appl., 134 (2021)], provided that the interface thickness is not too small.

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1. Introduction

Recently, a lot of researchers investigated numerical analysis of the following SPDEs under the homogeneous Dirichlet boundary condition in a finite time horizon:

$$
\begin{aligned}
&\mathrm{d}X(t,\xi) = \big(\Delta X(t,\xi) + f\big(X(t,\xi)\big)\big)\mathrm{d}t + g\big(X(t,\xi)\big)\mathrm{d}W(t,\xi),\\
&X(t,\xi) = 0, \qquad (t,\xi) \in \mathbb{R}_+ \times \partial\mathscr{O},\\
&X(0,\xi) = X_0(\xi), \quad \xi \in \mathscr{O},
\end{aligned}
\tag{1.1}
$$

where the physical domain $\mathscr{O} \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ is a bounded open set with smooth boundary. Here, f is assumed to be of monotone type with polynomial growth, g satisfies the usual Lipschitz condition in an infinite-dimensional setting, and W is an infinite-dimensional Wiener process, see Section 2.1. Eq. (1.1) includes the following semiclassical stochastic Allen-Cahn equation, arising from phase transition in materials science by stochastic perturbation, as a special case

$$
\mathrm{d}X = \Delta X \mathrm{d}t + \alpha^{-2}(X - X^3)\mathrm{d}t + \mathrm{d}W, \quad X(0) = X_0,
\tag{1.2}
$$

where $\alpha > 0$ is the interface thickness, see, e.g. $[2, 3, 8, 15, 19, 30\text{--}33]$ and references therein.

The numerical behavior in the infinite time horizon, especially the numerical ergodicity of SPDEs, including Eq. (1.1) (and Eq. (1.2)), is a natural and intriguing question. As a significant asymptotic behavior, the ergodicity characterizes the case of temporal average coinciding with spatial average, which has vital applications in quantum mechanics, fluid dynamics, financial mathematics, and many other fields [17, 20]. The spatial average, i.e. the mean of a given function for the stationary law, called the invariant measure of diffusion, is also known as the ergodic limit, which is desirable to compute in many practical applications. Then, one has to investigate a stochastic system over long time intervals, which is one of the main difficulties from the computational perspective. Generally, the explicit expression of the invariant measure for a nonlinear infinite-dimensional stochastic system is rarely available. Therefore, it is usually impossible to precisely compute the ergodic limit for a nonlinear SPDE driven by multiplicative noise; exceptional examples are gradient Langevin systems (driven by additive noise), see, e.g. [5, 23]. For this reason, it motivated and fascinated a lot of investigations in the recent decade for constructing numerical algorithms that can inherit the ergodicity of the original system and approximate the ergodic limit efficiently.

In finite-dimensional case, much progress has been made in the design and analysis of numerical approximations of the desired ergodic limits, see, e.g. [26,34,36] and references therein for numerical ergodicity of stochastic ordinary differential equations (SODEs). On the contrary, the construction and analysis of numerical ergodic limits for SPDEs are still in their early stages. Bréhier *et al.* [6,9,10] firstly studied Galerkin-based linear implicit Euler scheme and high order integrator to approximate the invariant measures of a parabolic SPDE driven by additive white noise; see also [13] for the ergodic limit of a spectral Galerkin modified implicit Euler scheme of the damping stochastic Schrödinger equation driven by additive trace-class noise. Recently, exponential Euler-type schemes of full discretization and tamed semi-discretization were used in [14] and [7], respectively, to approximate the invariant measure of a stochastic Allen-Cahn type equation driven by additive colored noise. Cui *et al.* [16] investigated the spectral Galerkin drift-implicit Euler (DIE) scheme to approximate the invariant measure of the same type of equation and proposed a question of whether the invariant measure of the temporal DIE scheme is unique. Almost all of the above literature focuses on the numerical ergodicity of SPDEs driven by additive noise, the numerical ergodicity in the multiplicative noise case is more subtle and challenging.

Apart from the approximation of the invariant measure, the strong approximation, i.e. in the moments' sense, is also an important issue. There exists a general theory of strong error analysis of numerical approximations for Lipschitz SPDEs in a finite time horizon, see, e.g. [1, 11, 21] and references therein. Recently, Liu and Qiao [32] give a theory of strong error analysis of numerical approximations for monotone SPDEs in a finite time horizon. Whether there is a theory of strong error analysis in the infinite time horizon is a natural question.

The above two questions on long-time behaviors of numerical approximations for Eq. (1.1) motivate the present study. On the one hand, we aim to establish the exponential ergodicity of the numerical schemes studied in [19,32], [33] for Eq. (1.1). It should be pointed out that there is no further restriction on the time step size for the DIE scheme (DIE) in addition to the usual requirement of the implicit scheme (see Theorem 3.1), see [26] for an upper bound requirement on the time step size for numerical ergodicity of monotone SODEs. Meanwhile, applying one of our main results, Theorem 3.1, we conclude that the DIE scheme (DIE) is unique ergodic (and exponentially mixing) and thus give an affirmative answer to the question proposed in [16] (see Remark 4.1), provided that the monotone coefficient of the drift function is not too large.

On the other hand, we develop a time-independent strong error analysis theory for these fully discrete schemes, see Theorem 3.3, which generalizes the main results of [32] to the infinite time horizon.

The paper is organized as follows. In Section 2, we give several necessary uniform estimates and exponential ergodicity of Eq. (1.1). These uniform estimates are utilized in Section 3 to establish the exponential ergodicity of the proposed semi-discrete and fully discrete schemes. In Section 3, we also derive time-independent strong error estimates for those above fully discrete schemes. Finally, the main results in Section 3 are applied to the stochastic Allen-Cahn equation (1.2) in Section 4.

2. Uniform a Priori Estimates and Exponential Ergodicity of Monotone SPDEs

In this section, we first present the required preliminaries and main assumptions that will be used throughout the paper. Then, we derive several necessary uniform a priori estimates of Eq. (1.1).

2.1. Main assumptions

Denote $a \vee b := \max(a, b)$ and $a \wedge b := \min(a, b)$ for $a, b \in \mathbb{R}$. For $r \in [2, \infty]$ and $\theta = -1$ or 1, we use the notations $(L^r = L^r(\mathscr{O}), \|\cdot\|_{L^r})$ and $(\dot{H}^\theta = \dot{H}^\theta(\mathscr{O}), \|\cdot\|_\theta)$ to denote the usual Lebesgue and Sobolev interpolation spaces, respectively. In particular, we denote $H := L^2, V := \dot{H}^1$, and $V^* := \dot{H}^{-1}$. The space H is equipped with Borel σ-algebra $\mathcal{B}(H)$ and with the inner product and norm given by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$, respectively. The norms in $\dot{H}^1$ and $\dot{H}^{-1} = (\dot{H}^1)^*$ and the duality product between them are denoted by $\|\cdot\|_1, \|\cdot\|_{-1}$, and $_1\langle\cdot, \cdot\rangle_{-1}$, respectively. We use $\mathcal{C}_b(H)$ and $\mathrm{Lip}_b(H)$ to denote the class of bounded, continuous, and Lipschitz continuous, respectively, functions on H. For $\phi \in \mathrm{Lip}_b(H)$, define

$$|\phi|_{\mathrm{Lip}} := \frac{\sup_{x \neq y} |\phi(x) - \phi(y)|}{\|x - y\|}.$$

Let $\mathbf{Q}$ be a self-adjoint and positive definite linear operator on H. Denote $(H_0 := Q^{1/2}H, \|\cdot\|_{H_0} := \|Q^{-1/2}\cdot\|)$ and denote by $(\mathcal{L}_2^0 := HS(H_0; H), \|\cdot\|_{\mathcal{L}_2^0})$ and $(\mathcal{L}_2^1 := HS(H_0; \dot{H}^1), \|\cdot\|_{\mathcal{L}_2^1})$ the space of Hilbert-Schmidt operators from H_0 to H and $\dot{H}^1$, respectively. Let W be an H-valued $\mathbf{Q}$-Wiener process on a filtered probability space $(\Omega, \mathscr{F}, \mathbb{F} := (\mathscr{F}_t)_{t \geq 0}, \mathbb{P})$, which has the Karhunen-Loève expansion

$$W(t) = \sum_{k \in \mathbb{N}} \sqrt{q_k} g_k \beta_k(t), \quad t \geq 0.$$

Here $\{g_k\}_{k=1}^\infty$ forming an orthonormal basis of H are the eigenvectors of $\mathbf{Q}$ subject to the eigenvalues $\{q_k\}_{k=1}^\infty$, and $\{\beta_k\}_{k=1}^\infty$ are mutually independent one-dimensional Brownian motions in $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$. We mainly focus on trace-class noise, i.e. $\mathbf{Q}$ is a trace-class operator, or equivalently, $\mathrm{Trace}(\mathbf{Q}) := \sum_{k \in \mathbb{N}} q_k < \infty$.

