HIGH-ORDER COMPACT ADI SCHEMES FOR 2D SEMI-LINEAR REACTION-DIFFUSION EQUATIONS WITH PIECEWISE CONTINUOUS ARGUMENT IN REACTION TERM*

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Abstract

This paper deals with the numerical solutions of two-dimensional (2D) semi-linear reaction-diffusion equations (SLRDEs) with piecewise continuous argument (PCA) in reaction term. A high-order compact difference method called I-type basic scheme is developed for solving the equations and it is proved under the suitable conditions that this method has the computational accuracy $\mathcal{O}(\tau^2 + h_x^4 + h_y^4)$, where τ, h_x and h_y are the calculation stepsizes of the method in t-, x- and y-direction, respectively. With the above method and Newton linearized technique, a II-type basic scheme is also suggested. Based on the both basic schemes, the corresponding I- and II-type alternating direction implicit (ADI) schemes are derived. Finally, with a series of numerical experiments, the computational accuracy and efficiency of the four numerical schemes are further illustrated.

Mathematics subject classification: 65M06, 65M12.
Key words: Semi-linear reaction-diffusion equations, Piecewise continuous argument, High-order compact difference methods, Alternating direction implicit schemes, Computational accuracy and efficiency.

1. Introduction

To describe the heat flow in a rod with both diffusion along the rod and heat loss/gain across the lateral sides of the rod, Wiener [16] first introduced the following initial-boundary value problems (IBVPs) of linear reaction-diffusion equations with PCA:

$$
\begin{cases}
u_t(x,t) = \hat{a} u_{xx}(x,t) + \hat{b} u(x, \lfloor t \rfloor), & x \in (a,b), \quad t \in (0,T], \\
u(x,0) = \varphi(x), & x \in [a,b], \\
u(a,t) = \psi_1(t), \quad u(b,t) = \psi_2(t), & t \in (0,T],
\end{cases}
\tag{1.1}
$$

where $u(x,t)$ denotes the temperature at point (x,t) in the rod and the lateral heat change is assumed to occur at time $\lfloor t \rfloor$, in which $\lfloor \cdot \rfloor$ is the greatest integer function. Subsequently, for depicting various diffusion phenomena, problems (1.1) were adapted into IBVPs of linear

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diffusion equations with PCA (cf. [9, 10])

$$\begin{cases} u_t(x,t) = \hat{a}_1 u_{xx}(x,t) + \hat{a}_2 u_{xx}(x, \lfloor t \rfloor), & x \in (a,b), \quad t \in (0,T], \\ u(x,0) = \varphi(x), & x \in [a,b], \\ u(a,t) = \psi_1(t), \quad u(b,t) = \psi_2(t), & t \in (0,T]. \end{cases} \tag{1.2}$$

IBVPs of linear forward-backward diffusion equations with PCA (cf. [15])

$$\begin{cases} u_t(x,t) = \hat{a}_1 u_{xx}(x,t) + \hat{a}_2 u_{xx}(x, \lfloor t \rfloor) + \hat{a}_3 u_{xx}(x, \lfloor t+1 \rfloor), & x \in (a,b), \quad t \in (0,T], \\ u(x,0) = \varphi(x), & x \in [a,b], \\ u(a,t) = \psi_1(t), \quad u(b,t) = \psi_2(t), & t \in (0,T]. \end{cases} \tag{1.3}$$

IBVPs of diffusion-convection equations with PCA (cf. [3])

$$\begin{cases} u_t(x,t) = \hat{a}_1 u_{xx}(x,t) + \hat{a}_2 u_x(x, \lfloor t \rfloor), & x \in (a,b), \quad t \in (0,T], \\ u(x,0) = \varphi(x), & x \in [a,b], \\ u(a,t) = \psi_1(t), \quad u(b,t) = \psi_2(t), & t \in (0,T]. \end{cases} \tag{1.4}$$

IBVPs of linear neutral reaction-diffusion equations with PCA (cf. [5, 6, 18])

$$\begin{cases} u_t(x,t) = \hat{a}_1 u_{xx}(x,t) + \hat{a}_2 u_{xx}(x, \lfloor t \rfloor) + \hat{b}_1 u(x,t) \\ \qquad\qquad + \hat{b}_2 u(x, \lfloor t \rfloor) + \hat{c} u_t(x, \lfloor t \rfloor), & x \in (a,b), \quad t \in (0,T], \\ u(x,0) = \varphi(x), & x \in [a,b], \\ u(a,t) = \psi_1(t), \quad u(b,t) = \psi_2(t), & t \in (0,T]. \end{cases} \tag{1.5}$$

