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AUGMENTED SUBSPACE SCHEME FOR EIGENVALUE PROBLEM BY WEAK GALERKIN FINITE ELEMENT METHOD*

Yue Feng, Zhijin Guan and Hehu Xie
LSEC, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China;
School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China
Emails: fengyue@amss.ac.cn, guanzhijin@lsec.cc.ac.cn, hhxie@lsec.cc.ac.cn
Chenguang Zhou[1)]
Department of Mathematics, Beijing University of Technology, Beijing 100124, China
Email: zhoucg@bjut.edu.cn

Abstract

This study proposes a class of augmented subspace schemes for the weak Galerkin (WG) finite element method used to solve eigenvalue problems. The augmented subspace is built with the conforming linear finite element space defined on the coarse mesh and the eigenfunction approximations in the WG finite element space defined on the fine mesh. Based on this augmented subspace, solving the eigenvalue problem in the fine WG finite element space can be reduced to the solution of the linear boundary value problem in the same WG finite element space and a low dimensional eigenvalue problem in the augmented subspace. The proposed augmented subspace techniques have the second order convergence rate with respect to the coarse mesh size, as demonstrated by the accompanying error estimates. Finally, a few numerical examples are provided to validate the proposed numerical techniques.

Mathematics subject classification: 65N30, 65N25, 65L15, 65B99.
Key words: Eigenvalue problem, Augmented subspace scheme, Weak Galerkin finite element method, Second order convergence rate.

1. Introduction

One of the most important tasks in contemporary scientific and engineering society is solving eigenvalue problems. The difficulty of solving eigenvalue problems is invariably higher than that of solving similar linear boundary value problems due to the increased computing and memory requirements. Large-scale eigenvalue problem solving in particular will provide formidable obstacles to scientific computing. Numerous eigensolvers have been developed so far, including the Jacobi-Davidson type technique [4], the preconditioned inverse iteration (PINVIT) method [5, 11, 14], the Krylov subspace type method (implicitly restarted Lanczos/Arnoldi method (IRLM/IRAM) [25]), and the generalized conjugate gradient eigensolver (GCGE) [16, 17, 38]. The orthogonalization processes involved in solving Rayleigh-Ritz problems are the common bottleneck in the design of effective parallel techniques for identifying a large number of eigenpairs, and they are included in all of these widely used approaches.

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1) Corresponding author

A class of augmented subspace methods and their multilevel correction methods has been proposed recently in [8,13,18,28–32] for the solution of eigenvalue problems. This kind of technique creates an augmented subspace using the low dimensional finite element space generated on the coarse grid, which is employed in each correction step. The notion of an augmented subspace gives rise to a class of augmented subspace techniques that need just the final finite element space on the finest mesh and the low dimension finite element space on the coarse mesh. Using the augmented subspace methods, the solution of the eigenvalue problem on the final level of mesh can be transformed to the solution of linear boundary value problems on the final level of mesh and the solution of the eigenvalue problem on the low dimensional augmented subspace. Even the coarse and finest meshes lack nested properties, these kinds of algorithms can still work [10]. The multilevel correction methods, which are based on the augmented subspace methods, provide ways to construct multigrid methods for eigenvalue problems [8,13,28,29,31]. In addition, the authors design an eigenpair-wise parallel eigensolver for the eigenvalue problems in [32]. A significant amount of the wall time in the parallel computation is saved by using this kind of parallel approach, which avoids performing orthogonalization and inner-products in the high dimensional space. However, the aforementioned references are mostly investigated using conforming finite element methods. There are few results on the augmented subspace approaches based on nonstandard finite element methods for solving eigenvalue problems.

The WG method, which is initially introduced and explored in [27], concerns the finite element methods utilized to solve partial differential equations in which the differential operators, such as gradient operator, divergence operator and curl operator, are approximated as distributions by weak forms. The WG approach employs generalized discrete weak derivatives and parameter-free stabilizers to weakly enforce continuity in the approximation space, in contrast to the standard finite element technique. Consequently, it ought to be more convenient to create high order precision discretization than the conforming finite element approach. Additionally, the WG approach can be easily implemented on polygonal meshes thanks to the relaxation of the continuity constraint, which also gives additional freedom for h- and p-adaptation. So far, the WG method has been applied to various partial differential equations, such as the parabolic equation [15,40], the biharmonic equation [21,26,39], the Brinkman equation [20,37], the Helmholtz equation [22,24] and the Maxwell equation [23]. The convergence analysis and several lower bound findings are produced in [34,35], where the WG approximation to the eigenvalue problems is studied. In [36], the authors adopt the WG method to obtain lower bounds of the Laplacian eigenvalue problem and the post-processing method based on interpolation is used to obtain upper bounds. In [6], the authors show that under certain conditions the WG method yields a guaranteed lower bound for the Laplacian eigenvalue, which is larger than a guaranteed lower bound for the Crouzeix-Raviart finite element space. Then, considering computational efficiency, the authors create a kind of two-grid or two-level schemes using the WG approach [35], and in [33], the shifted-inverse power technique is taken into consideration under the two-grid schemes. Based on the theoretical analysis presented in [35], it can be inferred that there is no independent relationship between the coarse and fine mesh sizes. As a result, the approaches cannot be used to develop an eigensolver for algebraic eigenvalue problems resulting from differential operator eigenvalue problems discretized by WG method.

This paper's contribution is to design an augmented subspace method for eigenvalue problems based on WG approximation. To the best of our knowledge, this is the first work aimed at the numerical analysis of the WG finite element discretization-based augmented subspace approach for eigenvalue problems. In contrast to the findings in [35], our approaches' selections for the coarse and fine mesh sizes are independent of one another. The algebraic eigenvalue problems that result from the WG approximation to the differential eigenvalue problems can then be solved by designing an eigensolver using the proposed techniques. Furthermore, we demonstrate the algebraic error estimate for the WG augmented subspace approaches as follows:

$$\big\|\bar{u}_h - u_h^{(\ell+1)}\big\|_{a,h} \leq CH^2\big\|\bar{u}_h - u_h^{(\ell)}\big\|_{a,h},$$

when the computing domain is convex.

This paper is organized as follows. We provide the WG approaches for the eigenvalue problems and deduce the associated error estimates in Section 2. These results give explicit dependence of the error estimates on the eigenvalue distribution which is another contribution of this paper. The majority of this work, Section 3, contains the augmented subspace techniques and the associated error estimates. A few numerical examples are given in Section 4 to validate the suggested augmented subspace algorithms' convergence rates. Lastly, the final section has a few closing thoughts.

2. Discretization by WG Finite Element Method

The WG finite element approach for the second order elliptic eigenvalue problem is presented in this section. The associated error estimates are offered. It should be noted that the letter C, with or without subscripts, symbolizes a generic positive constant for this purpose that may vary at various places in this work.

We consider the numerical method to solve the following second order elliptic eigenvalue problem: Find $(\lambda, u) \in \mathbb{R} \times H_0^1(\Omega)$ such that

$$\begin{cases} -\nabla \cdot (A\nabla u) = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ (A\nabla u, \nabla u) = 1, \end{cases} \tag{2.1}$$

where Ω denotes a convex bounded polygonal or polyhedral domain in $\mathbb{R}^d$, $d = 2, 3$, $A \in [L^\infty(\Omega)]^{d\times d}$ is a symmetric matrix-valued function on Ω with suitable regularity, and $(\cdot\,,\cdot)$ represents the standard L^2-inner product which is defined as $(v, w) = \int_\Omega vwd\Omega$. Assume that there exist positive constants c and C such that the matrix A satisfies the following property:

$$c\xi^\top\xi \leq \xi^\top A(x)\xi \leq C\xi^\top\xi, \quad \forall\, \xi \in \mathbb{R}^d, \quad x \in \Omega. \tag{2.2}$$

In order to construct a WG finite element method for the eigenvalue problem, (2.1) should be written as the following variational form: Find $(\lambda, u) \in \mathbb{R} \times V$ such that $a(u, u) = 1$ and

$$a(u, v) = \lambda b(u, v), \quad \forall\, v \in V, \tag{2.3}$$

where $V := H_0^1(\Omega)$ [1] and

$$a(u, v) = (A\nabla u, \nabla v), \quad b(u, v) = (u, v). \tag{2.4}$$

Furthermore, based on the bilinear forms $a(\cdot\,,\cdot)$ and $b(\cdot\,,\cdot)$, we can define the norms on the space V as follows:

$$\begin{aligned} \|v\|_a &= \sqrt{a(v, v)}, \quad && \forall\, v \in V, \\ \|w\|_b &= \sqrt{b(w, w)}, \quad && \forall\, w \in L^2(\Omega). \end{aligned} \tag{2.5}$$

It is well known that the eigenvalue problem (2.3) has an eigenvalue sequence $\{\lambda_j\}$ (cf. [2,7]),

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \cdots, \quad \lim_{k\to\infty} \lambda_k = \infty.$$

And the associated eigenfunctions are provided as

$$u_1, u_2, \cdots, u_k, \cdots.$$

Here $a(u_i, u_j) = \delta_{ij}$ (δ_{ij} denotes the Kronecker function).

Now, let us define the WG finite element space for the eigenvalue problem (2.3). First we generate a quasi-uniform mesh $\mathcal{T}_h$ of the computing domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$). Denote by $\mathcal{E}_h$ the set of all edges or faces of the mesh $\mathcal{T}_h$. For simplicity, in this paper, we only consider the triangle or tetrahedral mesh. The diameter of a cell $K \in \mathcal{T}_h$ is denoted by h_K and the mesh size h describes the maximal diameter of all cells $K \in \mathcal{T}_h$. For each cell $K \in \mathcal{T}_h$, we use K_0 and ∂K to denote the interior and the boundary of K. In the sense of geometry, K_0 is identical to K. Then we identify them if no ambiguity. Based on the mesh $\mathcal{T}_h$, we can construct the WG finite element space denoted by V_h as follows:

$$\begin{aligned} V_h = \big\{ v : v|_{K_0} \in \mathcal{P}_r(K_0) \ \text{ for } \ K \in \mathcal{T}_h; \ v|_e \in \mathcal{P}_{\bar{s}}(e) \ \text{ for } \ e \in \mathcal{E}_h, \\ \text{and } \ v|_e = 0 \ \text{ for } \ e \in \mathcal{E}_h \cap \partial\Omega \big\}, \end{aligned} \tag{2.6}$$

where $\mathcal{P}_r(K_0)$ denotes the set of polynomials of degree at most the integer $r \ge 0$, $\mathcal{P}_{\bar{s}}(e)$ is the set of polynomials of degree no more than the integer $\bar{s} \ge 0$. In this paper, we are concerned with the cases of $\bar{s} = r$ or $r + 1$. From the definition of V_h, it is easy to know that the function in V_h does not require any continuity across interior edges/faces. Actually, the function in V_h can be characterized by its value on the interior of each element and its value on edges/faces. Therefore, the functions in V_h can be represented with two components, $v = \{v_0, v_b\}$, where v_0 denotes the value of v on all K_0 and v_b denotes the value of v on $\mathcal{E}_h$. The polynomial space $\mathcal{P}_{\bar{s}}(e)$ consists of two choices: $\bar{s} = r$ or $r+1$ and the corresponding weak function space will sometimes be abbreviated as $V_{r,r}$ or $V_{r,r+1}$, respectively.

In order to define the WG method for the eigenvalue problem (2.3), we introduce the discrete weak gradient operator, which is defined on each element $K \in \mathcal{T}_h$. For the choices of V_h given above, i.e. using $V_{r,r}$ or $V_{r,r+1}$, suitable definitions of the weak gradient involve the Raviart-Thomas (RT) element or the Brezzi-Douglas-Marini (BDM) element [12], respectively. Let K be either a triangle or a tetrahedron and denote by $\widehat{\mathcal{P}}_t(K)$ the set of homogeneous polynomials of order t in the variable $\mathbf{x} = (x_1, \ldots, x_d)^\top$. Define the BDM element by $G_r(K) = [\mathcal{P}_{r+1}(K)]^d$ and the RT element by $G_r(K) = [\mathcal{P}_r(K)]^d + \widehat{\mathcal{P}}_r(K)\mathbf{x}$ for $r \ge 0$. Then, we can define a discrete space

$$\mathbf{\Sigma}_h = \big\{ \mathbf{q} \in \big(L^2(\Omega)\big)^d : \mathbf{q}|_K \in G_r(K) \ \text{ for } \ K \in \mathcal{T}_h \big\}.$$

In the definitions of V_h and $\mathbf{\Sigma}_h$, the RT element is coupled with $V_{r,r}$ while the BDM element is coupled with $V_{r,r+1}$. We should point out that $\mathbf{\Sigma}_h$ is not necessarily a subspace of $H(\text{div}, \Omega)$, since it does not require any continuity in the normal direction across edges/faces.

The discrete weak gradient of $v_h \in V_h$, denoted by $\nabla_w v_h$, is defined as the unique polynomial $(\nabla_w v_h)|_K \in G_r(K)$ satisfying the following equation:

$$(\nabla_w v_h, \mathbf{q})_K = -(v_0, \nabla \cdot \mathbf{q})_K + \langle v_b, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial K}, \quad \forall\, \mathbf{q} \in G_r(K), \tag{2.7}$$

where $\mathbf{n}$ is the unit outward normal vector on ∂K. Clearly, such a discrete weak gradient is always well-defined. Furthermore, if $v \in H^1(K)$, i.e. $v_b = v_0|_{\partial K}$, and $\nabla v \in G_r(K)$. Then we have $\nabla_w v = \nabla v$. Here we only consider the $V_{r,r}$-RT and $V_{r,r+1}$-BDM pairs on simplicial elements. Of course, there are many other different choices of discrete spaces in the WG method, defined on either simplicial meshes or general polytopal meshes [21, 26].

In order to define an interpolation operator for the WG finite element space, we define an L^2 projection from V onto V_h by setting $Q_h v \equiv \{Q_0 v, Q_b v\}$, where $Q_0 v|_{K_0}$ is the local L^2 projection of v to $\mathcal{P}_r(K_0)$, for $K \in \mathcal{T}_h$, and $Q_b v|_e$ is the local L^2 projection to $\mathcal{P}_{\bar{s}}(e)$, for $e \in \mathcal{E}_h$. We also introduce $\mathbb{Q}_h$ the L^2 projection operator onto $\boldsymbol{\Sigma}_h$. It is well known that the following operator identity holds [27]:

$$\mathbb{Q}_h \nabla v = \nabla_w Q_h v, \quad \forall\, v \in V. \tag{2.8}$$

For the $V_{r,r}$-RT and $V_{r,r+1}$-BDM pairs, the identity (2.8) shows that the discrete weak gradient is a good approximation to the classical gradient [27].

Then, the WG finite element method for the eigenvalue problem (2.3) can be designed as follows: Find $(\bar{\lambda}_h, \bar{u}_h) \in \mathbb{R} \times V_h$ such that $a_h(\bar{u}_h, \bar{u}_h) = 1$ and

$$a_h(\bar{u}_h, v_h) = \bar{\lambda}_h b_h(\bar{u}_h, v_h), \quad \forall\, v_h \in V_h, \tag{2.9}$$

where

$$a_h(u_h, v_h) = (A\nabla_w u_h, \nabla_w v_h)_{\mathcal{T}_h} = \sum_{K\in\mathcal{T}_h} (A\nabla_w u_h, \nabla_w v_h)_K, \tag{2.10}$$

$$b_h(u_h, v_h) = (u_0, v_0)_{\mathcal{T}_h} = \sum_{K\in\mathcal{T}_h} (u_0, v_0)_K. \tag{2.11}$$

Based on the bilinear form $a_h(\cdot\,,\cdot)$, we can define the following discrete norm on the space V_h:

$$\|v\|_{a,h} = \sqrt{a_h(v, v)}, \quad \forall\, v \in V_h. \tag{2.12}$$

In addition, we can also define the norm $\|\cdot\|_{b,h}$ by the bilinear form $b_h(\cdot\,,\cdot)$ on the space V_h,

$$\|w\|_{b,h} = \sqrt{b_h(w, w)}, \quad \forall\, w \in V_h. \tag{2.13}$$

From [2, 3], we obtain

$$0 < \bar{\lambda}_{1,h} \le \bar{\lambda}_{2,h} \le \cdots \le \bar{\lambda}_{k,h} \le \cdots \le \bar{\lambda}_{N_h,h},$$

and corresponding eigenfunctions

$$\bar{u}_{1,h}, \bar{u}_{2,h}, \cdots, \bar{u}_{k,h}, \cdots, \bar{u}_{N_h,h}, \tag{2.14}$$

where $a_h(\bar{u}_{i,h}, \bar{u}_{j,h}) = \delta_{ij}$, $1 \le i, j \le N_h$ (N_h is the dimension of the finite element space V_h).