Our main conditions on the coefficients of Eq. (1.1) are the following two assumptions.

Assumption 2.1. $f : \mathbb{R} \to \mathbb{R}$ is differentiable and there exist scalars $L_i \in \mathbb{R}, i = 1, \ldots, 5$, and $q \geq 1$ such that

$$\big(f(\xi) - f(\eta)\big)(\xi - \eta) \leq L_1(\xi - \eta)^2, \quad \xi, \eta \in \mathbb{R}, \tag{2.1}$$

$$f(\xi)\xi \leq L_2|\xi|^2 + L_3, \qquad\qquad \xi \in \mathbb{R}, \qquad\qquad (2.2)$$

$$|f'(\xi)| \leq L_4|\xi|^{q-1} + L_5, \qquad\qquad \xi \in \mathbb{R}. \qquad\qquad (2.3)$$

Example 2.1. A concrete example such that all the conditions in Assumption 2.1 holds is an odd polynomial (with order q and) with a negative leading coefficient.

By the condition (2.1) and the differentiability of f, it is clear that

$$f'(\xi) \leq L_1, \quad \xi \in \mathbb{R}. \qquad\qquad (2.4)$$

We also note that one can always assume that $L_1 \leq L_2$, as one can take $L_2 = L_1 + \varepsilon$ in (2.2) from (2.1) and Young inequality, here and in the rest of the paper, ε denotes an arbitrarily small positive constant which would be different in each appearance.

Define by $F : L^{q+1} \to L^{(q+1)'}$ the Nemytskii operator associated with f

$$F(x)(\xi) := f\big(x(\xi)\big), \quad x \in \dot{H}^1, \quad \xi \in \mathscr{O}, \qquad\qquad (2.5)$$

where $(q+1)'$ denotes the conjugation of $q+1$, i.e. $1/(q+1)' + 1/(q+1) = 1$. Then it follows from the monotonicity condition (2.1) and the coercivity condition (2.2) that the operator F defined in (2.5) has a continuous extension from L^{q+1} to $L^{(q+1)'}$ and satisfies

$$_{L^{q+1}}\langle x - y, F(x) - F(y)\rangle_{L^{(q+1)'}} \leq L_1\|x - y\|^2, \quad x, y \in L^{q+1}, \qquad\qquad (2.6)$$

$$_{L^{q+1}}\langle x, F(x)\rangle_{L^{(q+1)'}} \leq L_2\|x\|^2 + L_3, \qquad\qquad x \in L^{q+1}, \qquad\qquad (2.7)$$

where $_{L^{(q+1)'}}\langle \cdot, \cdot \rangle_{L^{q+1}}$ denotes the duality product between $L^{(q+1)'}$ and L^{q+1}.

Throughout, we assume that $q \geq 1$ when $d = 1, 2$ and $q \in [1, 3]$ when $d = 3$, so that the following frequently used Sobolev embeddings hold:

$$\dot{H}^1 \subset L^{2q} \subset L^{q+1} \subset H \subset L^{(q+1)'} \subset L^{(2q)'} \subset \dot{H}^{-1}. \qquad\qquad (2.8)$$

In this case, (2.6) and (2.7) can be expressed as

$$_1\langle x - y, F(x) - F(y)\rangle_{-1} \leq L_1\|x - y\|^2, \quad x, y \in \dot{H}^1, \qquad\qquad (2.9)$$

$$_1\langle x, F(x)\rangle_{-1} \leq L_2\|x\|^2 + L_3, \qquad\qquad x \in \dot{H}^1. \qquad\qquad (2.10)$$

The inequalities (2.9) and (2.10), in combination with the Poincaré inequality that

$$\|\nabla x\|^2 \geq \lambda_1\|x\|^2, \quad x \in \dot{H}^1, \qquad\qquad (2.11)$$

where λ_1 denotes the first eigenvalue of the Dirichlet Laplacian operator, denoted by A with domain $\mathrm{Dom}(A) = H^2 \cap \dot{H}^1$, implies that

$$_1\langle x - y, A(x - y) + F(x) - F(y)\rangle_{-1} \leq -(\lambda_1 - L_1)\|x - y\|^2, \quad x, y \in \dot{H}^1, \qquad (2.12)$$

and that

$$_1\langle x, Ax + F(x)\rangle_{-1} \leq -(\lambda_1 - L_2)\|x\|^2 + L_3, \quad x \in \dot{H}^1. \qquad\qquad (2.13)$$

Similarly, noting that $L_1 \leq L_2$, we have

$$\langle x, Ax + F(x)\rangle_1 \leq -(\lambda_1 - L_2)\|x\|_1^2 + L_3, \quad x \in \dot{H}^2. \qquad\qquad (2.14)$$

Denote by $G : H \to \mathcal{L}_2^0$ the Nemytskii operator associated with $g : \mathbb{R} \to \mathbb{R}$

$$G(x)g_k(\xi) := g\big(x(\xi)\big)g_k(\xi), \quad x \in H, \quad k \in \mathbb{N}, \quad \xi \in \mathscr{O}. \qquad\qquad (2.15)$$

Our primary condition on the diffusion operator G defined in (2.15) is the following usual Lipschitz continuity and linear growth conditions.

Assumption 2.2. There exist nonnegative constants L_6, L_7, and L_8 such that

$$\|G(x) - G(y)\|^2_{\mathcal{L}_2^0} \leq L_6\|x - y\|^2, \quad x, y \in H, \tag{2.16}$$

$$\|G(x)\|^2_{\mathcal{L}_2^0} \leq L_7\|x\|^2 + L_8, \qquad x \in H. \tag{2.17}$$

Moreover, $G(\dot{H}^1) \subset \mathcal{L}_2^1$ and there exist nonnegative constants L_9 and L_{10} such that

$$\|G(z)\|^2_{\mathcal{L}_2^1} \leq L_9\|z\|^2_1 + L_{10}, \quad z \in \dot{H}^1. \tag{2.18}$$

Remark 2.1. In the additive noise case $G \equiv \mathrm{Id}$, the identity operator in H, the conditions (2.16)-(2.18) reduce to the assumption $\|(-A)^{1/2}\mathbf{Q}^{1/2}\|_{HS(H;H)} < \infty$ which was imposed in [16, 35], see [32, Remark 2.2] for an example of multiplicative noise case.

With these preliminaries, (1.1) is equivalent to the following infinite-dimensional stochastic evolution equation:

$$\begin{aligned} \mathrm{d}X_t &= \big(AX_t + F(X_t)\big)\mathrm{d}t + G(X_t)\mathrm{d}W, \quad t \geq 0, \\ X(0) &= X_0, \end{aligned} \tag{2.19}$$

where the random initial datum X_0 is assumed to vanish on the boundary $\partial\mathscr{O}$ of the physical domain throughout the present paper.

2.2. Uniform moments estimates of the exact solution

The existence of a unique variational solution of Eq. (2.19) in H and $\dot{H}^1$ was shown in [12, 22] and [25], respectively, under similar conditions as in Assumptions 2.1-2.2. This solution is also the mild solution of (2.19) in [32], where the author also derived that the solution is indeed continuous a.s. and satisfies certain moments' estimates in a finite time horizon.

Taking advantage of the dissipativity conditions (2.13) and (2.14), we can show the following uniform stability estimate both in H and $\dot{H}^1$ of Eq. (2.19). Throughout, we assume without loss of generality that the initial datum X_0 of Eq. (2.19) is a deterministic function, X_t^x denotes the solution of Eq. (2.19) starting from $X_0 = x$. We will frequently use C to denote generic time-independent constants independent of various discrete parameters that would differ in each appearance.