IBVPs of I-type SLRDEs with PCA (cf. [2])

$$\begin{cases} u_t(x,t) = \hat{a} u_{xx}(x,t) + f\big(x,t,u(x,t),u(x, \lfloor t \rfloor)\big), & x \in (a,b), \quad t \in (0,T], \\ u(x,0) = \varphi(x), & x \in [a,b], \\ u(a,t) = \psi_1(t), \quad u(b,t) = \psi_2(t), & t \in (0,T]. \end{cases} \tag{1.6}$$

IBVPs of II-type SLRDEs with PCA (cf. [8])

$$\begin{cases} u_t(x,t) = \hat{a}_1 u_{xx}(x,t) + \hat{a}_2 u_{xx}(x, \lfloor t \rfloor) + f\big(x,t,u(x,t)\big), & x \in (a,b), \quad t \in (0,T], \\ u(x,0) = \varphi(x), & x \in [a,b], \\ u(a,t) = \psi_1(t), \quad u(b,t) = \psi_2(t), & t \in (0,T]. \end{cases} \tag{1.7}$$

For the above partial functional differential equations (PFDEs) with PCA, some effective numerical methods and the corresponding algorithm theory have been developed. For problem (1.2), Liang et $al.$ [9] and Liang et $al.$ [10], respectively, proposed θ-method and Galerkin method with asymptotical stability analysis. For problem (1.3), Wang and Wen [15] generalized the θ-method in [9] and its stability theory. For problem (1.4), Esmailzadeh et $al.$ [3] suggested an alternative θ-method and derived its asymptotical stability criterion. For problem (1.5), Zhang et $al.$ [18] considered a class of linear approximation methods based on block boundary value methods, and Han and Zhang [5, 6] constructed the one-parameter finite element methods with error and stability analysis. Furthermore, the authors of references [2, 8] extended the above numerical approach for linear problems to nonlinear problems (1.6) and (1.7), where Esmailzadeh et $al.$ [2] presented the θ-method for problem (1.6) and Hou and Zhang [8] derived a high-order compact difference method and its Richardson extrapolation scheme for problem (1.7).

It is worth noting that the above researches dealt only with the numerical methods solving 1D PFDEs with PCA. So far, the numerical approach on the high-dimensional PFDEs with PCA has not been concerned. Hence, in this paper, we plan to study the numerical methods for solving IBVPs of 2D SLRDEs with PCA

$$\begin{cases} u_t(x,y,t) = \alpha u_{xx}(x,y,t) + \beta u_{yy}(x,y,t) \\ \qquad + f\big(x,y,t,u(x,y,t),u(x,y,\lfloor t \rfloor)\big), & (x,y) \in \Omega, \quad t \in (0,T], \quad \text{(1.8a)} \\ u(x,y,0) = \varphi(x,y), & (x,y) \in \bar{\Omega}, \quad \text{(1.8b)} \\ u(x,y,t) = \psi(x,y,t), & (x,y) \in \partial\Omega, \quad t \in (0,T], \quad \text{(1.8c)} \end{cases}$$

where $\alpha, \beta > 0$ are the diffusion coefficients, $\bar{\Omega}$ (respectively $\partial\Omega$) is the closure (respectively boundary) of domain $\Omega := (a,b) \times (c,d)$, functions φ, ψ, f are continuous on their respective domains, and f is assumed to be smooth enough on $\Omega \times (0,T]$, with the possible exception of the points $(x,y,\lfloor t \rfloor)$, and satisfies the following Lipschitz condition for all $(x,y) \in \Omega$, $t \in (0,T]$ and $u_1, u_2, v_1, v_2 \in \mathbb{R}$:

$$|f(x,y,t,u_1,v_1) - f(x,y,t,u_2,v_2)| \leq \mathcal{L}_1|u_1 - u_2| + \mathcal{L}_2|v_1 - v_2|, \qquad \text{(1.9)}$$

where $\mathcal{L}_1, \mathcal{L}_2 > 0$ are two given constants. A function $u(x,y,t)$ is called the solution of problem (1.8) if it satisfies the following conditions:

i) $u(x,y,t)$ is continuous on $\bar{\Omega} \times [0,T]$.

ii) u_t, u_{xx} and u_{yy} exist and are continuous on $\Omega \times (0,T]$, with the possible exception of the points $(x,y,\lfloor t \rfloor)$ where one-sided derivatives exist.

iii) $u(x,y,t)$ satisfies the PFDE on $\Omega \times (0,T]$ with the possible exception of the points $(x,y,\lfloor t \rfloor)$, and the initial and boundary value conditions in (1.8).

As an instance of problems (1.8), we consider the heat flow phenomena on a sheet-like object. When the lateral heat change is assumed to occur not only at time t but also at time $\lfloor t \rfloor$ and has a complicated nonlinear relationship f between time, space and the temperature of an object, the heat flow phenomena can be modelled by (1.8), where $u(x,y,t)$ is the temperature at point (x,y,t) in the sheet-like object and f the heat exchange function.