For the following analysis in this paper, we define $\mu_i = 1/\lambda_i$ for $i = 1, 2, \ldots,$ and $\bar{\mu}_{i,h} = 1/\bar{\lambda}_{i,h}$ for $i = 1, \ldots, N_h$.

In order to state the error estimates for the eigenpair approximation by the WG finite element method, we define the WG finite element projection operator $\mathcal{P}_h : V \mapsto V_h$ as follows:

$$a_h(\mathcal{P}_h u, v_h) = \lambda b_h(u, v_h), \quad \forall\, v_h \in V_h. \tag{2.15}$$

It is obvious that the finite element projection operator $\mathcal{P}_h$ has the following error estimates.

Lemma 2.1 ([27]). *For $s \in (0,1]$, assume that the dual of the source problem corresponding to the eigenvalue problem (2.1) has H^{1+s}-regularity. Let the exact solution u be sufficiently smooth such that $u \in H^{m+1}(\Omega)$ with $0 \le m \le r+1$. Then the following error estimates hold:*

$$\|Q_h u - \mathcal{P}_h u\|_{a,h} \le C_1 h^m \|u\|_{m+1}, \tag{2.16}$$
$$\|Q_h u - \mathcal{P}_h u\|_{b,h} \le C_2 h^{m+s} \|u\|_{m+1}. \tag{2.17}$$

Before stating error estimates of the WG finite element method for the eigenvalue problem, we introduce the following lemma.

Lemma 2.2. *For any eigenpair (λ, u) of (2.3), the following equality holds:*

$$(\bar{\lambda}_{j,h} - \lambda) b_h(\mathcal{P}_h u, \bar{u}_{j,h}) = \lambda b_h(u - \mathcal{P}_h u, \bar{u}_{j,h}), \quad j = 1, \ldots, N_h.$$

Proof. Since $-\lambda b_h(\mathcal{P}_h u, \bar{u}_{j,h})$ appears on both sides, we only need to prove that

$$\bar{\lambda}_{j,h} b_h(\mathcal{P}_h u, \bar{u}_{j,h}) = \lambda b_h(u, \bar{u}_{j,h}).$$

From (2.3), (2.9) and (2.15), the following equalities hold:

$$\bar{\lambda}_{j,h} b_h(\mathcal{P}_h u, \bar{u}_{j,h}) = a_h(\mathcal{P}_h u, \bar{u}_{j,h}) = \lambda b(u, \bar{u}_{j,h}).$$

The proof is complete. □

Now, let us consider the error estimates for the first k eigenpair approximations associated with $\bar{\lambda}_{1,h} \le \cdots \le \bar{\lambda}_{k,h}$.

Theorem 2.1. *Let us define the spectral projection operator $\bar{F}_{k,h} : V_h \mapsto \text{span}\{\bar{u}_{1,h}, \cdots, \bar{u}_{k,h}\}$ as follows:*

$$a_h(\bar{F}_{k,h} w_h, \bar{u}_{i,h}) = a_h(w_h, \bar{u}_{i,h}), \quad i = 1, \ldots, k, \quad w_h \in V_h. \tag{2.18}$$

Then the associated exact eigenfunctions $u_1, \cdots, u_k$ of eigenvalue problem (2.3) have the following error estimates:

$$\begin{aligned} &\|Q_h u_i - \bar{F}_{k,h} Q_h u_i\|_{a,h} \\ &\le 2\|Q_h u_i - \mathcal{P}_h u_i\|_{a,h} + \frac{\sqrt{\bar{\mu}_{k+1,h}}}{\delta_{k,i,h}} \|Q_h u_i - \mathcal{P}_h u_i\|_{b,h}, \quad 1 \le i \le k, \end{aligned} \tag{2.19}$$

where $\delta_{k,i,h}$ is defined as

$$\delta_{k,i,h} = \min_{k < j \le N_h} \left| \frac{1}{\bar{\lambda}_{j,h}} - \frac{1}{\lambda_i} \right|. \tag{2.20}$$

Furthermore, these k exact eigenfunctions have the following error estimate in $\|\cdot\|_{b,h}$-norm:

$$\|Q_h u_i - \bar{F}_{k,h} Q_h u_i\|_{b,h} \le \left(2 + \frac{\bar{\mu}_{k+1,h}}{\delta_{k,i,h}}\right) \|Q_h u_i - \mathcal{P}_h u_i\|_{b,h}, \quad 1 \le i \le k. \tag{2.21}$$

Proof. Since $(I - \bar{F}_{k,h})\mathcal{P}_h u_i \in V_h$ and $(I - \bar{F}_{k,h})\mathcal{P}_h u_i \in \text{span}\{\bar{u}_{k+1,h}, \cdots, \bar{u}_{N_h,h}\}$, the following orthogonal expansion holds:

$$(I - \bar{F}_{k,h})\mathcal{P}_h u_i = \sum_{j=k+1}^{N_h} \alpha_j \bar{u}_{j,h}, \tag{2.22}$$

where $\alpha_j = a_h(\mathcal{P}_h u_i, \bar{u}_{j,h})$. From Lemma 2.2, we have

$$\begin{aligned}\alpha_j = a_h(\mathcal{P}_h u_i, \bar{u}_{j,h}) &= \bar{\lambda}_{j,h} b_h(\mathcal{P}_h u_i, \bar{u}_{j,h})\\ &= \frac{\bar{\lambda}_{j,h}\lambda_i}{\bar{\lambda}_{j,h}-\lambda_i} b_h(u_i - \mathcal{P}_h u_i, \bar{u}_{j,h})\\ &= \frac{1}{\mu_i - \bar{\mu}_{j,h}} b_h(u_i - \mathcal{P}_h u_i, \bar{u}_{j,h}).\end{aligned} \tag{2.23}$$

From the orthogonal property of eigenfunctions $\bar{u}_{1,h}, \cdots, \bar{u}_{N_h,h}$, we acquire

$$1 = a_h(\bar{u}_{j,h}, \bar{u}_{j,h}) = \bar{\lambda}_{j,h} b_h(\bar{u}_{j,h}, \bar{u}_{j,h}) = \bar{\lambda}_{j,h}\|\bar{u}_{j,h}\|_{b,h}^2,$$

which leads to the following property:

$$\|\bar{u}_{j,h}\|_{b,h}^2 = \frac{1}{\bar{\lambda}_{j,h}} = \bar{\mu}_{j,h}. \tag{2.24}$$

Because of (2.9) and the definitions of eigenfunctions $\bar{u}_{1,h}, \cdots, \bar{u}_{N_h,h}$, we obtain the following equalities:

$$a_h(\bar{u}_{j,h}, \bar{u}_{k,h}) = \delta_{jk}, \quad b_h\left(\frac{\bar{u}_{j,h}}{\|\bar{u}_{j,h}\|_{b,h}}, \frac{\bar{u}_{k,h}}{\|\bar{u}_{k,h}\|_{b,h}}\right) = \delta_{jk}, \quad 1 \le j,k \le N_h. \tag{2.25}$$

Then due to (2.22)-(2.25), we have the following estimates:

$$\begin{aligned}&\|(I-\bar{F}_{k,h})\mathcal{P}_h u_i\|_{a,h}^2\\ &= \left\|\sum_{j=k+1}^{N_h}\alpha_j \bar{u}_{j,h}\right\|_{a,h}^2 = \sum_{j=k+1}^{N_h}\alpha_j^2\\ &= \sum_{j=k+1}^{N_h}\left(\frac{1}{\mu_i-\bar{\mu}_{j,h}}\right)^2 b_h(u_i-\mathcal{P}_h u_i, \bar{u}_{j,h})^2\\ &\le \frac{1}{\delta_{k,i,h}^2}\sum_{j=k+1}^{N_h}\|\bar{u}_{j,h}\|_{b,h}^2 b_h\left(u_i-\mathcal{P}_h u_i, \frac{\bar{u}_{j,h}}{\|\bar{u}_{j,h}\|_{b,h}}\right)^2\\ &= \frac{1}{\delta_{k,i,h}^2}\sum_{j=k+1}^{N_h}\bar{\mu}_{j,h} b_h\left(u_i-\mathcal{P}_h u_i, \frac{\bar{u}_{j,h}}{\|\bar{u}_{j,h}\|_{b,h}}\right)^2\\ &\le \frac{\bar{\mu}_{k+1,h}}{\delta_{k,i,h}^2}\sum_{j=k+1}^{N_h} b_h\left(u_i-\mathcal{P}_h u_i, \frac{\bar{u}_{j,h}}{\|\bar{u}_{j,h}\|_{b,h}}\right)^2\\ &= \frac{\bar{\mu}_{k+1,h}}{\delta_{k,i,h}^2}\sum_{j=k+1}^{N_h} b_h\left(Q_h u_i-\mathcal{P}_h u_i, \frac{\bar{u}_{j,h}}{\|\bar{u}_{j,h}\|_{b,h}}\right)^2\\ &\le \frac{\bar{\mu}_{k+1,h}}{\delta_{k,i,h}^2}\|Q_h u_i - \mathcal{P}_h u_i\|_{b,h}^2,\end{aligned} \tag{2.26}$$

where the last inequality holds since $\bar{u}_{1,h}/\|\bar{u}_{1,h}\|_{b,h}, \cdots, \bar{u}_{j,h}/\|\bar{u}_{j,h}\|_{b,h}$ constitute an orthonormal basis for the space V_h in the sense of the inner product $b_h(\cdot,\cdot)$.

From (2.26), the following inequality holds:

$$\|(I-\bar{F}_{k,h})\mathcal{P}_h u_i\|_{a,h} \le \frac{\sqrt{\bar{\mu}_{k+1,h}}}{\delta_{k,i,h}}\|Q_h u_i - \mathcal{P}_h u_i\|_{b,h}. \tag{2.27}$$

From (2.27), $\|\bar{F}_{k,h}\|_{a,h} \le 1$ and the triangle inequality, it follows that

$$\begin{aligned}
&\|Q_h u_i - \bar{F}_{k,h} Q_h u_i\|_{a,h} \\
&= \|Q_h u_i - \mathcal{P}_h u_i\|_{a,h} + \|(I - \bar{F}_{k,h})\mathcal{P}_h u_i\|_{a,h} \\
&\quad + \|\bar{F}_{k,h}(\mathcal{P}_h - Q_h)u_i\|_{a,h} \\
&\le \|Q_h u_i - \mathcal{P}_h u_i\|_{a,h} + \|(I - \bar{F}_{k,h})\mathcal{P}_h u_i\|_{a,h} \\
&\quad + \|\bar{F}_{k,h}\|_{a,h}\|(\mathcal{P}_h - Q_h)u_i\|_{a,h} \\
&\le 2\|Q_h u_i - \mathcal{P}_h u_i\|_{a,h} + \frac{\sqrt{\bar{\mu}_{k+1,h}}}{\delta_{k,i,h}}\|Q_h u_i - \mathcal{P}_h u_i\|_{b,h}.
\end{aligned}$$

This is the desired result (2.19).

Similarly, with the help of (2.22)-(2.25), we have following estimates:

$$\begin{aligned}
&\|(I - \bar{F}_{k,h})\mathcal{P}_h u_i\|_{b,h}^2 \\
&= \left\| \sum_{j=k+1}^{N_h} \alpha_j \bar{u}_{j,h} \right\|_{b,h}^2 = \sum_{j=k+1}^{N_h} \alpha_j^2 \, \|\bar{u}_{j,h}\|_{b,h}^2 \\
&= \sum_{j=k+1}^{N_h} \left(\frac{1}{\mu_i - \bar{\mu}_{j,h}} \right)^2 b_h(u_i - \mathcal{P}_h u_i, \bar{u}_{j,h})^2 \|\bar{u}_{j,h}\|_{b,h}^2 \\
&\le \frac{1}{\delta_{k,i,h}^2} \sum_{j=k+1}^{N_h} \|\bar{u}_{j,h}\|_{b,h}^4 \; b_h\left(u_i - \mathcal{P}_h u_i, \frac{\bar{u}_{j,h}}{\|\bar{u}_{j,h}\|_{b,h}} \right)^2 \\
&= \frac{1}{\delta_{k,i,h}^2} \sum_{j=k+1}^{N_h} \bar{\mu}_{j,h}^2 b_h\left(Q_h u_i - \mathcal{P}_h u_i, \frac{\bar{u}_{j,h}}{\|\bar{u}_{j,h}\|_{b,h}} \right)^2 \\
&\le \frac{\bar{\mu}_{k+1,h}^2}{\delta_{k,i,h}^2} \|Q_h u_i - \mathcal{P}_h u_i\|_{b,h}^2,
\end{aligned}$$

which leads to the inequality

$$\|(I - \bar{F}_{k,h})\mathcal{P}_h u_i\|_{b,h} \le \frac{\bar{\mu}_{k+1,h}}{\delta_{k,i,h}} \|Q_h u_i - \mathcal{P}_h u_i\|_{b,h}. \tag{2.28}$$

From the definition of spectral projection (2.18), for any $w \in V_h$, we have

$$\bar{\lambda}_{i,h} b_h(\bar{F}_{k,h} w, \bar{u}_{i,h}) = a_h(\bar{F}_{k,h} w, \bar{u}_{i,h}) = a_h(w, \bar{u}_{i,h}) = \bar{\lambda}_{i,h} b_h(w, \bar{u}_{i,h}), \quad i = 1, \ldots, k.$$

This means the following equation holds:

$$b_h(\bar{F}_{k,h} w, \bar{u}_{i,h}) = b_h(w, \bar{u}_{i,h}), \quad i = 1, \ldots, k, \quad \forall\, w \in V_h,$$

which leads to $\|\bar{F}_{k,h}\|_{b,h} \le 1$. From (2.28), $\|\bar{F}_{k,h}\|_{b,h} \le 1$ and the triangle inequality, we find the following error estimates for the eigenfunction approximations in the $\|\cdot\|_{b,h}$-norm:

$$\begin{aligned}
&\|Q_h u_i - \bar{F}_{k,h} Q_h u_i\|_{b,h} \\
&\le \|Q_h u_i - \mathcal{P}_h u_i\|_{b,h} + \|(I - \bar{F}_{k,h})\mathcal{P}_h u_i\|_{b,h} + \|\bar{F}_{k,h}(\mathcal{P}_h u_i - Q_h u_i)\|_{b,h} \\
&\le (1 + \|\bar{F}_{k,h}\|_{b,h})\|\mathcal{P}_h u_i - Q_h u_i\|_{b,h} + \|(I - \bar{F}_{k,h})\mathcal{P}_h u_i\|_{b,h} \\
&\le \left(2 + \frac{\bar{\mu}_{k+1,h}}{\delta_{k,i,h}} \right) \|Q_h u_i - \mathcal{P}_h u_i\|_{b,h}.
\end{aligned}$$

This is the second desired result (2.21) and the proof is complete. $\square$

For the sake of simplicity in notation and to make sense of the estimates (2.19) and (2.21), we assume that the eigenvalue gap $\delta_{k,i,h}$ has a uniform lower bound, which is represented by $\delta_{k,i}$ (which can be understood as the "true" separation of the eigenvalues $\lambda_1, \cdots, \lambda_k$ from the unwanted eigenvalues) in the following sections of this paper. When the mesh size is sufficiently small, this assumption makes sense. Based on Theorem 2.1 and the convergence consequences of the WG finite element method for the boundary value problem, we then acquire the following convergence order.

Corollary 2.1. *Under the conditions of Lemma* 2.1*, Theorem* 2.1 *and* $\delta_{k,i,h}$ *having a uniform lower bound* $\delta_{k,i}$*, the following error estimates hold:*

$$\|Q_h u_i - \bar{F}_{k,h} Q_h u_i\|_{a,h} \le C_3 h^m \|u\|_{m+1}, \qquad 1 \le i \le k, \tag{2.29}$$

$$\|Q_h u_i - \bar{F}_{k,h} Q_h u_i\|_{b,h} \le C_4 h^{m+s} \|u\|_{m+1}, \quad 1 \le i \le k. \tag{2.30}$$

The following theorem gives the error estimates for the one eigenpair approximation and the proof is similar to that of Theorem 2.1.