Proposition 2.1. Let $p \geq 2, X_0 \in H$, Assumption 2.1 and conditions (2.16)-(2.17) hold. Assume that $L_2 + (p-1)L_7/2 < \lambda_1$. Then there exist positive constants γ_1 and C_{γ_1} such that

$$\mathbb{E}\|X_t\|^p \leq e^{-\gamma_1 t}\|X_0\|^p + C_{\gamma_1}, \quad t \geq 0. \tag{2.20}$$

Moreover, if $X_0 \in \dot{H}^1, L_2 + (p-1)L_9/2 < \lambda_1$, and condition (2.18) hold, then there exist positive constants γ_2 and C_{γ_2} such that

$$\mathbb{E}\|X_t\|_1^p \leq e^{-\gamma_2 t}\|X_0\|_1^p + C_{\gamma_2}, \quad t \geq 0. \tag{2.21}$$

Proof. Applying Itô formula (see, e.g. [27, Theorem 4.2.5]) to the functional $\|X\|^p$ leads to

$$\begin{aligned} \mathrm{d}\|X\|^p = &\left[\frac{1}{2}p(p-2)\|X\|^{p-4}\sum_{k\in\mathbb{N}_+}\left\langle X, G(X)\mathbf{Q}^{\frac{1}{2}}g_k\right\rangle^2 \right.\\ &\left. + \frac{p}{2}\|X\|^{p-2}\left(2\,_1\langle X, AX + F(X)\rangle_{-1} + \|G(X)\|^2_{\mathcal{L}_2^0}\right)\right]\mathrm{d}t \\ &+ p\|X\|^{p-2}\langle X, G(X)\mathrm{d}W\rangle. \end{aligned} \tag{2.22}$$

We omit the temporal variable when an integral appears here and after to lighten the notations. For $\gamma_1 \in \mathbb{R}$, it follows from the product rule that

$$
\begin{aligned}
e^{\gamma_1 t} \|X_t\|^p - \|X_0\|^p = & \int_0^t e^{\gamma_1 r} \Bigg[\frac{1}{2} p(p-2) \|X\|^{p-4} \sum_{k \in \mathbb{N}_+} \left\langle X, G(X) \mathbf{Q}^{\frac{1}{2}} g_k \right\rangle^2 \\
& + \frac{p}{2} \|X\|^{p-2} \left(2 \,_1\langle X, AX + F(X) \rangle_{-1} + \|G(X)\|_{\mathcal{L}_2^0}^2 \right) + \gamma_1 \|X\|^p \Bigg] \mathrm{d}r \\
& + p \int_0^t e^{\gamma_1 r} \|X\|^{p-2} \langle X, G(X) \mathrm{d}W \rangle.
\end{aligned}
\tag{2.23}
$$

By the condition (2.17) and Young inequality, we have

$$
\|X\|^{p-4} \sum_{k \in \mathbb{N}_+} \left\langle X, G(X) \mathbf{Q}^{\frac{1}{2}} g_k \right\rangle^2 \le (L_7 + \varepsilon) \|X\|^p + C_\varepsilon,
$$

$$
\|X\|^{p-2} \|G(X)\|_{\mathcal{L}_2^0}^2 \le (L_7 + \varepsilon) \|X\|^p + C_\varepsilon
$$

for $X \in H$. Combining the above two estimates and (2.13), taking expectation on Eq. (2.23), and noting that $\int_0^t e^{\gamma_1 r} \|X\|^{p-2} \langle X, G(X) \mathrm{d}W \rangle$ is a (real-valued) martingale, we obtain

$$
e^{\gamma_1 t} \mathbb{E}\|X_t\|^p \le \|X_0\|^p - \gamma_1^* \int_0^t e^{\gamma_1 r} \mathbb{E}\|X\|^p \mathrm{d}r + C_\varepsilon \int_0^t e^{\gamma_1 r} \mathrm{d}r
\tag{2.24}
$$

with

$$
\gamma_1^* := p \left(\lambda_1 - L_2 - \frac{p-1}{2}(L_7 + \varepsilon) \right) - \gamma_1.
$$

As $L_2 + (p-1)L_7/2 < \lambda_1$, one can take $\varepsilon > 0$ and $\gamma_1 > 0$ to be sufficiently small such that $\gamma_1^* > 0$. Then it follows that

$$
\mathbb{E}\|X_t\|^p \le e^{-\gamma_1 t} \|X_0\|^p + C_\varepsilon \int_0^t e^{-\gamma_1 r} \mathrm{d}r,
$$

from which we obtain (2.20).

Now let $X_0 \in \dot{H}^1$. Applying Itô formula to the functional $e^{\gamma_1 t} \|X_t\|_1^p$ leads to

$$
\begin{aligned}
e^{\gamma_2 t} \|X_t\|_1^p - \|X_0\|_1^p = & \int_0^t e^{\gamma_2 r} \Bigg[\frac{1}{2} p(p-2) \|X\|_1^{p-4} \sum_{k \in \mathbb{N}_+} \left\langle X, G(X) \mathbf{Q}^{\frac{1}{2}} g_k \right\rangle_1^2 \\
& + \frac{p}{2} \|X\|_1^{p-2} \left(2 \langle X, AX + F(X) \rangle_1 + \|G(X)\|_{\mathcal{L}_2^1}^2 \right) + \gamma_2 \|X\|_1^p \Bigg] \mathrm{d}r \\
& + p \int_0^t e^{\gamma_2 r} \|X\|_1^{p-2} \langle X, G(X) \mathrm{d}W \rangle_1.
\end{aligned}
$$

By the conditions (2.16) and (2.18), we have

$$
\|X\|_1^{p-4} \sum_{k \in \mathbb{N}_+} \left\langle X, G(X) \mathbf{Q}^{\frac{1}{2}} g_k \right\rangle_1^2 \le (L_9 + \varepsilon) \|X\|_1^p + C_\varepsilon, \quad u \in \dot{H}^1,
$$

$$
\|X\|_1^{p-2} \|G(X)\|_{\mathcal{L}_2^1}^2 \le (L_9 + \varepsilon) \|X\|_1^p + C_\varepsilon, \qquad u \in \dot{H}^1.
$$

Using techniques similar to those used to establish (2.24), together with the above two estimates, (2.14), and the fact that $\int_0^t e^{\gamma_2 r}\|X\|_1^{p-2}\langle X, G(X)\mathrm{d}W\rangle_1$ is a martingale, implies that

$$e^{\gamma_2 t}\mathbb{E}\|X_t\|_1^p \le \|X_0\|_1^p - \gamma_2^* \int_0^t e^{\gamma_2 r}\mathbb{E}\|X\|_1^p \mathrm{d}r + C_\varepsilon \int_0^t e^{\gamma_2 r}\mathrm{d}r$$

with

$$\gamma_2^* := p\left(\lambda_1 - L_2 - \frac{p-1}{2}(L_9 + \varepsilon)\right) - \gamma_2 > 0$$

for some sufficiently small and positive ε and γ_2, as $L_2 + (p-1)L_9/2 < \lambda_1$. Then it follows that

$$\mathbb{E}\|X_t\|_1^p \le e^{-\gamma_2 t}\|X_0\|^p + C_\varepsilon \int_0^t e^{-\gamma_2(t-r)}\mathrm{d}r,$$

from which we conclude (2.21). $\square$

As a by-product of the uniform stability estimate (2.21), we have the following uniform temporal Hölder continuity estimate, see [4, Proposition 5.4] for a similar result in finite-dimensional case. It will be used in Section 4 to derive the time-independent strong error estimates of the proposed implicit numerical schemes.

Corollary 2.1. *Let $p \ge 2, X_0 \in \dot{H}^1$, and Assumptions 2.1-2.2 hold. Assume that*

$$L_1 + \frac{1}{2}(p-1)L_9 < \lambda_1.$$

Then, there exists a constant C such that

$$\sup_{t>s\ge 0} \frac{\mathbb{E}\|X_t - X_s\|^p}{(t-s)^{p/2}} \le C\left(1 + \|X_0\|_1^{pq}\right). \tag{2.25}$$

Proof. For any $t > s \ge 0$, it was shown in [32, Theorem 3.2] that

$$\mathbb{E}\|X_t - X_s\|^p \le C(t-s)^{\frac{p}{2}}\left(1 + \|X_0\|_1^p + \sup_{t\ge 0}\mathbb{E}\|F(X_t)\|^p + \sup_{t\ge 0}\mathbb{E}\|G(X_t)\|_{\mathcal{L}_2^0}^p\right).$$

This estimate, together with the conditions (2.3), (2.17), the embeddings (2.8), and the stability estimate (2.21), implies the desired estimate (2.25). $\square$

2.3. Exponential ergodicity of the exact solution

It is clear that the solution of Eq. (2.19) is a Markov process, see, e.g. [27, Proposition 4.3.5], so we define the family of Markov semigroup $\{P_t\}_{t\ge 0}$ associated with Eq. (2.19) as

$$(P_t\phi)(x) := \mathbb{E}\phi\left(X_t^x\right), \quad x \in H, \quad t \ge 0.$$

A probability measure μ on H is called invariant for $\{P_t\}_{t\in\mathbb{I}}$ with either $\mathbb{I} = \mathbb{R}_+$ or $\mathbb{I} = \mathbb{N}$, if

$$\int_H P_t\phi(x)\mu(\mathrm{d}x) = \mu(\phi) =: \int_H \phi(x)\mu(\mathrm{d}x), \quad \phi \in \mathcal{C}_b(H), \quad t \in \mathbb{I}.$$

An invariant (probability) measure μ is called ergodic for $\{P_t\}_{t\ge 0}$ (or Eq. (2.19)) or $\{P_m\}_{m\in\mathbb{N}}$ if for all $\phi \in L^2(H;\mu)$, it holds, respectively, that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T P_t\phi\mathrm{d}t \left(\text{or } \lim_{m\to\infty} \frac{1}{m}\sum_{k=0}^m P_k\phi\right) = \mu(\phi) \quad \text{in } L^2(H;\mu).$$

The following unique ergodicity of Eq. (2.19), with an exponential convergence to the equilibrium, was proved using a remote initial condition method in [27, Theorem 4.3.9].