In order to solve 2D IBVPs (1.8) efficiently, we will employ the compact ADI technique. This technique has been verified to be very effective when it was applied to the other classes of high-dimensional delay partial differential equations. For example, Deng [1] dealt with the 2D constant-delay SLRDEs, Zhang et al. [20] extended the research to the case of multi-delay, Xie and Zhang [17] studied the 2D constant-delay SLRDEs with variable coefficients in diffusion term, Zhang et al. [19] considered the 2D constant-delay hyperbolic equations, Qin et al. [11] involved the semi-linear parabolic equations with distributed delay, and Tang et al. [14] covered the 2D wave equations with discrete and distributed time-variable delays. Nevertheless, these research only involved the case of non-vanishing delay of delay partial differential equations, where the delay quantity is given by $t - \hat{\tau}$ ($\hat{\tau} > 0$) and all values of the solution function on the initial interval need to be provided. This shows that IBVPs (1.8) are different from the above delay problems, whose initial condition only needs providing the value of the solution function at initial point $t = 0$. Namely, IBVPs (1.8) are of the problems of delay partial differential equations with vanishing delay. This implies that the compact ADI schemes derived for solving IBVPs (1.8) will be different from those for solving the IBVPs of delay partial differential equations with non-vanishing delay.

The rest of this paper is organized as follows. In Section 2, a high-order compact difference method solving problem (1.8) is derived. In Section 3, the global error of the compact difference method is analyzed, which shows that the presented method has the computational accuracy $\mathcal{O}(\tau^2 + h_x^4 + h_y^4)$, where τ, h_x and h_y are the calculation stepsizes of the method in t-, x- and y-direction, respectively. In Section 4, in order to improve the computational efficiency of the method, an I-type ADI scheme is proposed. Besides, by applying the Newton linearized technique, an alternative high-order compact difference method and its ADI scheme (i.e. II-type ADI scheme) are suggested. In Section 5, with a series of numerical experiments, the computational accuracy and efficiency of the derived four numerical schemes are further illustrated. In Section 6, we summarize the full-text and leave an open problem.

2. A High-order Compact Difference Method

In this section, for solving IBVPs (1.8), we will construct a high-order compact difference method. To this end, we divide the solution region $[a, b] \times [c, d] \times [0, T]$ with planes

$$x = x_i := a + ih_x, \quad 0 \le i \le M_x,$$

$$y = y_j := c + jh_y, \quad 0 \le j \le M_y,$$

$$t = t_n := n\tau, \qquad 0 \le n \le N := \frac{\lfloor T \rfloor}{\tau},$$

where

$$h_x = \frac{b-a}{M_x}, \quad h_y = \frac{d-c}{M_y}, \quad \tau = \frac{1}{m}$$

are the calculation stepsizes of the method in x-, y-, t-direction, respectively, and M_x, M_y, m are any given positive integers. Moreover, in the following, we will also use the non-grid planes

$$t = t_{n+\frac{1}{2}} := \left(n + \frac{1}{2}\right)\tau, \quad 0 \le n \le N-1.$$

Write

$$\Omega_h = \{(x_i, y_j) \mid 0 \le i \le M_x, \ 0 \le j \le M_y\},$$
$$\Omega_\tau = \{t_n \mid 0 \le n \le N\},$$
$$\Omega_{h\tau} = \Omega_h \times \Omega_\tau, \quad U_{ij}^n = u(x_i, y_j, t_n).$$

For any $w \in W := \{w_{ij}^n \mid 0 \le i \le M_x, \ 0 \le j \le M_y, \ 0 \le n \le N\}$, we define the following difference operators:

$$\mu_t w_{ij}^{n+\frac{1}{2}} = \frac{w_{ij}^n + w_{ij}^{n+1}}{2}, \qquad 0 \le n \le N-1,$$

$$\delta_t w_{ij}^{n+\frac{1}{2}} = \frac{1}{\tau}\left(w_{ij}^{n+1} - w_{ij}^n\right), \quad 0 \le n \le N-1,$$

$$\delta_x w_{ij}^n = \frac{w_{ij}^n - w_{i-1,j}^n}{h_x}, \qquad 1 \le i \le M_x,$$

$$\delta_x^2 w_{ij}^n = \frac{\delta_x w_{i+1,j}^n - \delta_x w_{ij}^n}{h_x}, \quad 1 \le i \le M_x - 1,$$

$$\delta_y w_{ij}^n = \frac{w_{ij}^n - w_{i,j-1}^n}{h_y}, \qquad 1 \le j \le M_y,$$