Theorem 2.2. *Let* (λ, u) *denote an exact eigenpair of the eigenvalue problem* (2.3)*. Assume the eigenpair approximation* $(\bar{\lambda}_{i,h}, \bar{u}_{i,h})$ *has the property that* $\bar{\mu}_{i,h} = 1/\bar{\lambda}_{i,h}$ *is the closest to* $\mu = 1/\lambda$*. The corresponding spectral projection operator* $E_{i,h} : V_h \mapsto \operatorname{span}\{\bar{u}_{i,h}\}$ *is defined as follows:*

$$a_h(E_{i,h} w, \bar{u}_{i,h}) = a_h(w, \bar{u}_{i,h}), \quad w \in V_h.$$

Then the following error estimate holds:

$$\|Q_h u - E_{i,h} Q_h u\|_{a,h} \le 2\|Q_h u - \mathcal{P}_h u\|_{a,h} + \frac{\sqrt{\bar{\mu}_{1,h}}}{\delta_{\lambda,h}} \|Q_h u - \mathcal{P}_h u\|_{b,h}, \tag{2.31}$$

where $\delta_{\lambda,h}$ *is defined as follows:*

$$\delta_{\lambda,h} := \min_{j \ne i} |\bar{\mu}_{j,h} - \mu| = \min_{j \ne i} \left| \frac{1}{\bar{\lambda}_{j,h}} - \frac{1}{\lambda} \right|. \tag{2.32}$$

Furthermore, the eigenfunction approximation $\bar{u}_{i,h}$ *has the following error estimate in* $\|\cdot\|_{b,h}$*-norm:*

$$\|Q_h u - E_{i,h} Q_h u\|_{b,h} \le \left(2 + \frac{\bar{\mu}_{1,h}}{\delta_{\lambda,h}}\right) \|Q_h u_i - \mathcal{P}_h u_i\|_{b,h}. \tag{2.33}$$

Proof. Since

$$(I - E_{i,h})\mathcal{P}_h u \in V_h, \quad (I - E_{i,h})\mathcal{P}_h u \in \operatorname{span}\{\bar{u}_{1,h}, \cdots, \bar{u}_{i-1,h}, \bar{u}_{i+1,h}, \cdots, \bar{u}_{N_h,h}\},$$

the following orthogonal expansion holds:

$$(I - E_{i,h})\mathcal{P}_h u = \sum_{j \ne i} \alpha_j \bar{u}_{j,h}, \tag{2.34}$$

where $\alpha_j = a_h(\mathcal{P}_h u, \bar{u}_{j,h})$ has the same equality (2.23). Then due to (2.23)-(2.25) and (2.34), we have the following estimates:

$$\|(I - E_{i,h})\mathcal{P}_h u\|_{a,h}^2 = \left\| \sum_{j \ne i} \alpha_j \bar{u}_{j,h} \right\|_{a,h}^2 = \sum_{j \ne i} \alpha_j^2$$

$$
\begin{aligned}
&= \sum_{j\neq i}\left(\frac{1}{\mu-\bar{\mu}_{j,h}}\right)^2 b_h(u-\mathcal{P}_hu,\bar{u}_{j,h})^2\\
&\leq \frac{1}{\delta_{\lambda,h}^2}\sum_{j\neq i}\|\bar{u}_{j,h}\|_{b,h}^2 b_h\bigg(u-\mathcal{P}_hu,\frac{\bar{u}_{j,h}}{\|\bar{u}_{j,h}\|_{b,h}}\bigg)^2\\
&= \frac{1}{\delta_{\lambda,h}^2}\sum_{j\neq i}\bar{\mu}_{j,h} b_h\bigg(u-\mathcal{P}_hu,\frac{\bar{u}_{j,h}}{\|\bar{u}_{j,h}\|_{b,h}}\bigg)^2\\
&\leq \frac{\bar{\mu}_{1,h}}{\delta_{\lambda,h}^2}\sum_{j\neq i} b_h\bigg(u-\mathcal{P}_hu,\frac{\bar{u}_{j,h}}{\|\bar{u}_{j,h}\|_{b,h}}\bigg)^2\\
&= \frac{\bar{\mu}_{1,h}}{\delta_{\lambda,h}^2}\sum_{j\neq i} b_h\bigg(Q_hu-\mathcal{P}_hu,\frac{\bar{u}_{j,h}}{\|\bar{u}_{j,h}\|_{b,h}}\bigg)^2\\
&\leq \frac{\bar{\mu}_{1,h}}{\delta_{\lambda,h}^2}\|Q_hu-\mathcal{P}_hu\|_{b,h}^2, \qquad (2.35)
\end{aligned}
$$

where the last inequality holds since $\bar{u}_{1,h}/\|\bar{u}_{1,h}\|_{b,h},\cdots,\bar{u}_{j,h}/\|\bar{u}_{j,h}\|_{b,h}$ constitute an orthonormal basis for the space V_h in the sense of the inner product $b_h(\cdot,\cdot)$.

From (2.35), the following inequality holds:

$$
\|(I-E_{i,h})\mathcal{P}_hu\|_{a,h}\leq \frac{\sqrt{\bar{\mu}_{1,h}}}{\delta_{\lambda,h}}\|Q_hu-\mathcal{P}_hu\|_{b,h}. \qquad (2.36)
$$

From (2.36), $\|E_{i,h}\|_{a,h}\leq 1$ and the triangle inequality, it follows that

$$
\begin{aligned}
&\|Q_hu-E_{i,h}Q_hu\|_{a,h}\\
&= \|Q_hu-\mathcal{P}_hu\|_{a,h}+\|(I-E_{i,h})\mathcal{P}_hu\|_{a,h}+\|E_{i,h}(\mathcal{P}_h-Q_h)u\|_{a,h}\\
&\leq \|Q_hu-\mathcal{P}_hu\|_{a,h}+\|(I-E_{i,h})\mathcal{P}_hu\|_{a,h}+\|E_{i,h}\|_{a,h}\|(\mathcal{P}_h-Q_h)u\|_{a,h}\\
&\leq 2\|Q_hu-\mathcal{P}_hu\|_{a,h}+\frac{\sqrt{\bar{\mu}_{1,h}}}{\delta_{\lambda,h}}\|Q_hu-\mathcal{P}_hu\|_{b,h}.
\end{aligned}
$$

This is the desired result (2.31).

Similarly, with the help of (2.23)-(2.25) and (2.34), we have the following estimates:

$$
\begin{aligned}
&\|(I-E_{i,h})\mathcal{P}_hu\|_{b,h}^2\\
&= \bigg\|\sum_{j\neq i}\alpha_j\bar{u}_{j,h}\bigg\|_{b,h}^2 = \sum_{j\neq i}\alpha_j^2\|\bar{u}_{j,h}\|_{b,h}^2\\
&= \sum_{j\neq i}\left(\frac{1}{\mu-\bar{\mu}_{j,h}}\right)^2 b_h(u-\mathcal{P}_hu,\bar{u}_{j,h})^2\|\bar{u}_{j,h}\|_{b,h}^2\\
&\leq \frac{1}{\delta_{\lambda,h}^2}\sum_{j\neq i}\|\bar{u}_{j,h}\|_{b,h}^4\ b_h\bigg(u-\mathcal{P}_hu,\frac{\bar{u}_{j,h}}{\|\bar{u}_{j,h}\|_{b,h}}\bigg)^2\\
&= \frac{1}{\delta_{\lambda,h}^2}\sum_{j\neq i}\bar{\mu}_{j,h}^2 b_h\bigg(Q_hu-\mathcal{P}_hu,\frac{\bar{u}_{j,h}}{\|\bar{u}_{j,h}\|_{b,h}}\bigg)^2\\
&\leq \frac{\bar{\mu}_{1,h}^2}{\delta_{\lambda,h}^2}\|Q_hu-\mathcal{P}_hu\|_{b,h}^2,
\end{aligned}
$$

which leads to the inequality

$$\|(I-E_{i,h})\mathcal{P}_h u\|_{b,h} \le \frac{\bar{\mu}_{1,h}}{\delta_{\lambda,h}}\|Q_h u-\mathcal{P}_h u\|_{b,h}. \tag{2.37}$$

Similarly to the proof of Theorem 2.1, we also have $\|E_{i,h}\|_{b,h} \le 1$. Then from (2.37) and the triangle inequality, we find the following error estimates for the eigenfunction approximations in the $\|\cdot\|_{b,h}$-norm:

$$\begin{aligned}
&\|Q_h u - E_{i,h}Q_h u\|_{b,h}\\
\le{}& \|Q_h u-\mathcal{P}_h u\|_{b,h} + \|(I-E_{i,h})\mathcal{P}_h u\|_{b,h} + \|E_{i,h}(\mathcal{P}_h u - Q_h u)\|_{b,h}\\
\le{}& (1+\|E_{i,h}\|_{b,h})\|\mathcal{P}_h u - Q_h u\|_{b,h} + \|(I-E_{i,h})\mathcal{P}_h u\|_{b,h}\\
\le{}& \left(2+\frac{\bar{\mu}_{1,h}}{\delta_{\lambda,h}}\right)\|Q_h u-\mathcal{P}_h u\|_{b,h}.
\end{aligned}$$

This is the second desired result (2.33) and the proof is complete. □

Likewise, for the sake of simplicity in notation and to make sense of the estimates (2.31) and (2.33), we assume that the eigenvalue gap $\delta_{\lambda,h}$ defined by (2.32) equally has a uniform lower bound, indicated by δ_λ, which can be understood as the "true" separation of the eigenvalue λ from others in the following sections of this paper. When the mesh size is small enough, this assumption is also reasonable. Next, we have the following convergence result for the eigenvalue problems using the WG finite element method, which is based on Theorem 2.2.

Corollary 2.2. *Under the conditions of Lemma* 2.1*, Theorem* 2.2 *and $\delta_{\lambda,h}$ having a uniform lower bound δ_λ, the following error estimates hold:*

$$\|Q_h u - E_{i,h}Q_h u\|_{a,h} \le C_5 h^m \|u\|_{m+1}, \tag{2.38}$$
$$\|Q_h u - E_{i,h}Q_h u\|_{b,h} \le C_6 h^{m+s} \|u\|_{m+1}. \tag{2.39}$$

Remark 2.1. The convergence analysis of the WG finite element method for eigenvalue problems has been provided in [34]. Compared with the results there, the convergence results in Theorems 2.1 and 2.2 are sharper and give the explicit dependence of the included constants on the eigenvalue distributions.

3. Augmented Subspace Method and Its Error Estimates

The augmented subspace techniques for the WG eigenvalue problem (2.9) are first laid out in this section. These schemes involve solving the eigenvalue problem on the augmented subspace $V_{H,h}$, which is generated by the coarse conforming linear finite element space W_H, and a WG finite element function in the fine finite element space V_h. They also involve solving the auxiliary linear boundary value problem in the fine finite element space V_h. Next, the related analysis of convergence for these augmented subspace schemes is addressed.

As in [30], we first create a coarse mesh $\mathcal{T}_H$, which is a quasi-uniform decomposition of Ω, consisting of simplicial elements, with the mesh size H. The corresponding conforming linear finite element space W_H is defined on the mesh $\mathcal{T}_H$. We denote by V_H the coarse space defined on $\mathcal{T}_H$, and by V_h the finite element space defined on the fine mesh $\mathcal{T}_h$. These notations allow us to design the augmented subspace technique and conduct the numerical experiments in the next section. The coarse conforming linear finite element space W_H is a subspace of the fine

WG finite element space V_h. This is because, for the sake of simplicity, we assume in this paper that the coarse mesh $\mathcal{T}_H$ and the fine mesh $\mathcal{T}_h$ have the nested property.

For the positive integer ℓ and given eigenfunction approximations $u_{1,h}^{(\ell)},\cdots,u_{k,h}^{(\ell)}$ which are the approximations for the first k eigenfunctions $\bar{u}_{1,h},\cdots,\bar{u}_{k,h}$ of (2.9), we can do the augmented subspace iteration step which is defined by Algorithm 3.1 to improve the accuracy of $u_{1,h}^{(\ell)},\cdots,u_{k,h}^{(\ell)}$.

For each ℓ, it is easy to know that the eigenvalue problems (3.1) and (3.3) have the following eigenvalues [2,3]:

$$0<\lambda_{1,h}^{(\ell)}\leq\lambda_{2,h}^{(\ell)}\leq\cdots\leq\lambda_{k,h}^{(\ell)}\leq\cdots\leq\lambda_{N_{H,h},h}^{(\ell)},$$

and corresponding eigenfunctions

$$u_{1,h}^{(\ell)},u_{2,h}^{(\ell)},\cdots,u_{k,h}^{(\ell)},\cdots,u_{N_{H,h},h}^{(\ell)}, \tag{3.4}$$

where

$$N_{H,h}=\dim V_{H,h}^{(\ell)}=N_H+k,\quad a_h\big(u_{i,h}^{(\ell)},u_{j,h}^{(\ell)}\big)=\delta_{ij},\quad 1\leq i,j\leq N_{H,h}.$$

From the min-max principle [2,3] and $V_{H,h}^{(\ell)}\subset V_h$, the eigenvalues $\lambda_{1,h}^{(\ell)},\cdots,\lambda_{N_{H,h},h}^{(\ell)}$ provide the upper bounds for the first $N_{H,h}$ eigenvalues of (2.9)

$$\bar{\lambda}_{i,h}\leq\lambda_{i,h}^{(\ell)},\quad \bar{\mu}_{i,h}\geq\mu_{i,h}^{(\ell)},\quad 1\leq i\leq N_{H,h}. \tag{3.5}$$

Since the low dimensional augmented subspace $V_{H,h}^{(\ell)}$ is a subspace of the WG finite element space V_h, the error estimates of the eigenfunction approximations $u_{1,h}^{(\ell)},\cdots,u_{k,h}^{(\ell)}$ to the eigenfunctions

Algorithm 3.1: Augmented Subspace Method for the First k Eigenpairs.

1. For $\ell=1$, we define $\widehat{u}_{i,h}^{(\ell)}=u_{i,h}^{(\ell)}$, $i=1,\ldots,k$, and the augmented subspace $V_{H,h}^{(\ell)}=W_H+\operatorname{span}\{\widehat{u}_{1,h}^{(\ell)},\cdots,\widehat{u}_{k,h}^{(\ell)}\}$. Then solve the following eigenvalue problem: Find $(\lambda_{i,h}^{(\ell)},u_{i,h}^{(\ell)})\in\mathbb{R}\times V_{H,h}^{(\ell)}$ such that $a_h(u_{i,h}^{(\ell)},u_{i,h}^{(\ell)})=1$ and
$$a_h\big(u_{i,h}^{(\ell)},v_{H,h}\big)=\lambda_{i,h}^{(\ell)}b_h\big(u_{i,h}^{(\ell)},v_{H,h}\big),\quad\forall\, v_{H,h}\in V_{H,h}^{(\ell)},\quad i=1,\ldots,k. \tag{3.1}$$
2. Solve the following linear boundary value problem: Find $\widehat{u}_{i,h}^{(\ell+1)}\in V_h$ such that
$$a_h\big(\widehat{u}_{i,h}^{(\ell+1)},v_h\big)=\lambda_{i,h}^{(\ell)}b_h\big(u_{i,h}^{(\ell)},v_h\big),\quad\forall\, v_h\in V_h,\quad i=1,\ldots,k. \tag{3.2}$$
3. Define the augmented subspace $V_{H,h}^{(\ell+1)}=W_H+\operatorname{span}\{\widehat{u}_{1,h}^{(\ell+1)},\cdots,\widehat{u}_{k,h}^{(\ell+1)}\}$ and solve the following eigenvalue problem: Find $(\lambda_{i,h}^{(\ell+1)},u_{i,h}^{(\ell+1)})\in\mathbb{R}\times V_{H,h}^{(\ell+1)}$ such that $a_h(u_{i,h}^{(\ell+1)},u_{i,h}^{(\ell+1)})=1$ and
$$a_h\big(u_{i,h}^{(\ell+1)},v_{H,h}\big)=\lambda_{i,h}^{(\ell+1)}b_h\big(u_{i,h}^{(\ell+1)},v_{H,h}\big),\quad\forall\, v_{H,h}\in V_{H,h}^{(\ell+1)},\quad i=1,\ldots,k. \tag{3.3}$$
Solve (3.3) to obtain $(\lambda_{1,h}^{(\ell+1)},u_{1,h}^{(\ell+1)}),\cdots,(\lambda_{k,h}^{(\ell+1)},u_{k,h}^{(\ell+1)})$.
4. Set $\ell=\ell+1$ and go to Step 2 for the next iteration until convergence.