Theorem 2.1. *Let Assumption* 2.1 *and conditions* (2.16)-(2.17) *hold. Assume that*

$$\left(L_1 + \frac{1}{2}L_6\right) \vee \left(L_2 + \frac{1}{2}L_7\right) < \lambda_1.$$

Then a unique invariant measure π in H exists for Eq. (2.19). *Moreover, for any $t \geq 0, x \in H$, and $\phi \in \mathrm{Lip}_b(H)$, there exist a positive constant γ_3 such that*

$$|P_t\phi(x) - \pi(\phi)| \leq \exp\left(-\frac{1}{2}\gamma_3 t\right) |\phi|_{\mathrm{Lip}} \int_H \|x - y\| \pi(\mathrm{d}y). \tag{2.26}$$

Remark 2.2. A Markov semigroup $\{P_t\}_{t\in\mathbb{I}}$ with $\mathbb{I} = \mathbb{R}_+$ or $\mathbb{I} = \mathbb{N}$ satisfying an exponential convergence to the equilibrium, like (2.26), is called exponentially mixing. It is clear that once $\{P_t\}$ is exponentially mixing for the invariant measure μ, then the following strong law of large numbers (implying the ergodicity) holds:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T P_t\phi(x)\mathrm{d}t \left(\text{or } \lim_{m\to\infty} \frac{1}{m} \sum_{k=0}^m P_k\phi(x)\right) = \mu(\phi), \quad \forall\, x \in H, \quad \mu\text{-a.s.}$$

3. Exponential Ergodicity and Uniform Estimates of Galerkin-based Euler Scheme

This section will first establish the exponential ergodicity of the drift-implicit Euler semi-discrete and Galerkin-based fully discrete schemes. Then, we will derive uniform, strong error estimates of this fully discrete scheme. At the end of this section, some remarks on the generalization to the Milstein-type scheme and its Galerkin discretization are given.

3.1. Exponential ergodicity of DIE scheme

Let $\tau \in (0,1)$. For an $m \in \mathbb{N}$, define $t_m := m\tau$. We discrete Eq. (2.19) in time with a DIE scheme, and the resulting temporal semi-discrete problem is to find an $\dot{H}^1$-valued Markov chain $\{X_m : m \in \mathbb{N}_+\}$ such that

$$X_{m+1} = X_m + \tau A X_{m+1} + \tau F(X_{m+1}) + G(X_m)\delta_m W, \tag{DIE}$$

starting from $X_0 \in H$, where $\delta_m W = W(t_{m+1}) - W(t_m), m \in \mathbb{N}$. This DIE scheme had been widely studied, see, e.g. [16,18,32,35], and references therein. Throughout this part we assume that $\tau \in (0,1)$ with $(L_1 - \lambda_1/(\lambda_1 + 1))\tau < 1$ so that (DIE) is well-posedness.

We begin with the following uniform moments' stability of the above DIE scheme (DIE). It should be pointed out that there is no further restriction on the time step size $\tau \in (0,1)$, see [26, Theorems 4.1, 4.2] for an upper bound requirement on τ for the numerical ergodicity of monotone SODEs.

Lemma 3.1. *Let $X_0 \in H$, Assumption* 2.1 *and conditions* (2.16)-(2.17) *hold, and $\{X_m : m \in \mathbb{N}\}$ be the solution of the DIE scheme* (DIE) *starting from X_0. Assume that $L_2 + L_7/2 < \lambda_1$. Then there exist positive constants γ_5, C_{γ_5}, and C such that for any $\tau \in (0,1)$,*

$$\mathbb{E}\|X_m\|^2 \leq e^{-\gamma_5 t_m}\|X_0\|^2 + C_{\gamma_5}, \quad m \in \mathbb{N}, \tag{3.1}$$

$$\sup_{m\in\mathbb{N}_+} \frac{1}{m} \sum_{k=0}^m \mathbb{E}\|X_k\|_1^2 \leq C\left(1 + \|X_0\|^2\right). \tag{3.2}$$

Proof. For simplicity, set $F_k = F(X_k)$ and $G_k = G(X_k)$ for $k \in \mathbb{N}$. Testing (DIE) using $\langle \cdot, \cdot \rangle$-inner product with X_{k+1} and using the elementary equality

$$2\langle x - y, x \rangle = \|x\|^2 - \|y\|^2 + \|x - y\|^2, \quad x, y \in H, \tag{3.3}$$

we have

$$
\begin{aligned}
&\|X_{k+1}\|^2 - \|X_k\|^2 + \|X_{k+1} - X_k\|^2 \\
&= 2 \, {}_1\langle X_{k+1}, \Delta X_{k+1} + F_{k+1} \rangle_{-1} \tau \\
&\quad + 2\langle X_{k+1} - X_k, G_k \delta_k W \rangle + 2\langle X_k, G_k \delta_k W \rangle.
\end{aligned}
\tag{3.4}
$$

By the estimate (2.13) and Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
\|X_{k+1}\|^2 - \|X_k\|^2 &\leq -2(\lambda_1 - L_2)\|X_{k+1}\|^2 \tau + \|G_k \delta_k W\|^2 \\
&\quad + 2\langle X_k, G_k \delta_k W \rangle + 2L_3 \tau.
\end{aligned}
$$

It follows that

$$\big(1 + 2(\lambda_1 - L_2)\tau\big)\|X_{k+1}\|^2 \leq \|X_k\|^2 + \|G_k \delta_k W\|^2 + 2\langle X_k, G_k \delta_k W \rangle + 2L_3 \tau.$$

Taking expectations on both sides, noting the fact that both G_k and X_k are independent of $\delta_k W$, using Itô isometry and (2.17), we get

$$\big(1 + 2(\lambda_1 - L_2)\tau\big)\mathbb{E}\|X_{k+1}\|^2 \leq (1 + L_7\tau)\mathbb{E}\|X_k\|^2 + (2L_3 + L_8)\tau,$$

from which we obtain

$$
\begin{aligned}
\mathbb{E}\|X_k\|^2 &\leq \left(\frac{1 + L_7\tau}{1 + 2(\lambda_1 - L_2)\tau} \right)^k \|X_0\|^2 \\
&\quad + \frac{(2L_3 + L_8)\tau}{1 + 2(\lambda_1 - L_2)\tau} \sum_{j=0}^{k-1} \left(\frac{1 + L_7\tau}{1 + 2(\lambda_1 - L_2)\tau} \right)^j, \quad k \in \mathbb{N}_+.
\end{aligned}
$$

Note that

$$a^k < e^{-(1-a)k}, \quad \forall a \in (0, 1), \tag{3.5}$$

and that $(1 + L_7\tau)/(1 + 2(\lambda_1 - L_2)\tau) < 1$ for any $\tau \in (0, 1)$, which is ensured by the condition $L_2 + L_7/2 < \lambda_1$, we conclude (3.1) with

$$\gamma_5 := \frac{2(\lambda_1 - L_2) - L_7}{1 + 2(\lambda_1 - L_2)\tau}, \quad C_{\gamma_5} := \frac{2L_3 + L_8}{2(\lambda_1 - L_2) - L_7}.$$

Subtracting $2\varepsilon \, {}_1\langle X_{k+1}, \Delta X_{k+1} \rangle_{-1} \tau$, with small $\varepsilon > 0$, from the first term on the right-hand side in Eq. (3.4), and using the arguments in the above, we obtain

$$
\begin{aligned}
&\big(1 + 2(\lambda_1 - L_2 - \varepsilon)\tau\big)\mathbb{E}\|X_{k+1}\|^2 + 2\varepsilon\mathbb{E}\|\nabla X_{k+1}\|^2 \tau \\
&\leq (1 + L_7\tau)\mathbb{E}\|X_k\|^2 + (2L_3 + L_8)\tau,
\end{aligned}
$$

from which we derive

$$\mathbb{E}\|X_k\|^2 + \frac{2\varepsilon}{1 + 2(\lambda_1 - L_2 - \varepsilon)\tau} \sum_{j=0}^{k} \mathbb{E}\|\nabla X_j\|^2 \tau$$

$$\leq \left(\frac{1 + L_7\tau}{1 + 2(\lambda_1 - L_2 - \varepsilon)\tau} \right)^k \|X_0\|^2$$
$$+ \frac{(2L_3 + L_8)\tau}{1 + 2(\lambda_1 - L_2)\tau} \sum_{j=0}^{k-1} \left(\frac{1 + L_7\tau}{1 + 2(\lambda_1 - L_2)\tau} \right)^j, \quad k \in \mathbb{N}_+.$$

Then we conclude (3.2) by the above estimate, the elementary estimate (3.5), and (3.1). $\quad\square$

In the discrete case, there is no Itô formula anymore. Whether $\{e^{\gamma t_m}\|X_m^x - X_m^y\|^2 : m \in \mathbb{N}\}$ for some γ, is discrete supermartingale is unknown, where $\{X_m^x : m \in \mathbb{N}\}$ and $\{X_m^y : m \in \mathbb{N}\}$ are denoted by the solution of the DIE scheme (DIE) starting from x and y, respectively. Fortunately, using the monotonicity conditions in Assumptions 2.1-2.2, one can show the following discrete uniform continuity dependence estimate similar to the finite-dimensional case in [24, Theorem 7.6] or [26, Lemma 4.2].