$$\delta_y^2 w_{ij}^n = \frac{\delta_y w_{i,j+1}^n - \delta_y w_{ij}^n}{h_y}, \quad 1 \le j \le M_y - 1,$$

$$\mathcal{A}w_{ij}^n = \begin{cases} \left(1 + \dfrac{h_x^2}{12}\delta_x^2\right) w_{ij}^n, & 1 \le i \le M_x - 1, \\ w_{ij}^n, & i = 0 \quad \text{or} \quad M_x, \end{cases}$$

$$\mathcal{B}w_{ij}^n = \begin{cases} \left(1 + \dfrac{h_y^2}{12}\delta_y^2\right) w_{ij}^n, & 1 \le j \le M_y - 1, \\ w_{ij}^n, & j = 0 \quad \text{or} \quad M_y. \end{cases}$$

Besides, the following lemma will play a key role in derivation of the method.

Lemma 2.1 (cf. [7]). *Assume that $G(x) \in C^6[x_{i-1}, x_{i+1}]$. Then*

$$\frac{1}{12}[G''(x_{i-1}) + 10G''(x_i) + G''(x_{i+1})]$$
$$- \frac{1}{h_x^2}[G(x_{i-1}) - 2G(x_i) + G(x_{i+1})]$$
$$= \frac{h_x^4}{240}G^{(6)}(\omega_i), \quad \omega_i \in (x_{i-1}, x_{i+1}).$$

Setting

$$(x, y, t) = \left(x_i, y_j, t_{n+\frac{1}{2}}\right), \quad 1 \le i \le M_x - 1, \quad 1 \le j \le M_y - 1, \quad 0 \le n \le N - 1$$

in the Eq. (1.8a) gives that

$$\frac{\partial u}{\partial t}\left(x_i, y_j, t_{n+\frac{1}{2}}\right) = \alpha \frac{\partial^2 u}{\partial x^2}\left(x_i, y_j, t_{n+\frac{1}{2}}\right) + \beta \frac{\partial^2 u}{\partial y^2}\left(x_i, y_j, t_{n+\frac{1}{2}}\right)$$
$$+ f\left(x_i, y_j, t_{n+\frac{1}{2}}, u\left(x_i, y_j, t_{n+\frac{1}{2}}\right), u\left(x_i, y_j, \lfloor t_{n+\frac{1}{2}} \rfloor\right)\right). \tag{2.1}$$

For $1 \le i \le M_x - 1, 1 \le j \le M_y - 1, 0 \le n \le N - 1$, applying Taylor expansion to each term of equality (2.1) yields that

$$\delta_t U_{ij}^{n+\frac{1}{2}} = \frac{\alpha}{2}\left[\frac{\partial^2 u}{\partial x^2}(x_i, y_j, t_n) + \frac{\partial^2 u}{\partial x^2}(x_i, y_j, t_{n+1})\right]$$
$$+ \frac{\beta}{2}\left[\frac{\partial^2 u}{\partial y^2}(x_i, y_j, t_n) + \frac{\partial^2 u}{\partial y^2}(x_i, y_j, t_{n+1})\right]$$
$$+ \frac{1}{2}\left[f\left(x_i, y_j, t_{n+\frac{1}{2}}, u(x_i, y_j, t_n), u\left(x_i, y_j, \lfloor t_{n+\frac{1}{2}} \rfloor\right)\right)\right.$$
$$\left. + f\left(x_i, y_j, t_{n+\frac{1}{2}}, u(x_i, y_j, t_{n+1}), u\left(x_i, y_j, \lfloor t_{n+\frac{1}{2}} \rfloor\right)\right)\right] + r_{ij}^{n+\frac{1}{2}}, \tag{2.2}$$

where derivatives f_u and g_u are defined by

$$f_u(x, y, t, u, v) = \frac{\partial f}{\partial u}(x, y, t, u, v),$$
$$g_u(x, y, t, u, v) = \frac{\partial^2 f}{\partial u^2}(x, y, t, u, v),$$

and

$$
\begin{aligned}
r_{ij}^{n+\frac{1}{2}} &= \frac{\tau^2}{24}\frac{\partial^3 u}{\partial t^3}(x_i, y_j, \varsigma_n) - \frac{\alpha\tau^2}{8}\frac{\partial^4 u}{\partial x^2\partial t^2}(x_i, y_j, \varpi_n) - \frac{\beta\tau^2}{8}\frac{\partial^4 u}{\partial y^2\partial t^2}(x_i, y_j, \zeta_n) \\
&\quad - \frac{\tau^2}{8}f_u\big(x_i, y_j, t_{n+\frac{1}{2}}, u(x_i, y_j, \rho_n), u\big(x_i, y_j, \lfloor t_{n+\frac{1}{2}}\rfloor\big)\big)\frac{\partial^2 u}{\partial t^2}(x_i, y_j, \rho_n) \\
&\quad - \frac{\tau^2}{8}g_u\big(x_i, y_j, t_{n+\frac{1}{2}}, u(x_i, y_j, \rho_n), u\big(x_i, y_j, \lfloor t_{n+\frac{1}{2}}\rfloor\big)\big)\left[\frac{\partial u}{\partial t}(x_i, y_j, \rho_n)\right]^2,
\end{aligned}
\tag{2.3}
$$

$$
\varsigma_n \in (t_n, t_{n+1}), \quad \varpi_n \in (t_n, t_{n+1}), \quad \zeta_n \in (t_n, t_{n+1}), \quad \rho_n \in (t_n, t_{n+1}).
$$