$\bar{u}_{1,h}, \cdots, \bar{u}_{k,h}$ can be deduced from the similar way of the conforming finite element method for eigenvalue problems [9].

In order to give the error estimates for the augmented subspace method defined by Algorithm 3.1, we define the subspace projection operator $\mathcal{P}_{H,h}^{(\ell)} : V_h \mapsto V_{H,h}^{(\ell)}$ as follows:

$$a_h\big(\mathcal{P}_{H,h}^{(\ell)} w_h, v_{H,h}\big) = a_h(w_h, v_{H,h}), \quad \forall\, v_{H,h} \in V_{H,h}^{(\ell)}, \quad w_h \in V_h. \tag{3.6}$$

For the error estimate of $\|w_h - \mathcal{P}_{H,h}^{(\ell)} w_h\|_{b,h}$, we define the following quantity:

$$\eta_a(W_H) = \sup_{\substack{f\in L^2(\Omega)\\ \|f\|_{b,h}=1}} \inf_{w_H\in W_H} \|T_h f - w_H\|_{a,h}, \tag{3.7}$$

where $T_h : L^2(\Omega) \mapsto V_h$ is defined as

$$a_h(T_h f, v) = b_h(f, v), \quad \forall\, v \in V_h, \quad f \in L^2(\Omega). \tag{3.8}$$

Then the projection operator $\mathcal{P}_{H,h}^{(\ell)}$ has the following error estimates:

$$\big\|w_h - \mathcal{P}_{H,h}^{(\ell)} w_h\big\|_{a,h} = \inf_{v_{H,h}\in V_{H,h}^{(\ell)}} \|w_h - v_{H,h}\|_{a,h}, \qquad w_h \in V_h, \tag{3.9}$$

$$\big\|w_h - \mathcal{P}_{H,h}^{(\ell)} w_h\big\|_{b,h} \le \eta_a(W_H)\big\|w_h - \mathcal{P}_{H,h}^{(\ell)} w_h\big\|_{a,h}, \quad w_h \in V_h. \tag{3.10}$$

Lemma 3.1. *Let us define the spectral projection operator $F_{k,h}^{(m)} : V_h \mapsto \mathrm{span}\{u_{1,h}^{(m)}, \cdots, u_{k,h}^{(m)}\}$ for any integer $m \ge 1$ as follows:*

$$a_h\big(F_{k,h}^{(m)} w, u_{i,h}^{(m)}\big) = a_h\big(w, u_{i,h}^{(m)}\big), \quad i = 1, \ldots, k, \quad w \in V_h. \tag{3.11}$$

Then the eigenfunctions $\bar{u}_{1,h}, \cdots, \bar{u}_{k,h}$ of (2.9) and the eigenfunction approximations $u_{1,h}^{(\ell+1)}$, $\cdots$, $u_{k,h}^{(\ell+1)}$ from Algorithm 3.1 with the integer $\ell \ge 1$ have the following error estimate:

$$\big\|\bar{u}_{i,h} - F_{k,h}^{(\ell)} \bar{u}_{i,h}\big\|_{a,h} \le \sqrt{1 + \frac{\bar{\mu}_{k+1,h}}{\big(\delta_{k,i,h}^{(\ell)}\big)^2}\eta_a^2(W_H)}\,\big\|\big(I - \mathcal{P}_{H,h}^{(\ell)}\big)\bar{u}_{i,h}\big\|_{a,h}, \tag{3.12}$$

where $\delta_{k,i,h}^{(\ell)}$ is defined as

$$\delta_{k,i,h}^{(\ell)} = \min_{k<j\le N_h} \left|\frac{1}{\lambda_{j,h}^{(\ell)}} - \frac{1}{\bar{\lambda}_{i,h}}\right|. \tag{3.13}$$

Furthermore, the following $\|\cdot\|_{b,h}$-norm error estimate holds:

$$\big\|\bar{u}_{i,h} - F_{k,h}^{(\ell)} \bar{u}_{i,h}\big\|_{b,h} \le \bar{\eta}_a(W_H)\big\|\bar{u}_{i,h} - F_{k,h}^{(\ell)} \bar{u}_{i,h}\big\|_{a,h}, \tag{3.14}$$

where

$$\bar{\eta}_a(W_H) = \left(1 + \frac{\bar{\mu}_{k+1,h}}{\delta_{k,i,h}^{(\ell)}}\right)\eta_a(W_H). \tag{3.15}$$

Proof. Since

$$\big(I - F_{k,h}^{(\ell)}\big)\mathcal{P}_{H,h}^{(\ell)} \bar{u}_{i,h} \in V_{H,h}^{(\ell)}, \quad \big(I - F_{k,h}^{(\ell)}\big)\mathcal{P}_{H,h}^{(\ell)} \bar{u}_{i,h} \in \mathrm{span}\{u_{k+1,h}^{(\ell)}, \cdots, u_{N_{H,h},h}^{(\ell)}\},$$

the following orthogonal expansion holds:

$$(I - F_{k,h}^{(\ell)})\mathcal{P}_{H,h}^{(\ell)}\bar{u}_{i,h} = \sum_{j=k+1}^{N_{H,h}} \alpha_j u_{j,h}^{(\ell)}, \tag{3.16}$$

where $\alpha_j = a_h(\mathcal{P}_{H,h}^{(\ell)}\bar{u}_{i,h}, u_{j,h}^{(\ell)})$. By virtue of Lemma 2.2, we have

$$\begin{aligned}
\alpha_j &= a_h\big(\mathcal{P}_{H,h}^{(\ell)}\bar{u}_{i,h}, u_{j,h}^{(\ell)}\big) = \lambda_{j,h}^{(\ell)} b_h\big(\mathcal{P}_{H,h}^{(\ell)}\bar{u}_{i,h}, u_{j,h}^{(\ell)}\big) \\
&= \frac{\lambda_{j,h}^{(\ell)}\bar{\lambda}_{i,h}}{\lambda_{j,h}^{(\ell)} - \bar{\lambda}_{i,h}} b_h\big(\bar{u}_{i,h} - \mathcal{P}_{H,h}^{(\ell)}\bar{u}_{i,h}, u_{j,h}^{(\ell)}\big) \\
&= \frac{1}{\bar{\mu}_{i,h} - \mu_{j,h}^{(\ell)}} b_h\big(\bar{u}_{i,h} - \mathcal{P}_{H,h}^{(\ell)}\bar{u}_{i,h}, u_{j,h}^{(\ell)}\big).
\end{aligned} \tag{3.17}$$

From the orthogonal property of eigenfunctions $u_{1,h}^{(\ell)}, \cdots, u_{N_{H,h},h}^{(\ell)}$, we have

$$1 = a_h\big(u_{j,h}^{(\ell)}, u_{j,h}^{(\ell)}\big) = \lambda_{j,h}^{(\ell)} b_h\big(u_{j,h}^{(\ell)}, u_{j,h}^{(\ell)}\big) = \lambda_{j,h}^{(\ell)} \|u_{j,h}^{(\ell)}\|_{b,h}^2,$$

which leads to the following property:

$$\|u_{j,h}^{(\ell)}\|_{b,h}^2 = \frac{1}{\lambda_{j,h}^{(\ell)}} = \mu_{j,h}^{(\ell)}. \tag{3.18}$$

Because of (3.1), (3.3) and the definitions of eigenfunctions $u_{1,h}^{(\ell)}, \cdots, u_{N_{H,h},h}^{(\ell)}$, we obtain the following equalities:

$$a_h\big(u_{j,h}^{(\ell)}, u_{k,h}^{(\ell)}\big) = \delta_{jk}, \quad b_h\left(\frac{u_{j,h}^{(\ell)}}{\|u_{j,h}^{(\ell)}\|_{b,h}}, \frac{u_{k,h}^{(\ell)}}{\|u_{k,h}^{(\ell)}\|_{b,h}}\right) = \delta_{jk}, \quad 1 \le j,k \le N_{H,h}. \tag{3.19}$$

Then due to (3.5) and (3.16)-(3.19), we have the following estimates:

$$\begin{aligned}
&\big\|(I - F_{k,h}^{(\ell)})\mathcal{P}_{H,h}^{(\ell)}\bar{u}_{i,h}\big\|_{a,h}^2 \\
&= \left\|\sum_{j=k+1}^{N_{H,h}} \alpha_j u_{j,h}^{(\ell)}\right\|_{a,h}^2 = \sum_{j=k+1}^{N_{H,h}} \alpha_j^2 \\
&= \sum_{j=k+1}^{N_{H,h}} \left(\frac{1}{\bar{\mu}_{i,h} - \mu_{j,h}^{(\ell)}}\right)^2 b_h\big(\bar{u}_{i,h} - \mathcal{P}_{H,h}^{(\ell)}\bar{u}_{i,h}, u_{j,h}^{(\ell)}\big)^2 \\
&\le \frac{1}{\big(\delta_{k,i,h}^{(\ell)}\big)^2} \sum_{j=k+1}^{N_{H,h}} \|u_{j,h}^{(\ell)}\|_{b,h}^2 b_h\left(\bar{u}_{i,h} - \mathcal{P}_{H,h}^{(\ell)}\bar{u}_{i,h}, \frac{u_{j,h}^{(\ell)}}{\|u_{j,h}^{(\ell)}\|_{b,h}}\right)^2 \\
&= \frac{1}{\big(\delta_{k,i,h}^{(\ell)}\big)^2} \sum_{j=k+1}^{N_{H,h}} \mu_{j,h}^{(\ell)} b_h\left(\bar{u}_{i,h} - \mathcal{P}_{H,h}^{(\ell)}\bar{u}_{i,h}, \frac{u_{j,h}^{(\ell)}}{\|u_{j,h}^{(\ell)}\|_{b,h}}\right)^2 \\
&\le \frac{\mu_{k+1,h}^{(\ell)}}{\big(\delta_{k,i,h}^{(\ell)}\big)^2} \sum_{j=k+1}^{N_{H,h}} b_h\left(\bar{u}_{i,h} - \mathcal{P}_{H,h}^{(\ell)}\bar{u}_{i,h}, \frac{u_{j,h}^{(\ell)}}{\|u_{j,h}^{(\ell)}\|_{b,h}}\right)^2 \\
&\le \frac{\mu_{k+1,h}^{(\ell)}}{\big(\delta_{k,i,h}^{(\ell)}\big)^2} \big\|\bar{u}_{i,h} - \mathcal{P}_{H,h}^{(\ell)}\bar{u}_{i,h}\big\|_{b,h}^2,
\end{aligned} \tag{3.20}$$

where the last inequality holds since $u_{1,h}^{(\ell)}/\|u_{1,h}^{(\ell)}\|_{b,h},\cdots,u_{N_{H,h},h}^{(\ell)}/\|u_{N_{H,h},h}^{(\ell)}\|_{b,h}$ constitute an orthonormal basis for the space $V_{H,h}^{(\ell)}$ in the sense of the inner product $b_h(\cdot,\cdot)$. Combining (3.5) and (3.20) leads to the following inequality:

$$\big\|(I-F_{k,h}^{(\ell)})\mathcal{P}_{H,h}^{(\ell)}\bar{u}_{i,h}\big\|_{a,h}^2\le\frac{\bar{\mu}_{k+1,h}}{\big(\delta_{k,i,h}^{(\ell)}\big)^2}\eta_a^2(W_H)\big\|(I-\mathcal{P}_{H,h}^{(\ell)})\bar{u}_{i,h}\big\|_{a,h}^2. \tag{3.21}$$

From (3.21) and the orthogonal property $a_h((I-\mathcal{P}_{H,h}^{(\ell)})\bar{u}_{i,h},(I-F_{k,h}^{(\ell)})\mathcal{P}_{H,h}^{(\ell)}\bar{u}_{i,h})=0$, it follows that

$$\begin{aligned}&\big\|\bar{u}_{i,h}-F_{k,h}^{(\ell)}\bar{u}_{i,h}\big\|_{a,h}^2\\&=\big\|\bar{u}_{i,h}-\mathcal{P}_{H,h}^{(\ell)}\bar{u}_{i,h}\big\|_{a,h}^2+\big\|(I-F_{k,h}^{(\ell)})\mathcal{P}_{H,h}^{(\ell)}\bar{u}_{i,h}\big\|_{a,h}^2\\&\le\left(1+\frac{\bar{\mu}_{k+1,h}}{\big(\delta_{k,i,h}^{(\ell)}\big)^2}\eta_a^2(W_H)\right)\big\|(I-\mathcal{P}_{H,h}^{(\ell)})\bar{u}_{i,h}\big\|_{a,h}^2.\end{aligned}$$

This is the desired result (3.12).

Similarly, with the help of (3.5) and (3.16)-(3.19), we have the following estimates:

$$\begin{aligned}&\big\|(I-F_{k,h}^{(\ell)})\mathcal{P}_{H,h}^{(\ell)}\bar{u}_{i,h}\big\|_{b,h}^2\\&=\Big\|\sum_{j=k+1}^{N_{H,h}}\alpha_j u_{j,h}^{(\ell)}\Big\|_{b,h}^2=\sum_{j=k+1}^{N_{H,h}}\alpha_j^2\|u_{j,h}^{(\ell)}\|_{b,h}^2\\&=\sum_{j=k+1}^{N_{H,h}}\left(\frac{1}{\bar{\mu}_{i,h}-\mu_{j,h}^{(\ell)}}\right)^2 b_h\big(\bar{u}_{i,h}-\mathcal{P}_{H,h}^{(\ell)}\bar{u}_{i,h},u_{j,h}^{(\ell)}\big)^2\|u_{j,h}^{(\ell)}\|_{b,h}^2\\&\le\frac{1}{\big(\delta_{k,i,h}^{(\ell)}\big)^2}\sum_{j=k+1}^{N_{H,h}}\|u_{j,h}^{(\ell)}\|_{b,h}^4\ b_h\left(\bar{u}_{i,h}-\mathcal{P}_{H,h}^{(\ell)}\bar{u}_{i,h},\frac{u_{j,h}^{(\ell)}}{\|u_{j,h}^{(\ell)}\|_{b,h}}\right)^2\\&=\frac{1}{\big(\delta_{k,i,h}^{(\ell)}\big)^2}\sum_{j=k+1}^{N_{H,h}}\big(\mu_{j,h}^{(\ell)}\big)^2 b_h\left(\bar{u}_{i,h}-\mathcal{P}_{H,h}^{(\ell)}\bar{u}_{i,h},\frac{u_{j,h}^{(\ell)}}{\|u_{j,h}^{(\ell)}\|_{b,h}}\right)^2\\&\le\frac{\big(\mu_{k+1,h}^{(\ell)}\big)^2}{\big(\delta_{k,i,h}^{(\ell)}\big)^2}\big\|\bar{u}_{i,h}-\mathcal{P}_{H,h}^{(\ell)}\bar{u}_{i,h}\big\|_{b,h}^2\\&\le\frac{\bar{\mu}_{k+1,h}^2}{\big(\delta_{k,i,h}^{(\ell)}\big)^2}\big\|\bar{u}_{i,h}-\mathcal{P}_{H,h}^{(\ell)}\bar{u}_{i,h}\big\|_{b,h}^2,\end{aligned}$$

which leads to the inequality

$$\big\|(I-F_{k,h}^{(\ell)})\mathcal{P}_{H,h}^{(\ell)}\bar{u}_{i,h}\big\|_{b,h}\le\frac{\bar{\mu}_{k+1,h}}{\delta_{k,i,h}^{(\ell)}}\big\|\bar{u}_{i,h}-\mathcal{P}_{H,h}^{(\ell)}\bar{u}_{i,h}\big\|_{b,h}. \tag{3.22}$$