Lemma 3.2. *Let $x, y \in H$, Assumption 2.1 and conditions (2.16)-(2.17) hold. Assume that $L_1 + L_6/2 < \lambda_1$. Then there exist a positive constant γ_6 such that for any $\tau \in (0, 1)$,*

$$\mathbb{E}\big\|X_m^x - X_m^y\big\|^2 \leq e^{-\gamma_6 t_m}\|x - y\|^2, \quad m \in \mathbb{N}. \tag{3.6}$$

Consequently, $\{P_m\}_{m\in\mathbb{N}_+}$ is Feller, i.e. if $\phi \in \mathcal{C}_b(H)$, then $P_m\phi \in \mathcal{C}_b(H)$ for all $m \in \mathbb{N}_+$ (or equivalently, $P_1\phi \in \mathcal{C}_b(H)$).

Proof. For $k \in \mathbb{N}$, set $F_k^x = F(X_k^x), F_k^y = F(X_k^y), G_k^x = G(X_k^x), G_k^y = G(X_k^y)$, and $E_k = X_k^x - X_k^y$. Using the same technique in the proof of Lemma 3.1 to derive (3.4), we have

$$\|E_{k+1}\|^2 - \|E_k\|^2 + \|E_{k+1} - E_k\|^2$$
$$= 2\,_1\big\langle E_{k+1}, \Delta E_{k+1} + F_{k+1}^x - F_{k+1}^y\big\rangle_{-1}\tau$$
$$+ 2\big\langle E_{k+1} - E_k, \big(G_k^x - G_k^y\big)\delta_k W\big\rangle + 2\big\langle E_k, \big(G_k^x - G_k^y\big)\delta_k W\big\rangle.$$

It follows from the estimate (2.12) and Cauchy-Schwarz inequality that

$$\|E_{k+1}\|^2 - \|E_k\|^2 \leq -2(\lambda_1 - L_1)\|E_{k+1}\|^2\tau + \big\|\big(G_k^x - G_k^y\big)\delta_k W\big\|^2$$
$$+ 2\big\langle E_k, \big(G_k^x - G_k^y\big)\delta_k W\big\rangle.$$

Taking expectations on both sides, noting the fact that both $G_k^x - G_k^y$ and X_k are independent of $\delta_k W$, using Itô isometry and (2.16), we obtain

$$\big(1 + 2(\lambda_1 - L_1)\tau\big)\mathbb{E}\|E_{k+1}\|^2 \leq (1 + L_6\tau)\mathbb{E}\|E_k\|^2,$$

from which we obtain

$$\mathbb{E}\|E_k\|^2 \leq \left(\frac{1 + L_6\tau}{1 + 2(\lambda_1 - L_1)\tau} \right)^k \|x - y\|^2, \quad k \in \mathbb{N}.$$

Then we conclude (3.6) with $\gamma_6 := (2(\lambda_1 - L_1) - L_6)/(2(\lambda_1 - L_1) + 1)$ by the above estimate and (3.5). $\quad\square$

We can show that the DIE scheme (DIE) is ergodic with a unique invariant measure. Moreover, it converges exponentially to the equilibrium.

Theorem 3.1. *Let Assumption* 2.1 *and conditions* (2.16)-(2.17) *hold. Assume that*

$$\left(L_1 + \frac{1}{2}L_6\right) \vee \left(L_2 + \frac{1}{2}L_7\right) < \lambda_1.$$

Then for any $\tau \in (0,1)$, the DIE scheme (DIE) *possesses a unique invariant measure π_τ in H which is exponentially mixing*

$$|P_m\phi(x) - \pi_\tau(\phi)| \le \exp\left(-\frac{1}{2}\gamma_6 t_m\right)|\phi|_{\mathrm{Lip}}\int_H \|x - y\|\pi_\tau(\mathrm{d}y) \tag{3.7}$$

for any $m \in \mathbb{N}, x \in H$, and $\phi \in \mathrm{Lip}_b(H)$, where γ_6 appeared at (3.6).

Proof. Define a sequence $\{\mu_m^\tau\}_{m\in\mathbb{N}_+}$ of probability measures by

$$\mu_m^\tau(B) := \frac{1}{m}\sum_{k=0}^{m} P_k\chi_B(0)dt, \quad B \in \mathcal{B}(H), \quad m \in \mathbb{N}_+.$$

By Chebyshev inequality and the temporal average estimate (3.2) (with $X_0 = 0$), for any $\varepsilon > 0$, there exists a positive scalar I_ε and a compact set $B = \{x \in H : \|x\|_1 \le I_\varepsilon\} \subset \mathcal{B}(H)$ such that

$$\mu_m^\tau(B^c) = \frac{1}{m}\sum_{k=0}^{m}\mathbb{P}\big(\|X_k^0\|_1 > I_\varepsilon\big)dt \le \frac{I_\varepsilon^{-2}}{m}\sum_{k=0}^{m}\mathbb{E}\|X_k^0\|_1^2 \le CI_\varepsilon^{-2} < \varepsilon,$$

from which we have that $\{\mu_m^\tau\}_{m\in\mathbb{N}_+}$ is tight. Then we conclude the existence of an invariant measure π_τ of $\{P_m\}_{m\in\mathbb{N}}$ with $\pi_\tau(\|\cdot\|^2) < \infty$, via Krylov-Bogoliubov theorem (see, e.g. [20, Theorem 1.2]), by the tightness of $\{\mu_m^\tau\}_{m\in\mathbb{N}_+}$ and the Feller property of $\{P_m\}_{m\in\mathbb{N}}$ in Lemma 3.2.

Now let μ be any invariant measure of $\{P_m\}_{m\ge0}$. By the property of invariant measure and continuous dependence estimate (3.6),

$$\begin{aligned}
|P_m\phi(x) - \mu(\phi)| &= \left|\int_H \big(P_m\phi(x) - P_m\phi(y)\big)\mu(\mathrm{d}y)\right| \\
&\le |\phi|_{\mathrm{Lip}}\int_H \mathbb{E}\|X_t^x - X_t^y\|\mu(\mathrm{d}y) \\
&\le \exp\left(-\frac{1}{2}\gamma_6 t_m\right)|\phi|_{\mathrm{Lip}}\int_H \|x - y\|\mu(\mathrm{d}y)
\end{aligned}$$

for any $x \in H$ and $m \ge 0$. Taking $m \to \infty$, we have that all such invariant measures coincide, i.e. $\mu = \pi_\tau$, and thus (3.7) holds. $\square$

3.2. Exponential ergodicity of DIEG scheme

Next, let us generalize the arguments in the previous two parts to spatial discretizations via Galerkin approximations of the temporal DIE scheme (DIE). Then, we derive the time-independent strong convergence rate for the fully discrete scheme towards the exact solution of Eq. (2.19).