Let $n = km + l$, where k and l are two nonnegative integers with $k \le \lfloor T\rfloor - 1$ and $l \le m - 1$. Then $\lfloor t_{n+1/2}\rfloor = t_{km}$ and thus (2.2) becomes that

$$
\begin{aligned}
\delta_t U_{ij}^{km+l+\frac{1}{2}} &= \frac{\alpha}{2}\left[\frac{\partial^2 u}{\partial x^2}(x_i, y_j, t_{km+l}) + \frac{\partial^2 u}{\partial x^2}(x_i, y_j, t_{km+l+1})\right] \\
&\quad + \frac{\beta}{2}\left[\frac{\partial^2 u}{\partial y^2}(x_i, y_j, t_{km+l}) + \frac{\partial^2 u}{\partial y^2}(x_i, y_j, t_{km+l+1})\right] \\
&\quad + \frac{1}{2}\Big[f\big(x_i, y_j, t_{km+l+\frac{1}{2}}, U_{ij}^{km+l}, U_{ij}^{km}\big) \\
&\qquad\quad + f\big(x_i, y_j, t_{km+l+\frac{1}{2}}, U_{ij}^{km+l+1}, U_{ij}^{km}\big)\Big] + r_{ij}^{km+l+\frac{1}{2}},
\end{aligned}
\tag{2.4}
$$

$$
1 \le i \le M_x - 1, \quad 1 \le j \le M_y - 1, \quad 0 \le k \le \lfloor T\rfloor - 1, \quad 0 \le l \le m - 1.
$$

Applying the operator $\mathcal{AB}$ to the both sides of (2.4) yields

$$
\begin{aligned}
\mathcal{AB}\delta_t U_{ij}^{km+l+\frac{1}{2}} &= \frac{\alpha}{2}\mathcal{AB}\left[\frac{\partial^2 u}{\partial x^2}(x_i, y_j, t_{km+l}) + \frac{\partial^2 u}{\partial x^2}(x_i, y_j, t_{km+l+1})\right] \\
&\quad + \frac{\beta}{2}\mathcal{AB}\left[\frac{\partial^2 u}{\partial y^2}(x_i, y_j, t_{km+l}) + \frac{\partial^2 u}{\partial y^2}(x_i, y_j, t_{km+l+1})\right] \\
&\quad + \frac{\mathcal{AB}}{2}\Big[f\big(x_i, y_j, t_{km+l+\frac{1}{2}}, U_{ij}^{km+l}, U_{ij}^{km}\big) \\
&\qquad\quad + f\big(x_i, y_j, t_{km+l+\frac{1}{2}}, U_{ij}^{km+l+1}, U_{ij}^{km}\big)\Big] + \mathcal{AB}r_{ij}^{km+l+\frac{1}{2}},
\end{aligned}
\tag{2.5}
$$

$$
1 \le i \le M_x - 1, \quad 1 \le j \le M_y - 1, \quad 0 \le k \le \lfloor T\rfloor - 1, \quad 0 \le l \le m - 1.
$$

Making use of Lemma 2.1 and Taylor expansion, we can obtain that

$$
\begin{aligned}
\mathcal{AB}\frac{\partial^2 u}{\partial x^2}(x_i, y_j, t_{km+\hat{l}}) &= \mathcal{B}\delta_x^2 U_{ij}^{km+\hat{l}} + \frac{h_x^4}{240}\mathcal{B}\frac{\partial^6 u}{\partial x^6}(\theta_i, y_j, t_{km+\hat{l}}), \\
\hat{l} &= l, l+1, \quad \theta_i \in (x_{i-1}, x_{i+1}),
\end{aligned}
\tag{2.6}
$$

$$
\begin{aligned}
\mathcal{AB}\frac{\partial^2 u}{\partial y^2}(x_i, y_j, t_{km+\hat{l}}) &= \mathcal{A}\delta_y^2 U_{ij}^{km+\hat{l}} + \frac{h_y^4}{240}\mathcal{A}\frac{\partial^6 u}{\partial y^6}(x_i, \nu_j, t_{km+\hat{l}}), \\
\hat{l} &= l, l+1, \quad \nu_j \in (y_{j-1}, y_{j+1}).
\end{aligned}
\tag{2.7}
$$