According to (3.10), (3.22) and the triangle inequality, we have the following error estimates for the eigenfunction approximations in the $\|\cdot\|_{b,h}$-norm:

$$\big\|\bar{u}_{i,h}-F_{k,h}^{(\ell)}\bar{u}_{i,h}\big\|_{b,h}\le\big\|\bar{u}_{i,h}-\mathcal{P}_{H,h}^{(\ell)}\bar{u}_{i,h}\big\|_{b,h}+\big\|(I-F_{k,h}^{(\ell)})\mathcal{P}_{H,h}^{(\ell)}\bar{u}_{i,h}\big\|_{b,h}$$

$$
\begin{aligned}
&\leq \left(1+\frac{\bar{\mu}_{k+1,h}}{\delta_{k,i,h}^{(\ell)}}\right)\big\|(I-\mathcal{P}_{H,h}^{(\ell)})\bar{u}_{i,h}\big\|_{b,h}\\
&\leq \left(1+\frac{\bar{\mu}_{k+1,h}}{\delta_{k,i,h}^{(\ell)}}\right)\eta_a(W_H)\big\|(I-\mathcal{P}_{H,h}^{(\ell)})\bar{u}_{i,h}\big\|_{a,h}\\
&\leq \left(1+\frac{\bar{\mu}_{k+1,h}}{\delta_{k,i,h}^{(\ell)}}\right)\eta_a(W_H)\big\|\bar{u}_{i,h}-F_{k,h}^{(\ell)}\bar{u}_{i,h}\big\|_{a,h}.
\end{aligned}
$$

This is the second desired result (3.14) and the proof is complete. □

Theorem 3.1. *Under the conditions of Lemma* 3.1, *Algorithm* 3.1 *has the following error estimate for $\ell \geq 1$:*

$$
\big\|\bar{u}_{i,h}-F_{k,h}^{(\ell+1)}\bar{u}_{i,h}\big\|_{a,h} \leq \gamma\big\|\bar{u}_{i,h}-F_{k,h}^{(\ell)}\bar{u}_{i,h}\big\|_{a,h}, \tag{3.23}
$$

where

$$
\gamma = \bar{\lambda}_{i,h}\sqrt{1+\frac{\eta_a^2(W_H)}{\bar{\lambda}_{k+1,h}\big(\delta_{k,i,h}^{(\ell+1)}\big)^2}}\left(1+\frac{\bar{\mu}_{k+1,h}}{\delta_{k,i,h}^{(\ell)}}\right)\eta_a^2(W_H). \tag{3.24}
$$

Proof. From Algorithm 3.1, it is easy to know that $u_{1,h}^{(\ell)},\cdots,u_{k,h}^{(\ell)}$ is the orthogonal basis for the space $\operatorname{span}\{u_{1,h}^{(\ell)},\cdots,u_{k,h}^{(\ell)}\}$. We define the $b_h(\cdot,\cdot)$-orthogonal projection operator $\pi_{k,h}^{(\ell)}$ to the space $\operatorname{span}\{u_{1,h}^{(\ell)},\cdots,u_{k,h}^{(\ell)}\}$. Then there exist k real numbers $q_1,\cdots,q_k\in\mathbb{R}$ such that $\pi_{k,h}^{(\ell)}\bar{u}_{i,h}$ has the following expansion:

$$
\pi_{k,h}^{(\ell)}\bar{u}_{i,h} = \sum_{j=1}^{k} q_j u_{j,h}^{(\ell)}. \tag{3.25}
$$

From (3.6) and the definition of $V_{H,h}^{(\ell+1)}$ in Step 3 of Algorithm 3.1, we obtain the orthogonal property of the projection operator $\mathcal{P}_{H,h}^{(\ell+1)}$, together with (3.2), (3.10), (3.14) and (3.25), the following inequalities hold:

$$
\begin{aligned}
&\big\|\bar{u}_{i,h}-\mathcal{P}_{H,h}^{(\ell+1)}\bar{u}_{i,h}\big\|_{a,h}^2\\
&= a_h\big(\bar{u}_{i,h}-\mathcal{P}_{H,h}^{(\ell+1)}\bar{u}_{i,h},\bar{u}_{i,h}-\mathcal{P}_{H,h}^{(\ell+1)}\bar{u}_{i,h}\big)\\
&= a_h\left(\bar{u}_{i,h}-\sum_{j=1}^{k}\bar{\lambda}_{i,h}\frac{q_j}{\lambda_{j,h}^{(\ell)}}\widehat{u}_{j,h}^{(\ell+1)},\bar{u}_{i,h}-\mathcal{P}_{H,h}^{(\ell+1)}\bar{u}_{i,h}\right)\\
&= \bar{\lambda}_{i,h}b_h\left(\bar{u}_{i,h}-\sum_{j=1}^{k}\frac{q_j}{\lambda_{j,h}^{(\ell)}}\lambda_{j,h}^{(\ell)}u_{j,h}^{(\ell)},\bar{u}_{i,h}-\mathcal{P}_{H,h}^{(\ell+1)}\bar{u}_{i,h}\right)\\
&= \bar{\lambda}_{i,h}b_h\left(\bar{u}_{i,h}-\sum_{j=1}^{k}q_j u_{j,h}^{(\ell)},\bar{u}_{i,h}-\mathcal{P}_{H,h}^{(\ell+1)}\bar{u}_{i,h}\right)\\
&= \bar{\lambda}_{i,h}b_h\big(\bar{u}_{i,h}-\pi_{k,h}^{(\ell)}\bar{u}_{i,h},\bar{u}_{i,h}-\mathcal{P}_{H,h}^{(\ell+1)}\bar{u}_{i,h}\big)\\
&\leq \bar{\lambda}_{i,h}\big\|\bar{u}_{i,h}\pi_{k,h}^{(\ell)}\bar{u}_{i,h}\big\|_{b,h}\big\|\bar{u}_{i,h}-\mathcal{P}_{H,h}^{(\ell+1)}\bar{u}_{i,h}\big\|_{b,h}\\
&\leq \bar{\lambda}_{i,h}\big\|\bar{u}_{i,h}-F_{k,h}^{(\ell)}\bar{u}_{i,h}\big\|_{b,h}\big\|\bar{u}_{i,h}-\mathcal{P}_{H,h}^{(\ell+1)}\bar{u}_{i,h}\big\|_{b,h}\\
&\leq \bar{\lambda}_{i,h}\eta_a(W_H)\big\|\bar{u}_{i,h}-F_{k,h}^{(\ell)}\bar{u}_{i,h}\big\|_{a,h}\eta_a(W_H)\big\|\bar{u}_{i,h}-\mathcal{P}_{H,h}^{(\ell+1)}\bar{u}_{i,h}\big\|_{a,h},
\end{aligned} \tag{3.26}
$$

where $\bar{\eta}_a(W_H)$ is defined in Lemma 3.1. Then from (3.26), it follows that

$$\big\|\bar{u}_{i,h}-\mathcal{P}_{H,h}^{(\ell+1)}\bar{u}_{i,h}\big\|_{a,h}\leq\bar{\lambda}_{i,h}\bar{\eta}_a(W_H)\eta_a(W_H)\big\|\bar{u}_{i,h}-F_{k,h}^{(\ell)}\bar{u}_{i,h}\big\|_{a,h}. \tag{3.27}$$

Since $u_{1,h}^{(\ell+1)},\cdots,u_{k,h}^{(\ell+1)}$ only come from (3.3) and Lemma 3.1, we have for $i=1,\ldots,k$,

$$\big\|\bar{u}_{i,h}-F_{k,h}^{(\ell+1)}\bar{u}_{i,h}\big\|_{a,h}\leq\sqrt{1+\frac{\eta_a^2(W_H)}{\bar{\lambda}_{k+1,h}\big(\delta_{k,i,h}^{(\ell+1)}\big)^2}}\big\|\big(I-\mathcal{P}_{H,h}^{(\ell+1)}\big)\bar{u}_{i,h}\big\|_{a,h}.$$

Together with (3.27), we arrive at

$$\big\|\bar{u}_{i,h}-F_{k,h}^{(\ell+1)}\bar{u}_{i,h}\big\|_{a,h}\leq\bar{\lambda}_{i,h}\sqrt{1+\frac{\eta_a^2(W_H)}{\bar{\lambda}_{k+1,h}\big(\delta_{k,i,h}^{(\ell+1)}\big)^2}}\bar{\eta}_a(W_H)\eta_a(W_H)\big\|\bar{u}_{i,h}-F_{k,h}^{(\ell)}\bar{u}_{i,h}\big\|_{a,h},$$

which is the desired result (3.23) and the proof is complete. □

Remark 3.1. According to Theorem 3.1, the augmented subspace techniques have a second order convergence rate, as indicated by the convergence result (3.23). Furthermore, we ought to lower the term $\eta_a(W_H)$, which is dependent on the coarse conforming linear finite element space W_H, in order to speed up the convergence rate. In other words, the convergence can be accelerated by expanding the space W_H.

Remark 3.2. Since the error estimates for the eigenvalue approximation can be simply inferred from the following error expansion, we are only concerned with the error estimates for the eigenfunction approximation in this paper

$$\begin{aligned}0\leq\widehat{\lambda}_i-\bar{\lambda}_{i,h}=&\frac{a_h(\bar{u}_{i,h}-\psi,\bar{u}_{i,h}-\psi)}{b_h(\psi,\psi)}-\bar{\lambda}_{i,h}\frac{b_h(\bar{u}_{i,h}-\psi,\bar{u}_{i,h}-\psi)}{b_h(\psi,\psi)}\\&+2\frac{a_h(\bar{u}_{i,h},\psi)-b_h(\bar{\lambda}_{i,h}\bar{u}_{i,h},\psi)}{b_h(\psi,\psi)},\end{aligned}$$

where ψ is the approximation to the eigenfunction $\bar{u}_{i,h}$ and

$$\widehat{\lambda}_i=\frac{a_h(\psi,\psi)}{b_h(\psi,\psi)}.$$

Since each linear equation can be solved separately, it follows that Step 2 of Algorithm 3.1 can be performed using the parallel computing approach. Therefore, a kind of parallel methods for eigenvalue problems can be designed using the augmented subspace approach. The eigenvalue problem (3.3) is solved in Step 3 of Algorithm 3.1. Thereinto, we must perform the inner products of the k vectors in the high dimensional space V_h in order to generate the matrices for (3.3). This is a very low scalable procedure for the parallel computing [17,32,38]. That is to say, a bottleneck for parallel computing does exist in the inner product calculation for many high dimensional vectors. Based on above, here we provide an additional version of the augmented subspace technique for a single eigenpair (possibly not the smallest eigenvalue corresponding to the eigenpair), which represents the single process version of this kind of parallel schemes, to get around this crucial bottleneck. Algorithm 3.2 defines the relevant numerical approach. This idea in relation to the conforming finite element technique has already been put out and examined in [32].

In Algorithm 3.2, we assume that the given eigenpair approximation $(\lambda_{i,h}^{(\ell)}, u_{i,h}^{(\ell)}) \in \mathbb{R} \times V_h$ is the closest to the eigenpair $(\bar{\lambda}_{i,h}, \bar{u}_{i,h})$ of (2.9). Based on this setting, we can give the following convergence result for the augmented subspace method defined by Algorithm 3.2.

For each ℓ, it is easy to know that the eigenvalue problems (3.28) and (3.30) also have the following eigenvalues [2, 3]:

$$0 < \lambda_{1,h}^{(\ell)} \le \lambda_{2,h}^{(\ell)} \le \cdots \le \lambda_{k,h}^{(\ell)} \le \cdots \le \lambda_{N_{H,h},h}^{(\ell)},$$

and corresponding eigenfunctions

$$u_{1,h}^{(\ell)}, u_{2,h}^{(\ell)}, \cdots, u_{k,h}^{(\ell)}, \cdots, u_{N_{H,h},h}^{(\ell)}, \tag{3.31}$$

where

$$N_{H,h} = \dim V_{H,h}^{(\ell)} = N_H + 1, \quad a_h\big(u_{i,h}^{(\ell)}, u_{j,h}^{(\ell)}\big) = \delta_{ij}, \quad 1 \le i, j \le N_{H,h}.$$

Since the WG finite element space V_h includes the low dimensional augmented subspace $V_{H,h}^{(\ell)}$ in Algorithm 3.2, Algorithm 3.2's error estimates can be derived similarly to those of Algorithm 3.1. During the process of error analysis, we also utilize the definitions (3.6) and (3.7) for the sake of simplicity in notation, and apply the property (3.5), the error estimates (3.9) and (3.10) for the eigenvalue problems (3.28) and (3.30).

Lemma 3.2. *Let $(\bar{\lambda}_h, \bar{u}_h)$ denote an eigenpair of the eigenvalue problem* (2.9)*. Assume the eigenpair approximation $(\lambda_{i,h}^{(\ell)}, u_{i,h}^{(\ell)})$ has the property that $\mu_{i,h}^{(\ell)}=1/\lambda_{i,h}^{(\ell)}$ is the closest to $\bar{\mu}_h=1/\bar{\lambda}_h$. And the spectral projection operator $E_{i,h}^{(\ell)} : V_h \mapsto \mathrm{span}\{u_{i,h}^{(\ell)}\}$ is defined as follows:*

$$a_h\big(E_{i,h}^{(\ell)} w, u_{i,h}^{(\ell)}\big) = a_h\big(w, u_{i,h}^{(\ell)}\big), \quad w \in V_h.$$

Algorithm 3.2: Augmented Subspace Method for One Eigenpair.

1. For $\ell = 1$, we define $\widehat{u}_{i,h}^{(\ell)} = u_{i,h}^{(\ell)}$, and the augmented subspace $V_{H,h}^{(\ell)} = W_H + \mathrm{span}\{\widehat{u}_{i,h}^{(\ell)}\}$. Then solve the following eigenvalue problem: Find $(\lambda_{i,h}^{(\ell)}, u_{i,h}^{(\ell)}) \in \mathbb{R} \times V_{H,h}^{(\ell)}$ such that $a_h(u_{i,h}^{(\ell)}, u_{i,h}^{(\ell)}) = 1$ and
$$a_h\big(u_{i,h}^{(\ell)}, v_{H,h}\big) = \lambda_{i,h}^{(\ell)} b_h\big(u_{i,h}^{(\ell)}, v_{H,h}\big), \quad \forall\, v_{H,h} \in V_{H,h}^{(\ell)}. \tag{3.28}$$

2. Solve the following linear boundary value problem: Find $\widehat{u}_{i,h}^{(\ell+1)} \in V_h$ such that
$$a_h\big(\widehat{u}_{i,h}^{(\ell+1)}, v_h\big) = \lambda_{i,h}^{(\ell)} b_h\big(u_{i,h}^{(\ell)}, v_h\big), \quad \forall\, v_h \in V_h. \tag{3.29}$$

3. Define the augmented subspace $V_{H,h}^{(\ell+1)} = W_H + \mathrm{span}\{\widehat{u}_{i,h}^{(\ell+1)}\}$ and solve the following eigenvalue problem: Find $(\lambda_{i,h}^{(\ell+1)}, u_{i,h}^{(\ell+1)}) \in \mathbb{R} \times V_{H,h}^{(\ell+1)}$ such that $a_h(u_{i,h}^{(\ell+1)}, u_{i,h}^{(\ell+1)}) = 1$ and
$$a_h\big(u_{i,h}^{(\ell+1)}, v_{H,h}\big) = \lambda_{i,h}^{(\ell+1)} b_h\big(u_{i,h}^{(\ell+1)}, v_{H,h}\big), \quad \forall\, v_{H,h} \in V_{H,h}^{(\ell+1)}. \tag{3.30}$$
Solve (3.30) and the output $(\lambda_{i,h}^{(\ell+1)}, u_{i,h}^{(\ell+1)})$ is chosen such that $u_{i,h}^{(\ell+1)}$ has the largest component in $\mathrm{span}\{\widehat{u}_{i,h}^{(\ell+1)}\}$ among all eigenfunctions of (3.30).