Let $h \in (0,1), \mathcal{T}_h$ be a regular family of partitions of $\mathscr{O}$ with maximal length h, and $V_h \subset \dot{H}^1$ be the space of continuous functions on $\bar{\mathscr{O}}$ which are piecewise linear over $\mathcal{T}_h$ and vanish on the

boundary $\partial\mathcal{O}$. Let $A_h : V_h \to V_h$ and $\mathcal{P}_h : \dot{H}^{-1} \to V_h$ be the discrete Laplacian and generalized orthogonal projection operators, respectively, defined by

$$\langle v^h, A_h x^h \rangle = -\langle \nabla x^h, \nabla v^h \rangle, \quad x^h, v^h \in V_h,$$
$$\langle v^h, \mathcal{P}_h z \rangle =_1 \langle v^h, z \rangle_{-1}, \quad z \in \dot{H}^{-1}, \quad v^h \in V_h.$$

We discretize (DIE) in space with the Galerkin method, called the DIEG scheme, and the resulting fully discrete problem is to find a V^h-valued discrete process $\{X_m^h : m \in \mathbb{N}\}$ such that

$$X_{m+1}^h = X_m^h + \tau A_h X_{m+1}^h + \tau \mathcal{P}_h F(X_{m+1}^h) + \mathcal{P}_h G(X_m^h)\delta_m W, \quad \text{(DIEG)}$$
$$X_0^h = \mathcal{P}_h X_0,$$

$m \in \mathbb{N}$, which had been widely studied, see, e.g. [18, 32, 35]. It is clear that the DIEG scheme (DIEG) is equivalent to the scheme

$$X_{m+1}^h = S_{h,\tau} X_m^h + \tau S_{h,\tau} \mathcal{P}_h F(X_{m+1}^h) + S_{h,\tau} \mathcal{P}_h G(X_m^h)\delta_m W, \quad m \in \mathbb{N}$$

with initial datum $X_0^h = \mathcal{P}_h X_0$, where $S_{h,\tau} := (\mathrm{Id} - \tau A_h)^{-1}$ is a space-time approximation of the continuous semigroup S in one step. Iterating the above equation for m-times, we obtain

$$X_{m+1}^h = S_{h,\tau}^{m+1} X_0^h + \tau \sum_{i=0}^{m} S_{h,\tau}^{m+1-i} \mathcal{P}_h F(X_{i+1}^h)$$
$$+ \sum_{i=0}^{m} S_{h,\tau}^{m+1-i} \mathcal{P}_h G(X_i^h)\delta_i W, \quad m \in \mathbb{N}.$$

Throughout this part we assume that $\tau \in (0,1)$ with $(L_1 - \lambda_1)\tau < 1$ so that (DIEG) is well-posedness.

As in Lemmas 3.1 and 3.2, we can show the uniform stability and continuous dependence on initial data independent of h, where the resulting constants appeared at Lemmas 3.1 and 3.2. The proof is analogous to that of Lemmas 3.1 and 3.2, noting that $\mathcal{P}_h$ is an orthogonal projection operator also contracted, so we omit the details.

Lemma 3.3. *Let $X_0, x, y \in H$, Assumption 2.1 and conditions (2.16)-(2.17) hold. Denote by $\{X_m^h : m \in \mathbb{N}\}, \{X_m^{h,x} : m \in \mathbb{N}\}$, and $\{X_m^{h,y} : m \in \mathbb{N}\}$ the solutions of the DIEG scheme (DIEG) starting from X_0, x, and y, respectively.*

i) *Assume that $L_2 + L_7/2 < \lambda_1$. Then for any $\tau \in (0,1)$,*

$$\mathbb{E}\|X_m^h\|^2 \le e^{-\gamma_5 t_m} \|X_0\|^2 + C_{\gamma_5}, \quad m \in \mathbb{N}. \tag{3.8}$$

ii) *Assume that $L_1 + L_6/2 < \lambda_1$. Then for any $\tau \in (0,1)$,*

$$\mathbb{E}\|X_m^{h,x} - X_m^{h,y}\|^2 \le e^{-\gamma_6 t_m} \|X - Y\|^2, \quad m \in \mathbb{N}. \tag{3.9}$$

Consequently, the corresponding fully discrete Markov semigroup, denoted by $\{P_m^h\}_{m\in\mathbb{N}}$, of the DIEG scheme (DIEG) is Feller.

We can show that the DIEG scheme (DIEG) is exponentially mixing. On the contrary to the case of DIE scheme (DIE), the dimension of state space V_h of the DIEG scheme (DIEG) is finite.

Theorem 3.2. *Let Assumption* 2.1 *and conditions* (2.16)-(2.17) *hold. Assume that*

$$\left(L_1 + \frac{1}{2}L_6\right) \vee \left(L_2 + \frac{1}{2}L_7\right) < \lambda_1.$$

Then for any $\tau \in (0,1)$ and $h \in (0,1)$, the DIEG scheme (DIEG) *possesses a unique invariant measure π_τ^h in V_h which is exponentially mixing*

$$\left|P_m^h \phi(x) - \pi_\tau^h(\phi)\right| \le \exp\left(-\frac{1}{2}\gamma_6 t_m\right) |\phi|_{\mathrm{Lip}} \int_H \|x - y\|\pi_\tau^h(\mathrm{d}y) \tag{3.10}$$

for any $m \in \mathbb{N}, x \in H$, and $\phi \in \mathrm{Lip}_b(H)$, where γ_6 appeared at (3.6).

Proof. Consider the sequence $\{\mu_m^\tau\}_{m\in\mathbb{N}_+}$ of probability measures

$$\mu_m^{\tau,h}(B) := \frac{1}{m}\sum_{k=0}^{m} P_k^h \chi_B(0)dt, \quad B \in \mathcal{B}(V_h), \quad m \in \mathbb{N}_+.$$

By Chebyshev inequality and the uniform estimate (3.8), for any $\varepsilon > 0$, there exists a positive scalar J_ε and a compact set $B = \{x \in V_h : \|x\| \le J_\varepsilon\} \subset \mathcal{B}(H)$ such that

$$\mu_m^{\tau,h}(B^c) = \frac{1}{m}\sum_{k=0}^{m} \mathbb{P}\big(\big\|X_k^{h,0}\big\| > J_\varepsilon\big)dt \le J_\varepsilon^{-2}\sup_{k\in\mathbb{N}}\mathbb{E}\big\|X_k^{h,0}\big\|^2 \le CJ_\varepsilon^{-2} < \varepsilon,$$

from which we have that $\{\mu_t\}_{T\ge 1}$ is tight. The exponential mixing estimate (3.10) and the existence of a unique invariant measure follow from the continuous dependence estimate (3.9) and similar arguments in the proof of Theorem 3.1. $\qquad\square$

3.3. Uniform strong error estimate of DIEG scheme

In this part, we aim to derive time-independent strong error estimates of the proposed DIEG scheme (DIEG) towards the exact solution of Eq. (2.19).

We will combine semigroup and variational theories as in [32]. To this end, for $m \in \mathbb{N}$, let us denote by $E_{h,\tau}(t) = S(t) - S_{h,\tau}^{m+1}\mathcal{P}_h$ for $t \in (t_m, t_{m+1}]$, and define the auxiliary process (see [32, 35])

$$\widetilde{X}_{m+1}^h = S_{h,\tau}^{m+1}X_0^h + \tau\sum_{i=0}^{m} S_{h,\tau}^{m+1-i}\mathcal{P}_h F\big(X(t_{i+1})\big) + \sum_{i=0}^{m} S_{h,\tau}^{m+1-i}\mathcal{P}_h G\big(X(t_i)\big)\delta_i W. \tag{3.11}$$

In view of the well-known uniform boundedness of $S_{h,\tau}$, the conditions (2.3) and (2.18), and the uniform stability (2.21), it is not difficult to show that for any $p \ge 2$,

$$\sup_{k\in\mathbb{N}}\mathbb{E}\big\|\widetilde{X}_k^h\big\|_1^p \le C\Big(1 + \sup_{t\ge 0}\mathbb{E}\|X_t\|_1^{pq}\Big) \le C\big(1 + \|X_0\|_1^{pq}\big). \tag{3.12}$$

We begin with the following uniform strong error estimate between the exact solution u of Eq. (2.19) and the auxiliary process $\{\widetilde{X}_m^h\}_{m\in\mathbb{N}}$ defined by (3.11). It is proved using the idea in [32, Lemma 4.2] and the uniform moments' bound (2.21).

Lemma 3.4. *Let $X_0 \in \dot{H}^1$ and Assumptions 2.1-2.2 hold. Assume that $p \geq 2$ and*

$$L_1 + \frac{1}{2}\big(p(2q-1)-1\big)L_9 < \lambda_1.$$

Then there exist a positive constant C such that for any $\tau \in (0,1)$,

$$\sup_{m \in \mathbb{N}} \mathbb{E}\big\|X(t_m) - \widetilde{X}_m^h\big\|^p \leq C\big(1 + \|X_0\|_1^{p(2q-1)}\big)\big(h^p + \tau^{\frac{p}{2}}\big). \tag{3.13}$$