Substituting (2.6)-(2.7) into (2.5) and performing an identical transformation to the right-side of the generative equality by adding and subtracting the small term $-\alpha\beta\tau^2\delta_x^2\delta_y^2\delta_t U_{ij}^{km+l+1/2}/4$ derive that

$$
\mathcal{AB}\delta_t U_{ij}^{km+l+\frac{1}{2}} = (\alpha\mathcal{B}\delta_x^2 + \beta\mathcal{A}\delta_y^2)\mu_t U_{ij}^{km+l+\frac{1}{2}} - \frac{1}{4}\alpha\beta\tau^2\delta_x^2\delta_y^2\delta_t U_{ij}^{km+l+\frac{1}{2}}
$$

$$+ \frac{\mathcal{AB}}{2}\Big[f\big(x_i, y_j, t_{km+l+\frac{1}{2}}, U_{ij}^{km+l}, U_{ij}^{km}\big)$$

$$+ f\big(x_i, y_j, t_{km+l+\frac{1}{2}}, U_{ij}^{km+l+1}, U_{ij}^{km}\big)\Big] + R_{ij}^{km+l+\frac{1}{2}}, \qquad (2.8)$$

$$1 \le i \le M_x - 1, \quad 1 \le j \le M_y - 1, \quad 0 \le k \le \lfloor T \rfloor - 1, \quad 0 \le l \le m - 1,$$

where

$$R_{ij}^{km+l+\frac{1}{2}} = \mathcal{AB}r_{ij}^{km+l+\frac{1}{2}} + \frac{\alpha h_x^4}{480}\mathcal{B}\left[\frac{\partial^6 u}{\partial x^6}(\theta_i, y_j, t_{km+l}) + \frac{\partial^6 u}{\partial x^6}(\theta_i, y_j, t_{km+l+1})\right]$$

$$+ \frac{\beta h_y^4}{480}\mathcal{A}\left[\frac{\partial^6 u}{\partial y^6}(x_i, \nu_j, t_{km+l}) + \frac{\partial^6 u}{\partial y^6}(x_i, \nu_j, t_{km+l+1})\right]$$

$$+ \frac{1}{4}\alpha\beta\tau^2\delta_x^2\delta_y^2\delta_t U_{ij}^{km+l+\frac{1}{2}}. \qquad (2.9)$$

Write $\mathcal{D}_n = [n, n+1]$. When $u(x, y, t) \in C^{6,3}(\bar{\Omega} \times \mathcal{D}_n)$ $(0 \le n \le N-1)$, it can be inferred from (2.3), (2.9) and Taylor expansion that there exists a constant $c_0 > 0$ independent of τ, h_x, h_y such that

$$\big|R_{ij}^{km+l+\frac{1}{2}}\big| \le c_0\big(\tau^2 + h_x^4 + h_y^4\big),$$

$$1 \le i \le M_x - 1, \quad 1 \le j \le M_y - 1, \quad 0 \le k \le \lfloor T \rfloor - 1, \quad 0 \le l \le m - 1. \qquad (2.10)$$

In equality (2.8), dropping local truncation error $R_{ij}^{km+l+1/2}$ and replacing $U_{ij}^{km+\hat{l}}$ with its corresponding approximation $u_{ij}^{km+\hat{l}}$, $\hat{l} = 0, l, l+1/2, l+1$, a high-order compact difference method for IBVPs (1.8) can be derived as follows:

$$\mathcal{AB}\delta_t u_{ij}^{km+l+\frac{1}{2}} = \big(\alpha\mathcal{B}\delta_x^2 + \beta\mathcal{A}\delta_y^2\big)\mu_t u_{ij}^{km+l+\frac{1}{2}} - \frac{1}{4}\alpha\beta\tau^2\delta_x^2\delta_y^2\delta_t u_{ij}^{km+l+\frac{1}{2}}$$

$$+ \frac{\mathcal{AB}}{2}\Big[f\big(x_i, y_j, t_{km+l+\frac{1}{2}}, u_{ij}^{km+l}, u_{ij}^{km}\big)$$

$$+ f\big(x_i, y_j, t_{km+l+\frac{1}{2}}, u_{ij}^{km+l+1}, u_{ij}^{km}\big)\Big], \qquad (2.11)$$

$$1 \le i \le M_x - 1, \quad 1 \le j \le M_y - 1, \quad 0 \le k \le \lfloor T \rfloor - 1, \quad 0 \le l \le m - 1,$$

whose initial and boundary values are determined by

$$u_{ij}^0 = \varphi(x_i, y_j), \qquad 0 \le i \le M_x, \quad 0 \le j \le M_y,$$

$$u_{ij}^n = \psi(x_i, y_j, t_n), \quad (i, j) \in \Gamma, \qquad 1 \le n \le N, \qquad (2.12)$$

where

$$\Gamma = \big\{(i, 0), (i, M_y), (0, j), (M_x, j) \,\big|\, 0 \le i \le M_x, \, 0 \le j \le M_y\big\}.$$