4. Set $\ell = \ell + 1$ and go to Step 2 for the next iteration until convergence.

Then, the following Algorithm 3.2's error estimates hold:

$$\big\|\bar{u}_h - E_{i,h}^{(\ell)}\bar{u}_h\big\|_{a,h} \le \bar{\lambda}_{i,h}\sqrt{1+\frac{\eta_a^2(W_H)}{\bar{\lambda}_{1,h}\big(\delta_{\lambda,h}^{(\ell)}\big)^2}}\big\|\bar{u}_h - \mathcal{P}_{H,h}^{(\ell)}\bar{u}_h\big\|_{a,h}, \tag{3.32}$$

$$\big\|\bar{u}_h - E_{i,h}^{(\ell)}\bar{u}_h\big\|_{b,h} \le \bar{\eta}_a(W_H)\big\|\bar{u}_h - E_{i,h}^{(\ell)}\bar{u}_h\big\|_{a,h}, \tag{3.33}$$

where $\delta_{\lambda,h}^{(\ell)}$ *and* $\bar{\eta}_a(W_H)$ *are defined as*

$$\delta_{\lambda,h}^{(\ell)} = \min_{j\neq i}\left|\frac{1}{\lambda_{j,h}^{(\ell)}} - \frac{1}{\bar{\lambda}_h}\right|, \quad \bar{\eta}_a(W_H) = \left(1+\frac{1}{\bar{\lambda}_{1,h}\delta_{\lambda,h}^{(\ell)}}\right)\eta_a(W_H). \tag{3.34}$$

Proof. Since

$$\begin{aligned}&\big(I - E_{i,h}^{(\ell)}\big)\mathcal{P}_{H,h}^{(\ell)}\bar{u}_h \in V_{H,h}^{(\ell)},\\ &\big(I - E_{i,h}^{(\ell)}\big)\mathcal{P}_{H,h}^{(\ell)}\bar{u}_h \in \operatorname{span}\big\{u_{1,h}^{(\ell)},\cdots,u_{i-1,h}^{(\ell)},u_{i+1,h}^{(\ell)},\cdots,u_{N_{H,h},h}^{(\ell)}\big\},\end{aligned}$$

the following orthogonal expansion holds:

$$\big(I - E_{i,h}^{(\ell)}\big)\mathcal{P}_{H,h}^{(\ell)}\bar{u}_h = \sum_{j\neq i}\alpha_j u_{j,h}^{(\ell)}, \tag{3.35}$$

where $\alpha_j = a_h(\mathcal{P}_{H,h}^{(\ell)}\bar{u}_h, u_{j,h}^{(\ell)})$. From Lemma 2.2, we have the same equality (3.17).

From the orthogonal property of eigenfunctions $u_{1,h}^{(\ell)},\cdots,u_{N_{H,h},h}^{(\ell)}$, we acquire

$$1 = a_h\big(u_{j,h}^{(\ell)},u_{j,h}^{(\ell)}\big) = \lambda_{j,h}^{(\ell)} b_h\big(u_{j,h}^{(\ell)},u_{j,h}^{(\ell)}\big) = \lambda_{j,h}^{(\ell)}\big\|u_{j,h}^{(\ell)}\big\|_{b,h}^2,$$

which leads to the following property:

$$\big\|u_{j,h}^{(\ell)}\big\|_{b,h}^2 = \frac{1}{\lambda_{j,h}^{(\ell)}} = \mu_{j,h}^{(\ell)}. \tag{3.36}$$

Because of (3.28), (3.30) and the definition of eigenfunctions $u_{1,h}^{(\ell)},\cdots,u_{N_{H,h},h}^{(\ell)}$, we obtain the following equalities:

$$a_h\big(u_{j,h}^{(\ell)},u_{k,h}^{(\ell)}\big) = \delta_{jk}, \quad b_h\left(\frac{u_{j,h}^{(\ell)}}{\big\|u_{j,h}^{(\ell)}\big\|_{b,h}}, \frac{u_{k,h}^{(\ell)}}{\big\|u_{k,h}^{(\ell)}\big\|_{b,h}}\right) = \delta_{jk}, \quad 1\le j,k\le N_{H,h}. \tag{3.37}$$

Then due to (3.5), (3.17), (3.19), (3.35) and (3.36), we have the following estimates:

$$\begin{aligned}&\big\|\big(I - E_{i,h}^{(\ell)}\big)\mathcal{P}_{H,h}^{(\ell)}\bar{u}_h\big\|_{a,h}^2\\ =&\left\|\sum_{j\neq i}\alpha_j u_{j,h}^{(\ell)}\right\|_{a,h}^2 = \sum_{j\neq i}\alpha_j^2\\ =&\sum_{j\neq i}\left(\frac{1}{\bar{\mu}_h - \mu_{j,h}^{(\ell)}}\right)^2 b_h\big(\bar{u}_h - \mathcal{P}_{H,h}^{(\ell)}\bar{u}_h, u_{j,h}^{(\ell)}\big)^2\\ \le&\frac{1}{\big(\delta_{\lambda,h}^{(\ell)}\big)^2}\sum_{j\neq i}\big\|u_{j,h}^{(\ell)}\big\|_{b,h}^2 b_h\left(\bar{u}_h - \mathcal{P}_{H,h}^{(\ell)}\bar{u}_h, \frac{u_{j,h}^{(\ell)}}{\big\|u_{j,h}^{(\ell)}\big\|_{b,h}}\right)^2\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\big(\delta^{(\ell)}_{\lambda,h}\big)^2}\sum_{j\neq i}\mu^{(\ell)}_{j,h} b_h\left(\bar{u}_h - \mathcal{P}^{(\ell)}_{H,h}\bar{u}_h, \frac{u^{(\ell)}_{j,h}}{\|u^{(\ell)}_{j,h}\|_{b,h}}\right)^2 \\
&\le \frac{\mu^{(\ell)}_{1,h}}{\big(\delta^{(\ell)}_{\lambda,h}\big)^2}\sum_{j\neq i} b_h\left(\bar{u}_h - \mathcal{P}^{(\ell)}_{H,h}\bar{u}_h, \frac{u^{(\ell)}_{j,h}}{\|u^{(\ell)}_{j,h}\|_{b,h}}\right)^2 \\
&\le \frac{\mu^{(\ell)}_{1,h}}{\big(\delta^{(\ell)}_{\lambda,h}\big)^2}\big\|\bar{u}_h - \mathcal{P}^{(\ell)}_{H,h}\bar{u}_h\big\|^2_{b,h},
\end{aligned} \tag{3.38}
$$

where the last inequality holds since $u^{(\ell)}_{1,h}/\|u^{(\ell)}_{1,h}\|_{b,h}, \cdots, u^{(\ell)}_{N_{H,h},h}/\|u^{(\ell)}_{N_{H,h},h}\|_{b,h}$ constitute an orthonormal basis for the space $V^{(\ell)}_{H,h}$ in the sense of the inner product $b_h(\cdot\,,\cdot)$.

Combining (3.5) and (3.38) leads to the following inequality:

$$
\big\|\big(I - E^{(\ell)}_{i,h}\big)\mathcal{P}^{(\ell)}_{H,h}\bar{u}_h\big\|^2_{a,h} \le \frac{\bar{\mu}_{1,h}}{\big(\delta^{(\ell)}_{\lambda,h}\big)^2}\eta_a^2(W_H)\big\|\big(I - \mathcal{P}^{(\ell)}_{H,h}\big)\bar{u}_h\big\|^2_{a,h}. \tag{3.39}
$$

From (3.39) and the orthogonal property $a_h((I-\mathcal{P}^{(\ell)}_{H,h})\bar{u}_h, (I - E^{(\ell)}_{i,h})\mathcal{P}^{(\ell)}_{H,h}\bar{u}_h) = 0$, it follows that

$$
\begin{aligned}
\big\|\bar{u}_h - E^{(\ell)}_{i,h}\bar{u}_h\big\|^2_{a,h} &= \big\|\bar{u}_h - \mathcal{P}^{(\ell)}_{H,h}\bar{u}_h\big\|^2_{a,h} + \big\|\big(I - E^{(\ell)}_{i,h}\big)\mathcal{P}^{(\ell)}_{H,h}\bar{u}_h\big\|^2_{a,h} \\
&\le \left(1 + \frac{\bar{\mu}_{1,h}}{(\delta^{(\ell)}_{\lambda,h})^2}\eta_a^2(W_H)\right)\big\|\big(I - \mathcal{P}^{(\ell)}_{H,h}\big)\bar{u}_h\big\|^2_{a,h}.
\end{aligned}
$$

This is the desired result (3.32).

Similarly, with the help of (3.5), (3.17), (3.35)-(3.37), we have the following estimates:

$$
\begin{aligned}
&\big\|\big(I - E^{(\ell)}_{i,h}\big)\mathcal{P}^{(\ell)}_{H,h}\bar{u}_h\big\|^2_{b,h} \\
&= \Big\|\sum_{j\neq i}\alpha_j u^{(\ell)}_{j,h}\Big\|^2_{b,h} = \sum_{j\neq i}\alpha_j^2\big\|u^{(\ell)}_{j,h}\big\|^2_{b,h} \\
&= \sum_{j\neq i}\left(\frac{1}{\bar{\mu}_h - \mu^{(\ell)}_{j,h}}\right)^2 b_h\big(\bar{u}_h - \mathcal{P}^{(\ell)}_{H,h}\bar{u}_h, u^{(\ell)}_{j,h}\big)^2\big\|u^{(\ell)}_{j,h}\big\|^2_{b,h} \\
&\le \frac{1}{\big(\delta^{(\ell)}_{\lambda,h}\big)^2}\sum_{j\neq i}\big\|u^{(\ell)}_{j,h}\big\|^4_{b,h}\ b_h\left(\bar{u}_h - \mathcal{P}^{(\ell)}_{H,h}\bar{u}_h, \frac{u^{(\ell)}_{j,h}}{\|u^{(\ell)}_{j,h}\|_{b,h}}\right)^2 \\
&= \frac{1}{\big(\delta^{(\ell)}_{\lambda,h}\big)^2}\sum_{j\neq i}\big(\mu^{(\ell)}_{j,h}\big)^2 b_h\left(\bar{u}_h - \mathcal{P}^{(\ell)}_{H,h}\bar{u}_h, \frac{u^{(\ell)}_{j,h}}{\|u^{(\ell)}_{j,h}\|_{b,h}}\right)^2 \\
&\le \frac{\big(\mu^{(\ell)}_{1,h}\big)^2}{\big(\delta^{(\ell)}_{\lambda,h}\big)^2}\big\|\bar{u}_h - \mathcal{P}^{(\ell)}_{H,h}\bar{u}_h\big\|^2_{b,h} \le \frac{\bar{\mu}^2_{1,h}}{\big(\delta^{(\ell)}_{\lambda,h}\big)^2}\big\|\bar{u}_h - \mathcal{P}^{(\ell)}_{H,h}\bar{u}_h\big\|^2_{b,h},
\end{aligned}
$$

which leads to the inequality

$$
\big\|\big(I - E^{(\ell)}_{i,h}\big)\mathcal{P}^{(\ell)}_{H,h}\bar{u}_h\big\|_{b,h} \le \frac{\bar{\mu}_{1,h}}{\delta^{(\ell)}_{\lambda,h}}\big\|\bar{u}_h - \mathcal{P}^{(\ell)}_{H,h}\bar{u}_h\big\|_{b,h}. \tag{3.40}
$$

From (3.10), (3.40) and the triangle inequality, we have the following error estimates for the eigenfunction approximations in the $\|\cdot\|_{b,h}$-norm:

$$\begin{aligned}\big\|\bar{u}_h - E_{i,h}^{(\ell)}\bar{u}_h\big\|_{b,h} &\le \big\|\bar{u}_h - \mathcal{P}_{H,h}^{(\ell)}\bar{u}_h\big\|_{b,h} + \big\|(I - E_{i,h}^{(\ell)})\mathcal{P}_{H,h}^{(\ell)}\bar{u}_h\big\|_{b,h}\\ &\le \left(1+\frac{\bar{\mu}_{1,h}}{\delta_{\lambda,h}^{(\ell)}}\right)\big\|(I-\mathcal{P}_{H,h}^{(\ell)})\bar{u}_h\big\|_{b,h}\\ &\le \left(1+\frac{\bar{\mu}_{1,h}}{\delta_{\lambda,h}^{(\ell)}}\right)\eta_a(W_H)\big\|(I-\mathcal{P}_{H,h}^{(\ell)})\bar{u}_h\big\|_{a,h}\\ &\le \left(1+\frac{\bar{\mu}_{1,h}}{\delta_{\lambda,h}^{(\ell)}}\right)\eta_a(W_H)\big\|\bar{u}_h - E_{i,h}^{(\ell)}\bar{u}_h\big\|_{a,h}.\end{aligned}$$

This is the second desired result (3.33) and the proof is complete. □

Theorem 3.2. *Under the conditions of Lemma* 3.2*, Algorithm* 3.2 *has the following error estimate for* $\ell \ge 1$*:*

$$\big\|\bar{u}_h - E_{i,h}^{(\ell+1)}\bar{u}_h\big\|_{a,h} \le \bar{\lambda}_{i,h}\sqrt{1+\frac{\eta_a^2(W_H)}{\bar{\lambda}_{1,h}\big(\delta_{\lambda,h}^{(\ell+1)}\big)^2}}\left(1+\frac{1}{\bar{\lambda}_{1,h}\delta_{\lambda,h}^{(\ell)}}\right)\eta_a^2(W_H)\big\|\bar{u}_h - E_{i,h}^{(\ell)}\bar{u}_h\big\|_{a,h}. \tag{3.41}$$

Proof. We define the $b(\cdot,\cdot)$-orthogonal projection operator $\pi_h^{(\ell)}$ to the space $\operatorname{span}\{u_{i,h}^{(\ell)}\}$. Then, there exists a real number $q \in \mathbb{R}$ such that $\pi_h^{(\ell)}\bar{u}_h = q u_{i,h}^{(\ell)}$. According to the orthogonal property of the projection operator $\mathcal{P}_{H,h}^{(\ell+1)}$, (3.10), (3.29) and (3.33), we obtain

$$\begin{aligned}&\big\|\bar{u}_h - \mathcal{P}_{H,h}^{(\ell+1)}\bar{u}_h\big\|_{a,h}^2\\ &= a_h\big(\bar{u}_h - \mathcal{P}_{H,h}^{(\ell+1)}\bar{u}_h, \bar{u}_h - \mathcal{P}_{H,h}^{(\ell+1)}\bar{u}_h\big)\\ &= a_h\left(\bar{u}_h - \frac{\bar{\lambda}_{i,h}}{\lambda_{i,h}^{(\ell)}}q\widehat{u}_{i,h}^{(\ell+1)}, \bar{u}_h - \mathcal{P}_{H,h}^{(\ell+1)}\bar{u}_h\right)\\ &= a_h\big(\bar{u}_h, \bar{u}_h - \mathcal{P}_{H,h}^{(\ell+1)}\bar{u}_h\big) - \frac{\bar{\lambda}_{i,h}}{\lambda_{i,h}^{(\ell)}}q a_h\big(\widehat{u}_{i,h}^{(\ell+1)}, \bar{u}_h - \mathcal{P}_{H,h}^{(\ell+1)}\bar{u}_h\big)\\ &= \bar{\lambda}_h b_h\big(\bar{u}_h, \bar{u}_h - \mathcal{P}_{H,h}^{(\ell+1)}\bar{u}_h\big) - \bar{\lambda}_{i,h} b_h\big(q u_{i,h}^{(\ell)}, \bar{u}_h - \mathcal{P}_{H,h}^{(\ell+1)}\bar{u}_h\big)\\ &= \bar{\lambda}_h b_h\big(\bar{u}_h - \pi_h^{(\ell)}\bar{u}_h, \bar{u}_h - \mathcal{P}_{H,h}^{(\ell+1)}\bar{u}_h\big)\\ &\le \bar{\lambda}_h\big\|\bar{u}_h - \pi_h^{(\ell)}\bar{u}_h\big\|_{b,h}\big\|\bar{u}_h - \mathcal{P}_{H,h}^{(\ell+1)}\bar{u}_h\big\|_{b,h}\\ &\le \bar{\lambda}_h\big\|\bar{u}_h - E_{i,h}^{(\ell)}\bar{u}_h\big\|_{b,h}\big\|\bar{u}_h - \mathcal{P}_{H,h}^{(\ell+1)}\bar{u}_h\big\|_{b,h}\\ &\le \bar{\lambda}_h\left(1+\frac{1}{\bar{\lambda}_{1,h}\delta_{\lambda,h}^{(\ell)}}\right)\eta_a(W_H)\big\|\bar{u}_h - E_{i,h}^{(\ell)}\bar{u}_h\big\|_{a,h}\eta_a(W_H)\big\|\bar{u}_h - \mathcal{P}_{H,h}^{(\ell+1)}\bar{u}_h\big\|_{a,h}\\ &\le \bar{\lambda}_h\left(1+\frac{1}{\bar{\lambda}_{1,h}\delta_{\lambda,h}^{(\ell)}}\right)\eta_a^2(W_H)\big\|\bar{u}_h - E_{i,h}^{(\ell)}\bar{u}_h\big\|_{a,h}\big\|\bar{u}_h - \mathcal{P}_{H,h}^{(\ell+1)}\bar{u}_h\big\|_{a,h}.\end{aligned} \tag{3.42}$$