Proof. Let $m \in \mathbb{N}$. In [32, Lemma 4.2], the error

$$J^{m+1} := \big(\mathbb{E}\big\|X(t_{m+1}) - \widetilde{X}_{m+1}^h\big\|^p\big)^{\frac{1}{p}}$$

is divided into three parts $J_i^{m+1}, i = 1, 2, 3$, respectively, which satisfies

$$J_1^{m+1} = \big(\mathbb{E}\big\|E_{h,\tau}(t_{m+1})X_0\big\|^p\big)^{\frac{1}{p}} \leq C\big(h + \tau^{\frac{1}{2}}\big)\|X_0\|_1,$$

$$J_2^{m+1} = \left(\mathbb{E}\left\|\int_0^{t_{m+1}} S_{t_{m+1}-r}F(X_r)\mathrm{d}r - \tau\sum_{i=0}^m S_{h,\tau}^{m+1-i}\mathcal{P}_h F\big(X(t_{i+1})\big)\right\|^p\right)^{\frac{1}{p}}$$

$$\leq C\left(1 + \sup_{t \geq 0}\big(\mathbb{E}\|X_t\|_1^{2p(q-1)}\big)^{\frac{1}{2p}}\right)\left(\sup_{t \neq s}\frac{\mathbb{E}\|X_t - X_s\|^{2p}}{|t-s|^p}\right)^{\frac{1}{2p}}\tau^{\frac{1}{2}}$$

$$+ C\big(h + \tau^{\frac{1}{2}}\big)\sup_{t \geq 0}\big(\mathbb{E}\|F(X_t)\|^p\big)^{\frac{1}{p}},$$

$$J_3^{m+1} = \left(\mathbb{E}\left\|\int_0^{t_{m+1}} S_{t_{m+1}-r}G(X_r)\mathrm{d}W_r - \sum_{i=0}^m S_{h,\tau}^{m+1-i}\mathcal{P}_h G\big(X(t_i)\big)\delta_i W\right\|^p\right)^{\frac{1}{p}}$$

$$\leq C\left(\sup_{t \neq s}\frac{\mathbb{E}\|X_t - X_s\|^{2p}}{|t-s|^p}\right)^{\frac{1}{2p}}\tau^{\frac{1}{2}} + C\big(h + \tau^{\frac{1}{2}}\big)\left(1 + \sup_{t \geq 0}\big(\mathbb{E}\|X_t\|_1^p\big)^{\frac{1}{p}}\right).$$

Putting the above three estimates together results in

$$J^{m+1} \leq C\Bigg(1 + \|X_0\|_1 + \left(1 + \sup_{t \geq 0}\big(\mathbb{E}\|X_t\|_1^{2p(q-1)}\big)^{\frac{1}{2p}}\right)\left(\sup_{t \neq s}\frac{\mathbb{E}\|X_t - X_s\|^{2p}}{|t-s|^p}\right)^{\frac{1}{2p}}$$

$$+ \sup_{t \geq 0}\big(\mathbb{E}\|F(X_t)\|^p\big)^{\frac{1}{p}} + \sup_{t \geq 0}\big(\mathbb{E}\|X_t\|_1^p\big)^{\frac{1}{p}}\Bigg)\big(h + \tau^{\frac{1}{2}}\big).$$

This inequality, in combination with the estimate on J^0 that

$$J^0 = \|X_0 - \mathcal{P}_h X_0\|_{L_\omega^p L^2} \leq Ch\|X_0\|_1,$$

the condition (2.3), the embeddings (2.8), the uniform stability estimate (2.21), and the uniform Hölder estimate (2.25), completes the proof of (3.13). $\qquad\square$

Now, we are at the position of deriving the uniform strong error estimate between the exact solution u of Eq. (2.19) and the DIEG scheme (DIEG).

Theorem 3.3. *Let $X_0 \in \dot{H}^1$, Assumptions 2.1-2.2 hold. Assume that*

$$L_1 + \frac{1}{2}\big(2(q^2 + q - 1) - 1\big)L_9 < \lambda_1.$$

Then for any $\tau \in (0,1)$, there exist a positive constant C such that

$$\sup_{k \in \mathbb{N}} \mathbb{E}\big\|X_{t_k} - X_k^h\big\|^2 \leq C\big(1 + \|X_0\|_1^{2(q^2+q-1)}\big)(h^2 + \tau). \tag{3.14}$$

Proof. For $k \in \mathbb{N}$, denote $e_k^h := \widetilde{X}_k^h - X_k^h$. From (3.11), it is clear that

$$\widetilde{X}_{m+1}^h = \widetilde{X}_m^h + \tau A_h \widetilde{X}_{m+1}^h + \tau \mathcal{P}_h F(X_{t_{m+1}}) + \mathcal{P}_h G(X_{t_m})\delta_m W.$$

Then, by (DIEG) and the above equality, we have

$$e_{k+1}^h - e_k^h = \left[\Delta_h e_{k+1}^h + \mathcal{P}_h\big(F(X_{t_{k+1}}) - F(X_{k+1}^h)\big)\right]\tau$$
$$+ \mathcal{P}_h\big(G(X_{t_k}) - G(X_k^h)\big)\delta_k W.$$

Testing with e_{k+1}^h and using the inequality (3.3), the condition (2.12), Cauchy-Schwarz inequality, and the elementary inequality $\|x + y\|^2 \leq (1 + \varepsilon)\|x\|^2 + (1 + \varepsilon^{-1})\|y\|^2$ for $x, y \in H$, we obtain

$$\left\|e_{k+1}^h\right\|^2 - \left\|e_k^h\right\|^2 + \left\|e_{k+1}^h - e_k^h\right\|^2$$
$$= 2\,_1\big\langle e_{k+1}^h, F(X_{t_{k+1}}) - F\big(\widetilde{X}_{k+1}^h\big)\big\rangle_{-1}\tau$$
$$+ 2\,_1\big\langle e_{k+1}^h, \Delta_h e_{k+1}^h + F\big(\widetilde{X}_{k+1}^h\big) - F(X_{k+1}^h)\big\rangle_{-1}\tau$$
$$+ 2\big\langle e_{k+1}^h - e_k^h, \big(G(X_{t_k}) - G(X_k^h)\big)\delta_k W\big\rangle$$
$$+ 2\big\langle e_k^h, \big(G(X_{t_k}) - G(X_k^h)\big)\delta_k W\big\rangle$$
$$\leq -2(\lambda_1 - L_1 - \varepsilon)\tau\left\|e_{k+1}^h\right\|^2 + \left\|e_{k+1}^h - e_k^h\right\|^2$$
$$+ 2\big\langle e_k^h, \big(G(X_{t_k}) - G(X_k^h)\big)\delta_k W\big\rangle + C_\varepsilon\tau\left\|F(X_{t_{k+1}}) - F\big(\widetilde{X}_{k+1}^h\big)\right\|_{-1}^2$$
$$+ (1 + \varepsilon^{-1})\left\|\big(G(X_{t_k}) - G\big(\widetilde{X}_k^h\big)\big)\delta_k W\right\|^2$$
$$+ (1 + \varepsilon)\left\|\big(G\big(\widetilde{X}_k^h\big) - G(X_k^h)\big)\delta_k W\right\|^2.$$

It follows that

$$\big(1 - 2(\lambda_1 - L_1 - \varepsilon)\tau\big)\left\|e_{k+1}^h\right\|^2$$
$$\leq \left\|e_k^h\right\|^2 + 2\big\langle e_k^h, \big(G(X_{t_k}) - G\big(X_k^h\big)\big)\delta_k W\big\rangle$$
$$+ (1 + \varepsilon^{-1})\left\|\big(G(X_{t_k}) - G\big(\widetilde{X}_k^h\big)\big)\delta_k W\right\|^2$$
$$+ C_\varepsilon\tau\left\|F(X_{t_{k+1}}) - F\big(\widetilde{X}_{k+1}^h\big)\right\|_{-1}^2$$
$$+ (1 + \varepsilon)\left\|\big(G\big(\widetilde{X}_k^h\big) - G(X_k^h)\big)\delta_k W\right\|^2.$$

Now, taking expectation on both sides of the above inequality and using the fact

$$\mathbb{E}\big\langle e_k^h, \big(G(X_{t_k}) - G\big(X_k^h\big)\big)\delta_k W\big\rangle = 0,$$

and the estimates

$$\sup_{k \in \mathbb{N}} \mathbb{E}\left\|F(X_{t_{k+1}}) - F\big(\widetilde{X}_{k+1}^h\big)\right\|_{-1}^2 \leq C\big(1 + \|X_0\|_1^{2(q^2+q-1)}\big)(h^2 + \tau),$$
$$\sup_{k \in \mathbb{N}} \mathbb{E}\left\|G(X_{t_k}) - G\big(\widetilde{X}_k^h\big)\right\|_{\mathcal{L}_2^0}^2 \leq C\big(1 + \|X_0\|_1^{2(2q-1)}\big)(h^2 + \tau),$$

which follow from (3.13) and the conditions (2.3) and (2.16) (see [28, Theorem 4.1] for details), the embeddings (2.8), and the uniform stability estimates (2.20) and (3.12), we get

$$\big(1 + 2(\lambda_1 - L_1 - \varepsilon)\tau\big)\mathbb{E}\left\|e_{k+1}^h\right\|^2$$
$$\leq \big(1 + L_6(1 + \varepsilon)\tau\big)\mathbb{E}\left\|e_k^h\right\|^2 + C_\varepsilon\big(1 + \|X_0\|_1^{2(2q-1)}\big)(h^2 + \tau)\tau,$$

from which we obtain

$$\mathbb{E}\|e_k^h\|^2 \leq \left(\frac{1 + L_6(1 + \varepsilon)\tau}{1 + 2(\lambda_1 - L_1 - \varepsilon)\tau}\right)^k \|e_0^h\|^2$$

$$+ \frac{C_\varepsilon\big(1 + \|X_0\|_1^{2(2q-1)}\big)(h^2 + \tau)\tau}{1 + 2(\lambda_1 - L_1 - \varepsilon)\tau} \sum_{j=0}^{k-1} \left(\frac{1 + L_6(1 + \varepsilon)\tau}{1 + 2(\lambda_1 - L_1 - \varepsilon)\tau}\right)^j$$

for $k \in \mathbb{N}$. Now we can conclude (3.14) by the above estimate, the inequality (3.5), the estimate (3.13), and the estimate

$$\mathbb{E}\|X_{t_k} - X_k^h\|^2 \leq 2\mathbb{E}\|X_{t_k} - \widetilde{X}_k^h\|^2 + 2\mathbb{E}\|e_k^h\|^2.$$

The proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Finally, we can apply the previous arguments to the following DIE Milstein (DIEM) scheme and its Galerkin full discretization under an additional infinite-dimensional commutativity condition, see [32, Assumption 5.2 and Example 5.1], which is frequently used to construct temporal high order scheme

$$Y_{m+1} = Y_m + \tau A Y_{m+1} + \tau F(X_{m+1}) + G(X_m)\delta_m W$$

$$+ DG(Y_m)G(Y_m)\left[\int_{t_m}^{t_{m+1}} \big(W(r) - W(t_m)\big)\mathrm{d}W_r\right], \quad Y_0 := X_0. \qquad \text{(DIEM)}$$

The proofs are analogous to the previous arguments, so we omit the details.

Remark 3.1. Similar to the DIE scheme (DIE), we have the following uniform moments' stability and continuity dependence estimates for (DIEM): There exist positive constants $\gamma_7, C_{\gamma_7}, C, \gamma_8$ and τ_0 such that for any $X_0, x, y \in H, m \in \mathbb{N}$, and $\tau \in (0, \tau_0)$,

$$\mathbb{E}\|Y_m\|^2 \leq e^{-\gamma_7 t_m}\|X_0\|^2 + C_{\gamma_7},$$

$$\sup_{n \in \mathbb{N}_+} \frac{1}{n}\sum_{k=0}^{n} \mathbb{E}\|Y_k\|_1^2 \leq C\big(1 + \|X_0\|^2\big),$$

$$\mathbb{E}\|Y_m^x - Y_m^y\|^2 \leq e^{-\gamma_8 t_m}\|x - y\|^2.$$

In this case, there is an upper bound restriction on the time step size τ. We note that such restriction also appeared in [36] where the authors derived numerical ergodicity of the Milstein scheme for Lipschitz SODEs, as one needs to control the last term on the right-hand side of (DIEM). Moreover, we have the exponential mixing property of (DIEM): For any $\tau \in (0, \tau_0)$, (DIEM) (whose Markov semigroup is denoted by $\hat{P}_m$) possesses a unique invariant measure $\hat{\pi}_\tau$ in H such that for any $m \in \mathbb{N}, x \in H$, and $\phi \in \mathrm{Lip}_b(H)$,

$$\big|\hat{P}_m\phi(x) - \hat{\pi}_\tau(\phi)\big| \leq \exp\left(-\frac{1}{2}\gamma_8 t_m\right)|\phi|_{\mathrm{Lip}}\int_H \|x - y\|\hat{\pi}_\tau(\mathrm{d}y).$$

Using the method in [28, Theorem 4.1] and the arguments in the proof of Theorem 3.3, one can show a similar uniform strong error estimate for (DIEM) and its Galerkin full discretization. In particular, if $X_0 \in \dot{H}^{1+\gamma}$ with some $\gamma \in [0, 1]$, then the DIEM Galerkin scheme (the Galerkin scheme of (DIEM), denoted by $\{Y_k^h\}_{k \in \mathbb{N}}$) possesses higher strong order of convergence in both spatial and temporal directions for any $p \geq 2$,

$$\sup_{k \in \mathbb{N}} \mathbb{E}\|X_{t_k} - Y_k^h\|^p \leq C\big(1 + \|X_0\|_{1+\gamma}^{p(q^2+q-1)}\big)\big(h^{1+\gamma} + \tau^{\frac{1+\gamma}{2}}\big)^p,$$

provided more regular and growth conditions (see [32, Assumptions 5.1 and 5.2]) on the coefficients of Eq. (2.19) hold true.

4. Applications to Stochastic Allen-Cahn Equation

In the last section, let us apply the exponential ergodicity and uniform strong error analysis results, developed in Section 3, to the semiclassical stochastic Allen-Cahn equation (1.2).

We note that as the driven noise in Eq. (1.2) is additive, the DIE scheme (DIE) and DIEG scheme (DIEG) coincide with the DIEM scheme (DIEM) and its Galerkin full discretization, respectively. Therefore, in this part, it suffices to consider the DIE scheme (DIE) and DIEG scheme (DIEG).

Let us first check the conditions in the main Assumptions 2.1 and 2.2 for Eq. (1.2) with $f(x) = \alpha^{-2}(x - x^3)$ and $G \equiv \text{Id}$. At first, the growth order of f' in (2.4) is $q - 1 = 2$, so that $q = 3$. Secondly, it is not difficult to show that $L_1 = \alpha^{-2}$ in (2.1) and $L_4 = 2\alpha^{-2}$ and $L_5 = \alpha^{-2}$ in (2.3). For (2.2), note that by Young inequality,

$$\alpha^{-2}(x - x^3)x = -\alpha^{-2}x^4 + \alpha^{-2}x^2 \leq -Cx^2 + C_\alpha$$

for any positive C and certain positive constant C_α. So, one can take L_2 to be any negative scalar. We emphasize that the choice of L_2 to be any negative scalar is only needed in this section to ensure the existence of the invariant measures of Eq. (1.2) and its DIEG scheme (DIEG).

For the conditions on G of Assumption 2.2, $L_6 = L_7 = L_9 = 0, L_8 = \text{Trace}(\mathbf{Q})$, and $L_{10} = \|(-A)^{1/2}\mathbf{Q}^{1/2}\|^2_{HS(H;H)}$. Therefore, $L_2 + L_7/2 = L_2 < \lambda_1$ is always valid as one can take L_2 to be any negative scalar, so that the main conditions in Proposition 2.1 and Lemma 3.1 hold and thus Eq. (1.2) and its DIEG scheme (DIEG) always possess at least one invariant measure, respectively. Meanwhile, $L_1 + L_6/2 = \alpha^{-2} < \lambda_1$ for $\alpha > \lambda_1^{-1/2}$, so that the main conditions in Theorems 2.1 and 3.1 hold and thus Eq. (1.2) and its DIEG scheme (DIEG) are both exponentially ergodic, respectively, provided that the interface thickness is not too small.

Finally, when $\alpha > \lambda_1^{-1/2}$, we have $L_1 + 21L_9/2 = \alpha^{-2} < \lambda_1$, so that the main condition in Theorem 3.3 hold. Therefore, we have exponential ergodicity and uniform strong error estimate for the stochastic Allen-Cahn equation (1.2).

Theorem 4.1. *Eq.* (1.2) *as well as its DIE scheme* (DIE) *and DIEG scheme* (DIEG) *always possess at least one invariant measure, respectively. They are all exponentially ergodic, provided that $\alpha > \lambda_1^{-1/2}$. Moreover, there exist a positive constant C such that for any $\tau \in (0,1)$,*

$$\sup_{k \in \mathbb{N}} \mathbb{E}\|X_{t_k} - X_k^h\|^2 \leq C\big(1 + \|X_0\|_1^{22}\big)(h^2 + \tau).$$

Remark 4.1. The authors in [16, p. 88] proposed a question whether the invariant measure of the DIE scheme (DIE), applied to Eq. (1.1) with polynomial drift of order 3 with negative leading coefficients (as in Remark 2.1), driven by additive noise, is unique. In this case, L_1 is just the bound $\sup_{\xi \in \mathbb{R}} f'(x)$ and $L_6 = 0$. Applying Theorem 3.1, we conclude that, in this case, the DIE scheme (DIE) is exponentially ergodic and thus gives an affirmative answer to the question proposed in [16], provided that $\sup_{\xi \in \mathbb{R}} f'(x) < \lambda_1$. We also note that this restriction can be removed in the non-degenerate noise case using the methodology developed in the recent paper [29].

Remark 4.2. Combining the uniform ergodicity of the DIEG scheme (DIEG) with the uniform weak error estimate in [16], one can show the error estimate between the fully discrete invariant measure π_τ^h of (DIEG) and the exact invariant measure π of Eq. (1.2).

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