It is remarkable that method (2.11) also can be written in a more compact form

$$\left(\mathcal{A} - \frac{\alpha\tau}{2}\delta_x^2\right)\left(\mathcal{B} - \frac{\beta\tau}{2}\delta_y^2\right)u_{ij}^{km+l+1}$$

$$= \left(\mathcal{A} + \frac{\alpha\tau}{2}\delta_x^2\right)\left(\mathcal{B} + \frac{\beta\tau}{2}\delta_y^2\right)u_{ij}^{km+l}$$

$$+ \frac{\tau\mathcal{AB}}{2}\Big[f\big(x_i, y_j, t_{km+l+\frac{1}{2}}, u_{ij}^{km+l}, u_{ij}^{km}\big)$$

$$+ f\big(x_i, y_j, t_{km+l+\frac{1}{2}}, u_{ij}^{km+l+1}, u_{ij}^{km}\big)\Big], \qquad (2.13)$$

$$1 \le i \le M_x - 1, 1 \le j \le M_y - 1, \quad 0 \le k \le \lfloor T \rfloor - 1, \quad 0 \le l \le m - 1,$$

which can be implemented by the usual Newton iteration method.

3. Error Analysis of the Compact Difference Method

This section will deal with error analysis of the compact difference method (2.11)-(2.12). For this, we first need to introduce some basic notations and preliminary results.

Let $V = \{v \mid v = \{v_{ij} \mid 0 \leq i \leq M_x, \ 0 \leq j \leq M_y$ and $v_{ij} = 0$ for $(i,j) \in \Gamma\}\}$ be a grid function space defined on Ω_h. For all $v \in V$, we define the following norms:

$$\|v\| = \sqrt{h_x h_y \sum_{i=1}^{M_x-1} \sum_{j=1}^{M_y-1} (v_{ij})^2}, \qquad \|\delta_x v\| = \sqrt{h_x h_y \sum_{i=0}^{M_x-1} \sum_{j=1}^{M_y-1} (\delta_x v_{ij})^2},$$

$$\|\delta_y v\| = \sqrt{h_x h_y \sum_{i=1}^{M_x-1} \sum_{j=0}^{M_y-1} (\delta_y v_{ij})^2}, \quad \|\delta_x^2 v\| = \sqrt{h_x h_y \sum_{i=1}^{M_x-1} \sum_{j=1}^{M_y-1} (\delta_x^2 v_{ij})^2},$$

$$\|\delta_y^2 v\| = \sqrt{h_x h_y \sum_{i=1}^{M_x-1} \sum_{j=1}^{M_y-1} (\delta_y^2 v_{ij})^2}, \quad \|\delta_x \delta_y v\| = \sqrt{h_x h_y \sum_{i=0}^{M_x-1} \sum_{j=0}^{M_y-1} (\delta_x \delta_y v_{ij})^2},$$

$$|v|_1^2 = \|\delta_x v\|^2 + \|\delta_y v\|^2, \qquad\qquad \|v\|_1^2 = \|v\|^2 + |v|_1^2.$$

The following lemmas will be quite useful for the subsequent error analysis.

Lemma 3.1 (cf. [4]). *Let* $v \in V$. *Then the following inequalities hold:*

(i) $h_x h_y \displaystyle\sum_{i=1}^{M_x-1} \sum_{j=1}^{M_y-1} (\mathcal{A}v_{ij})v_{ij} \geq \frac{2h_x h_y}{3} \sum_{i=1}^{M_x-1} \sum_{j=1}^{M_y-1} (v_{ij})^2,$

(ii) $h_x h_y \displaystyle\sum_{i=1}^{M_x-1} \sum_{j=1}^{M_y-1} (\mathcal{B}v_{ij})v_{ij} \geq \frac{2h_x h_y}{3} \sum_{i=1}^{M_x-1} \sum_{j=1}^{M_y-1} (v_{ij})^2,$

(iii) $h_x h_y \displaystyle\sum_{i=1}^{M_x-1} \sum_{j=1}^{M_y-1} (\mathcal{AB}v_{ij})v_{ij} \leq h_x h_y \sum_{i=1}^{M_x-1} \sum_{j=1}^{M_y-1} (v_{ij})^2,$