Since the approximation $u_{i,h}^{(\ell+1)}$ only comes from (3.28) or (3.30), together with Lemma 3.2, we have

$$\|\bar{u}_h - E_{i,h}^{(\ell+1)}\bar{u}_h\|_{a,h} \le \sqrt{1+\frac{\eta_a^2(W_H)}{\bar{\lambda}_{1,h}\big(\delta_{\lambda,h}^{(\ell+1)}\big)^2}}\,\|(I-\mathcal{P}_{H,h}^{(\ell+1)})\bar{u}_h\|_{a,h}. \tag{3.43}$$

From (3.42), there holds

$$\|\bar{u}_h - \mathcal{P}_{H,h}^{(\ell+1)}\bar{u}_h\|_{a,h} \le \bar{\lambda}_h\left(1+\frac{1}{\bar{\lambda}_{1,h}\delta_{\lambda,h}^{(\ell)}}\right)\eta_a^2(W_H)\|\bar{u}_h - E_{i,h}^{(\ell)}\bar{u}_h\|_{a,h}. \tag{3.44}$$

Combining (3.43) with (3.44), we have the following estimate:

$$\|\bar{u}_h - E_{i,h}^{(\ell+1)}\bar{u}_h\|_{a,h} \le \bar{\lambda}_h\sqrt{1+\frac{\eta_a^2(W_H)}{\bar{\lambda}_{1,h}\big(\delta_{\lambda,h}^{(\ell+1)}\big)^2}}\left(1+\frac{1}{\bar{\lambda}_{1,h}\delta_{\lambda,h}^{(\ell)}}\right)\eta_a^2(W_H)\|\bar{u}_h - E_{i,h}^{(\ell)}\bar{u}_h\|_{a,h}.$$

This is the desired result (3.41) and the proof is complete. □

4. Applications to Laplace Eigenvalue Problem

This section will demonstrate the applications of the augmented subspace techniques introduced in Section 3 to the Laplace eigenvalue problem and provide the associated convergence rates. It is noteworthy that the finest WG finite element space has little bearing on the mesh size selection of the coarse mesh $\mathcal{T}_H$ in augmented subspace techniques. Compared to the two-grid WG finite element technique in [33, 35] where the choices of coarse and fine mesh sizes are not free each other, this represents a significant distinction.

Here, we are concerned with the following Laplace eigenvalue problem: Find $(\lambda, u) \in \mathbb{R}\times H_0^1(\Omega)$ such that

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega,\\ |u|_1^2 = 1, \end{cases} \tag{4.1}$$

where $|\cdot|_1$ represents the H^1-type semi-norm and the computing domain is set to be the unit square $\Omega = (0,1)\times(0,1)$. Then, in (2.3), the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are defined as follows:

$$a(u,v) = \int_\Omega \nabla u\cdot\nabla v d\Omega, \quad b(u,v) = \int_\Omega uv d\Omega.$$

The norms $\|\cdot\|_{a,h}$ and $\|\cdot\|_{b,h}$ defined in (2.12) and (2.13) are equivalent to the H^1-type semi-norm $|\cdot|_1$ and the L^2 norm $\|\cdot\|$, respectively. In order to utilize the WG finite element method, we employ the meshes defined in Section 2.

Here, the problem (4.1) is treated using the augmented subspace techniques specified by Algorithms 3.1 and 3.2. In this section, the regular refinement is used to create the fine mesh $\mathcal{T}_h$ from the coarse mesh $\mathcal{T}_H$. The WG finite element space on the fine mesh $\mathcal{T}_h$ is set to V_h, and the coarse conforming linear finite element space on the coarse mesh $\mathcal{T}_H$ is set to W_H. We consider the computational domain Ω to be convex for the sake of simplicity.

In order to give the explicit convergence rate of the augmented subspace methods defined by Algorithms 3.1 and 3.2, we need to estimate the quantity $\eta_a(W_H)$ in (3.7). For this aim, we define the conforming linear finite element projection operator $\mathcal{P}_H : H_0^1(\Omega)\mapsto W_H$ as follows:

$$a(\mathcal{P}_H w, v_H) = a(w, v_H), \quad \forall\, v_H\in W_H, \quad w\in H_0^1(\Omega). \tag{4.2}$$

It is well known that the following error estimate holds:

$$\|Tf - \mathcal{P}_H Tf\|_1 \le CH\|Tf\|_2 \le CH\|f\|_{b,h}, \tag{4.3}$$

where $T : L^2(\Omega) \mapsto H_0^1(\Omega)$ is defined as

$$a(Tf, v) = b(f, v), \quad \forall\, v \in H_0^1(\Omega). \tag{4.4}$$

In order to deduce the estimate for the term $\eta_a(W_H)$, we define the norm $\|\cdot\|_{1,h}$ as follows:

$$\|v\|_{1,h}^2 = \sum_{K\in\mathcal{T}_h} \big(\|\nabla v_0\|_{0,K}^2 + h_K^{-1}\|v_0 - v_b\|_{\partial K}^2\big).$$

Obviously, the norm $\|\cdot\|_{1,h}$ coincides with $\|\cdot\|_1$ on the Sobolev space $H_0^1(\Omega)$. Furthermore, there is the following equivalence between $\|\cdot\|_{1,h}$ and $\|\cdot\|_{a,h}$ on the WG finite element space V_h.

Lemma 4.1 ([19]). *For any $v_h \in V_h$, the following inequalities hold:*

$$C_7\|v_h\|_{1,h} \le \|v_h\|_{a,h} \le C_8\|v_h\|_{1,h}, \tag{4.5}$$

where C_7 and C_8 are two constants independent of the mesh size h.

Based on above, $\|T_h f - \mathcal{P}_H Tf\|_{a,h}$ has the following inequalities:

$$\begin{aligned}
&\|T_h f - \mathcal{P}_H Tf\|_{a,h}\\
\le\ &\|T_h f - Q_h Tf\|_{a,h} + \|Q_h Tf - \mathcal{P}_H Tf\|_{a,h}\\
\le\ &\|T_h f - Q_h Tf\|_{a,h} + C\|Q_h Tf - \mathcal{P}_H Tf\|_{1,h}\\
\le\ &\|T_h f - Q_h Tf\|_{a,h} + C\|Q_h Tf - Tf\|_{1,h} + C\|Tf - \mathcal{P}_H Tf\|_{1,h}\\
\le\ &\|T_h f - Q_h Tf\|_{a,h} + C\|Q_h Tf - Tf\|_{1,h} + C\|Tf - \mathcal{P}_H Tf\|_1\\
\le\ &C(h + h + H)\|Tf\|_2 \le CH\|f\|_{b,h}.
\end{aligned} \tag{4.6}$$

From the definition of $\eta_a(W_H)$ in (3.7) and (4.6), we can obtain the following estimates:

$$\begin{aligned}
\eta_a(W_H) &= \sup_{\substack{f\in L^2(\Omega)\\ \|f\|_{b,h}=1}} \inf_{w_H\in W_H} \|T_h f - w_H\|_{a,h}\\
&\le \sup_{\substack{f\in L^2(\Omega)\\ \|f\|_{b,h}=1}} \|T_h f - \mathcal{P}_H Tf\|_{a,h}\\
&\le \sup_{\substack{f\in L^2(\Omega)\\ \|f\|_{b,h}=1}} CH\|f\|_{b,h} = CH.
\end{aligned} \tag{4.7}$$

Based on Theorems 3.1 and 3.2, the convergence results for the augmented subspace method can be concluded with the following inequalities:

$$\big\|\bar{u}_{i,h} - F_{k,h}^{(\ell+1)}\bar{u}_{i,h}\big\|_{a,h} \le C\big(CH\big)^{2\ell}\big\|\bar{u}_{i,h} - F_{k,h}^{(1)}\bar{u}_{i,h}\big\|_{a,h}, \quad i = 1, \ldots, k, \tag{4.8}$$

$$\big\|\bar{u}_{i,h} - F_{k,h}^{(\ell+1)}\bar{u}_{i,h}\big\|_{b,h} \le CH\big\|\bar{u}_{i,h} - F_{k,h}^{(\ell+1)}\bar{u}_{i,h}\big\|_{a,h}, \quad i = 1, \ldots, k, \tag{4.9}$$

$$\big\|\bar{u}_h - E_{i,h}^{(\ell+1)}\bar{u}_h\big\|_{a,h} \le C\big(CH\big)^{2\ell}\big\|\bar{u}_h - E_{i,h}^{(1)}\bar{u}_h\big\|_{a,h}, \tag{4.10}$$

$$\big\|\bar{u}_h - E_{i,h}^{(\ell+1)}\bar{u}_h\big\|_{b,h} \le CH\big\|\bar{u}_h - E_{i,h}^{(\ell+1)}\bar{u}_h\big\|_{a,h}. \tag{4.11}$$

The goal of this section is to validate these convergence findings using a few numerical examples. The exact WG finite element eigenfunction can be found by directly solving the eigenvalue problem on the fine WG finite element space V_h. Let this be noted. To aid with comprehension, the nomenclature in all of the following figures denotes the exact WG finite element eigenfunctions and the augmented subspace approximations, respectively, with and without the "dir" superscript.

4.1. Augmented subspace method for P_0/P_0 WG finite element space

For the WG finite element space P_0/P_0, we examine the performance of the augmented subspace approach described by Algorithms 3.1 and 3.2 in the first subsection. Here, W_H is defined as the conforming linear finite element space on the coarse mesh $\mathcal{T}_H$ in all the numerical cases. The P_0/P_0 WG finite element space V_h defined on the fine mesh $\mathcal{T}_h$ can be written as follows:

$$\begin{aligned} V_h = \{v : v|_{K_0} \in \mathcal{P}_0(K_0) \ \text{ for } \ K \in \mathcal{T}_h;\ v|_e \in \mathcal{P}_0(e) \ \text{ for } \ e \in \mathcal{E}_h, \\ \text{and } \ v|_e = 0 \ \text{ for } \ e \in \mathcal{E}_h \cap \partial\Omega\}. \end{aligned}$$

The fine mesh $\mathcal{T}_h$ is obtained from the coarse mesh $\mathcal{T}_H$ by the regular refinement. Here, we set the size $h = \sqrt{2}/256$ for the fine mesh $\mathcal{T}_h$.

We verify the convergence results for algebraic error with various sizes H by examining the numerical errors corresponding to the results in (4.8)-(4.11). The purpose of this is to determine how the mesh size H affects the convergence rate. In this case, the quasi-uniform mesh $\mathcal{T}_H$ is specified as the coarse mesh.

Under the boundary condition restriction, the initial eigenfunction approximation is specified to be rand vectors in this case. Next, we employ the augmented subspace approach, as specified by Algorithms 3.1 and 3.2, to carry out the iteration steps. The convergence behaviors for the first eigenfunction using the augmented subspace techniques are displayed in Fig. 4.1, and they correspond to the coarse mesh sizes $H = \sqrt{2}/8, \sqrt{2}/16, \sqrt{2}/32$ and $\sqrt{2}/64$, respectively. The rates of convergence associated with $\|\cdot\|_{a,h}$ and $\|\cdot\|_{b,h}$ are, respectively, $0.048945, 0.012834, 0.00279122, 0.00058513$ and $0.052177, 0.01405, 0.0032556, 0.00076374$. As a consequence, the results (4.8)-(4.11) hold and validate the second order convergence speed of the augmented subspace technique described by Algorithms 3.1 and 3.2.

Next, we evaluate Algorithm 3.1 in terms of its ability to compute the first 4 eigenpairs. The corresponding convergence behaviors for the smallest 4 eigenfunctions by Algorithm 3.1 are displayed in Fig. 4.2. The conforming linear finite element space on the mesh with size $H = \sqrt{2}/8, \sqrt{2}/16, \sqrt{2}/32$ and $\sqrt{2}/64$, respectively, forms the coarse space W_H. Employing the 4-th eigenfunction as an example, we can determine the related convergence rates, which indicate the second convergence order of Algorithm 3.1, to be $0.3353, 0.11061, 0.029854$ and 0.0054112. Furthermore, we are able to observe from Fig. 4.2 that the 4-th eigenfunction's convergence rate is slower than the one of the 1-st eigenfunction, which is in accordance with Theorem 3.1.

Assessing Algorithm 3.2's performance in determining the single 4-th eigenpair is the next objective. Since the smallest eigenpair is not the goal, the eigenvalue problem (4.1) is solved on the coarse WG finite element space V_H to provide the initial eigenfunction approximation. The augmented subspace approach, which is specified by Algorithm 3.2, is then used to carry out the iteration phases. The coarse space is the linear finite element space on the mesh with size

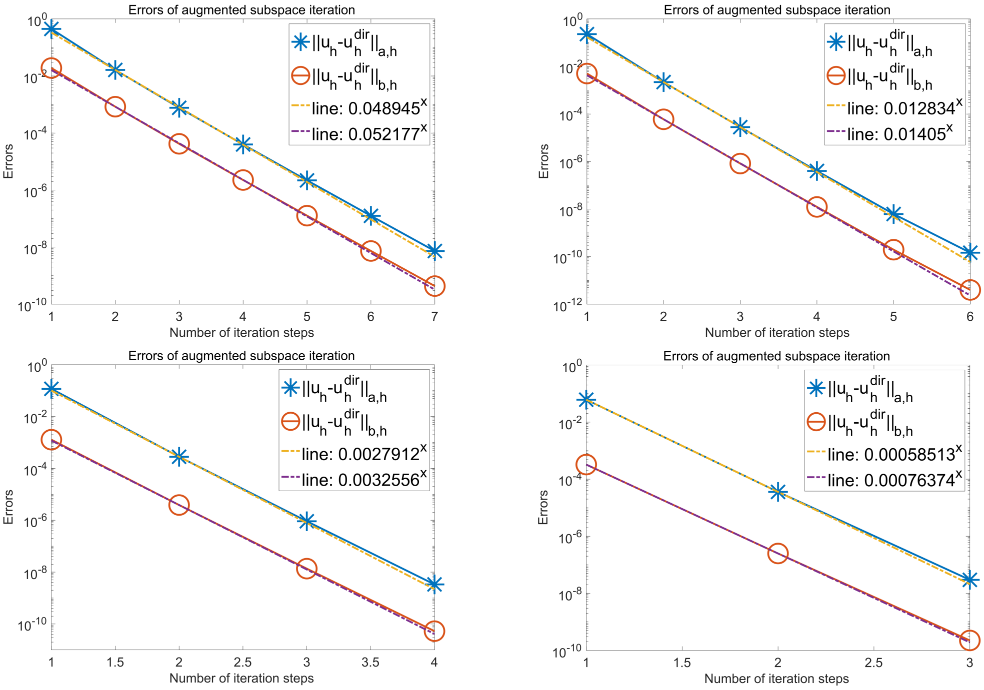

Fig. 4.1. The convergence behaviors for the first eigenfunction by Algorithms 3.1 and 3.2 corresponding to the P_0/P_0 WG finite element method and the coarse mesh size $H = \sqrt{2}/8, \sqrt{2}/16, \sqrt{2}/32$ and $\sqrt{2}/64$, respectively.

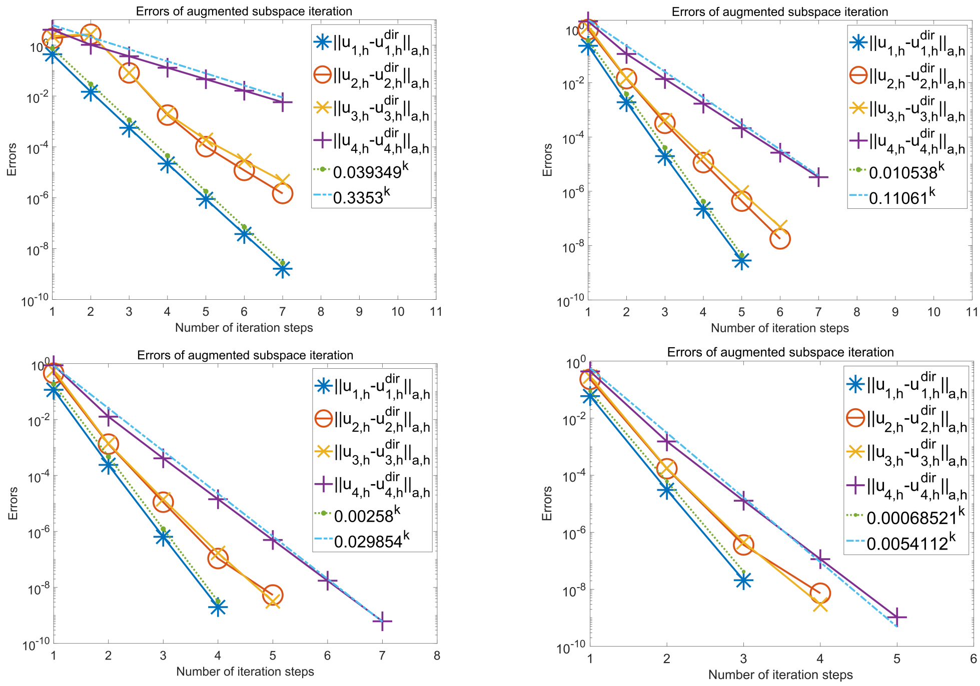

Fig. 4.2. The convergence behaviors for the smallest 4 eigenfunctions by Algorithm 3.1 with the P_0/P_0 WG finite element method and the coarse space being the linear finite element space on the mesh with size $H = \sqrt{2}/8, \sqrt{2}/16, \sqrt{2}/32$ and $\sqrt{2}/64$, respectively.

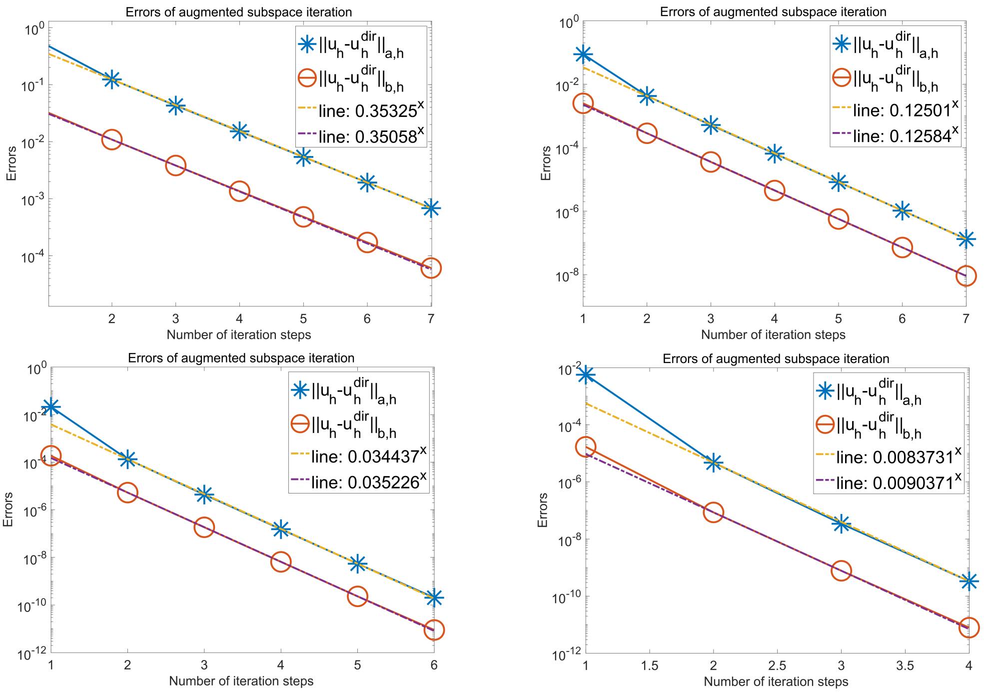

Fig. 4.3. The convergence behaviors for the only 4-th eigenfunction by Algorithm 3.2 with the P_0/P_0 WG finite element method and the coarse space being the linear finite element space on the mesh with size $H = \sqrt{2}/8, \sqrt{2}/16, \sqrt{2}/32$ and $\sqrt{2}/64$, respectively.

$H = \sqrt{2}/8, \sqrt{2}/16, \sqrt{2}/32$ and $\sqrt{2}/64$, respectively. The corresponding convergence behaviors for the only 4-th eigenfunction by Algorithm 3.2 are depicted in Fig. 4.3. The norms $\|\cdot\|_{a,h}$ and $\|\cdot\|_{b,h}$ in Fig. 4.3 correspond to the convergence rates, which are $0.35325, 0.12501, 0.034437$ and 0.0083731, and $0.35058, 0.12584, 0.035226$ and 0.0090371, respectively. According to these findings, the augmented subspace approach described by Algorithm 3.2 has a second order speed of convergence, validating the findings of (4.10)-(4.11).

4.2. Augmented subspace method for P_1/P_1 WG finite element space

We examine the augmented subspace method's performance for the WG finite element space P_1/P_1, as described by Algorithms 3.1 and 3.2, in the second subsection. Also, W_H is designated as the conforming linear finite element space on the coarse mesh $\mathcal{T}_H$ in these numerical tests. Here, V_h is the P_1/P_1 WG finite element space defined on the fine mesh $\mathcal{T}_h$, which is generated by the regular refinement from the coarse mesh $\mathcal{T}_H$.

Here, we set the size $h = \sqrt{2}/256$ for the fine mesh $\mathcal{T}_h$ and the WG finite element space V_h is defined as follows:

$$\begin{aligned} V_h = \{v : v|_{K_0} \in \mathcal{P}_1(K_0) \ \text{ for } \ K \in \mathcal{T}_h;\ v|_e \in \mathcal{P}_1(e) \ \text{ for } \ e \in \mathcal{E}_h, \\ \text{and } \ v|_e = 0 \ \text{ for } \ e \in \mathcal{E}_h \cap \partial\Omega\}. \end{aligned}$$

We check the algebraic errors corresponding to the conforming linear finite element space W_H with different sizes H. This helps to confirm the convergence results described in (4.8)-(4.11). Here, also determining how the convergence rate varies with the mesh size H is the goal. In this case, the shape-regular, quasi-uniform mesh $\mathcal{T}_H$ is specified as the coarse mesh.

In a similar way, under the boundary condition restriction, the initial eigenfunction approximation is made to be rand vectors. The convergence results for the first eigenfunction using the augmented subspace techniques are displayed in Fig. 4.4, which corresponds to the coarse mesh size $H = \sqrt{2}/8, \sqrt{2}/16, \sqrt{2}/32$ and $\sqrt{2}/64$, respectively. $\|\cdot\|_{a,h}$ and $\|\cdot\|_{b,h}$ have respective convergence rates of $0.053287, 0.013798, 0.0036045, 0.00075399$ and $0.05535, 0.014936, 0.0038268, 0.00090686$. The findings support the results (4.8)-(4.11) by demonstrating the second order convergence speed of the augmented subspace technique specified in Algorithms 3.1 and 3.2.

Next, we examine Algorithm 3.1's performance in terms of computing the first 4 eigenpairs. The corresponding convergence behaviors for the smallest 4 eigenfunctions by Algorithm 3.1 are presented in Fig. 4.5. The conforming linear finite element space on the mesh with size $H = \sqrt{2}/8, \sqrt{2}/16, \sqrt{2}/32$ and $\sqrt{2}/64$, respectively, constitutes the coarse space. By employing the 4-th eigenfunction as an example, we can get the related convergence rates $0.29933, 0.10565, 0.029315$ and 0.0065776, which reflect the second convergence order of Algorithm 3.1.

The final objective is evaluating the effectiveness of Algorithm 3.2 in determining the only 4-th eigenpair. Similarly, the coarse WG finite element space V_H is used to solve the eigenvalue problem (4.1) to get the initial eigenfunction approximation. The corresponding convergence behaviors for the only 4-th eigenfunction by Algorithm 3.2 are displayed in Fig. 4.6. The conforming linear finite element space on the mesh with size $H = \sqrt{2}/8, \sqrt{2}/16, \sqrt{2}/32$ and $\sqrt{2}/64$, respectively, is the coarse space. The convergence rates associated with $\|\cdot\|_{a,h}$ and $\|\cdot\|_{b,h}$ are $0.33464, 0.1179, 0.027908, 0.0030174$ and $0.35213, 0.12511, 0.034041, 0.0084659$, respectively, as depicted in Fig. 4.6. The results (4.10)-(4.11) are likewise validated by these findings.

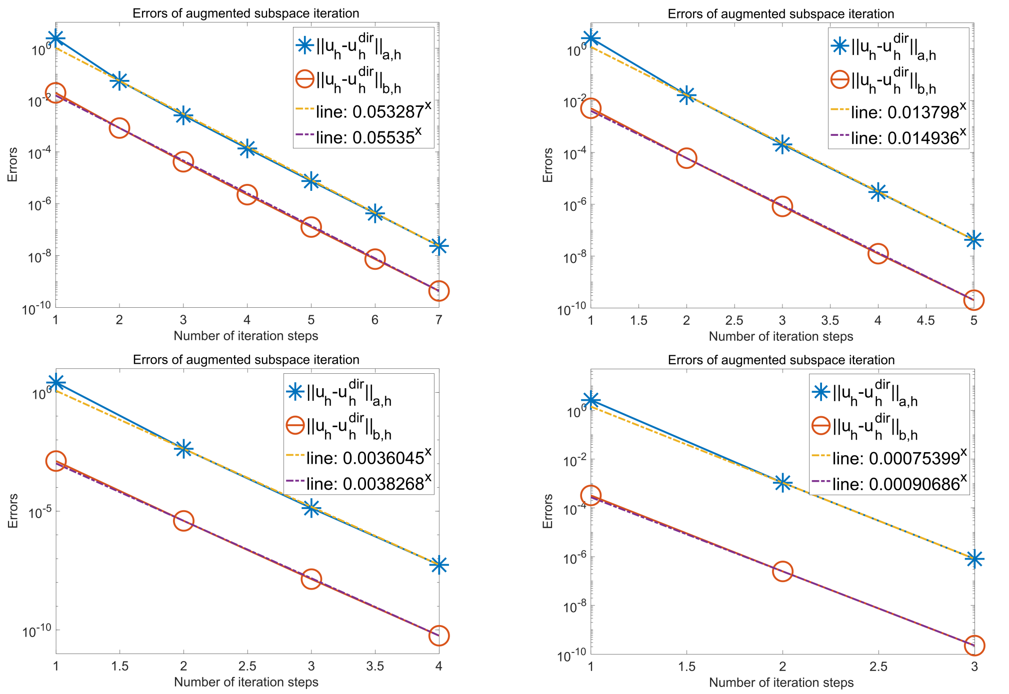

Fig. 4.4. The convergence behaviors for the first eigenfunction by Algorithms 3.1 and 3.2 corresponding to the P_1/P_1 WG finite element method and the coarse mesh size $H = \sqrt{2}/8, \sqrt{2}/16, \sqrt{2}/32$ and $\sqrt{2}/64$, respectively.

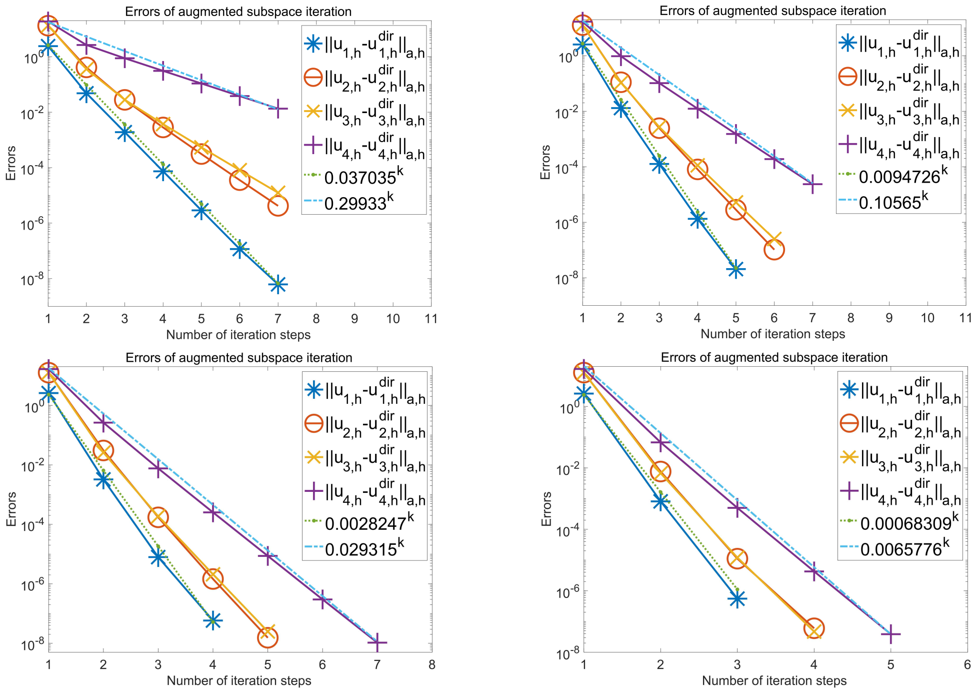

Fig. 4.5. The convergence behaviors for the smallest 4 eigenfunctions by Algorithm 3.1 with the P_1/P_1 WG finite element method and the coarse space being the linear finite element space on the mesh with size $H = \sqrt{2}/8, \sqrt{2}/16, \sqrt{2}/32$ and $\sqrt{2}/64$, respectively.

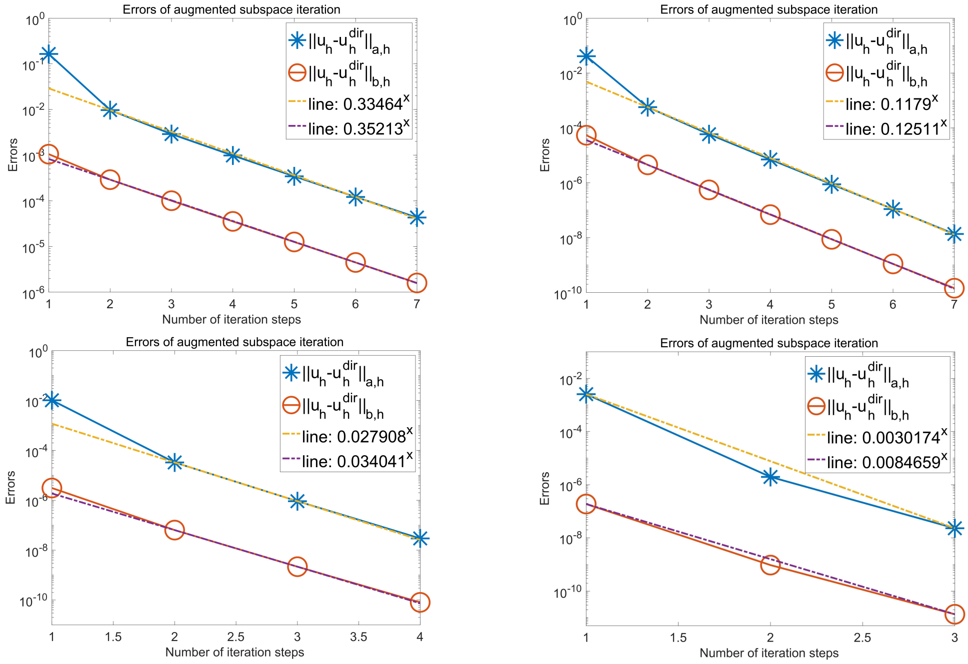

Fig. 4.6. The convergence behaviors for the only 4-th eigenfunction by Algorithm 3.2 with the P_1/P_1 WG finite element method and the coarse space being the linear finite element space on the mesh with size $H = \sqrt{2}/8, \sqrt{2}/16, \sqrt{2}/32$ and $\sqrt{2}/64$, respectively.

5. Concluding Remarks

In this study, two augmented subspace strategies for addressing the eigenvalue problems using the WG finite element method are proposed, with the assistance of conforming linear finite element space on the coarse mesh. We derive the associated error estimates, which demonstrate that the WG method's augmented subspace scheme has a second convergence order in relation to the coarse mesh size.

We can develop a sort of eigensolver for algebraic eigenvalue problems, which originate from the discretization of differential eigenvalue problems using the WG finite element technique, based on these provided augmented subspace approaches. Moreover, the methods presented here provide a means of designing the parallel eigensolver for the WG finite element technique, which will be the subject of our next research project.

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