(iv) $h_x h_y \displaystyle\sum_{i=1}^{M_x-1} \sum_{j=1}^{M_y-1} (\mathcal{AB}v_{ij})v_{ij} \geq \frac{4h_x h_y}{9} \sum_{i=1}^{M_x-1} \sum_{j=1}^{M_y-1} (v_{ij})^2.$

Lemma 3.2 (cf. [4]). *Let* $v, w \in V$. *Then the following equalities hold:*

(i) $h_x h_y \displaystyle\sum_{i=1}^{M_x-1} \sum_{j=1}^{M_y-1} (\delta_x^2 v_{ij})w_{ij} = -h_x h_y \sum_{i=0}^{M_x-1} \sum_{j=1}^{M_y-1} (\delta_x v_{ij})\delta_x w_{ij},$

(ii) $h_x h_y \displaystyle\sum_{i=1}^{M_x-1} \sum_{j=1}^{M_y-1} (\delta_y^2 v_{ij})w_{ij} = -h_x h_y \sum_{i=1}^{M_x-1} \sum_{j=0}^{M_y-1} (\delta_y v_{ij})\delta_y w_{ij}.$

Lemma 3.3 (cf. [4]). *Let* $v \in V$. *Then the following inequalities hold:*

(i) $\|\delta_x v\|^2 \leq \dfrac{4}{h_x^2}\|v\|^2,$ *(ii)* $\|\delta_y v\|^2 \leq \dfrac{4}{h_y^2}\|v\|^2,$

(iii) $\|v\|^2 \leq \dfrac{(b-a)^2}{4}\|\delta_x v\|^2,$ *(iv)* $\|v\|^2 \leq \dfrac{(d-c)^2}{4}\|\delta_y v\|^2.$

Lemma 3.4 (cf. [13]). *Let $A, B \geq 0$ be two constants such that nonnegative sequence $\{F_k\}$ satisfies*

$$F_{k+1} \leq A + B\tau \sum_{i=0}^{k} F_i, \quad \forall\, k \geq 0.$$

Then $F_{k+1} \leq A \exp(Bk\tau)$ for all $k \geq 0$.

Below, we set out to study the global error of method (2.11)-(2.12), where the global error of the method on n-th layer is indicated by

$$e^n = \begin{cases} \{e_{ij}^0 := 0 \mid 0 \leq i \leq M_x, \ 0 \leq j \leq M_y\}, & n = 0, \\ \{e_{ij}^n := U_{ij}^n - u_{ij}^n \mid 0 \leq i \leq M_x, \ 0 \leq j \leq M_y, \ \text{and } e_{ij}^n = 0 \text{ for } (i,j) \in \Gamma\}, & 1 \leq n \leq N. \end{cases}$$

An error estimate of method (2.11)-(2.12) can be stated as follows.

Theorem 3.1. *Assume that $u(x,y,t) \in C^{6,3}(\bar{\Omega} \times \mathcal{D}_n)$ $(0 \leq n \leq N-1)$ and Lipschitz condition (1.9) holds. Then the global error of method (2.11)-(2.12) satisfies the following estimates:*

$$\|e^n\| \leq c_1\big(\tau^2 + h_x^4 + h_y^4\big), \quad \|e^n\|_1 \leq c_2\big(\tau^2 + h_x^4 + h_y^4\big), \quad 1 \leq n \leq N, \tag{3.1}$$

where $c_1, c_2 > 0$ are two constants independent of stepsizes τ, h_x and h_y.

Proof. Subtracting (2.11) from (2.8) yields that

$$\begin{aligned}
\mathcal{AB}\delta_t e_{ij}^{km+l+\frac{1}{2}} = {}& \big(\alpha\mathcal{B}\delta_x^2 + \beta\mathcal{A}\delta_y^2\big)\mu_t e_{ij}^{km+l+\frac{1}{2}} - \frac{1}{4}\alpha\beta\tau^2 \delta_x^2 \delta_y^2 \delta_t e_{ij}^{km+l+\frac{1}{2}} \\
& + \frac{\mathcal{AB}}{2}\Big[f\big(x_i, y_j, t_{km+l+\frac{1}{2}}, U_{ij}^{km+l}, U_{ij}^{km}\big) \\
& \qquad - f\big(x_i, y_j, t_{km+l+\frac{1}{2}}, u_{ij}^{km+l}, u_{ij}^{km}\big) \\
& \qquad + f\big(x_i, y_j, t_{km+l+\frac{1}{2}}, U_{ij}^{km+l+1}, U_{ij}^{km}\big) \\
& \qquad - f\big(x_i, y_j, t_{km+l+\frac{1}{2}}, u_{ij}^{km+l+1}, u_{ij}^{km}\big) \Big] + R_{ij}^{km+l+\frac{1}{2}},
\end{aligned} \tag{3.2}$$

$$1 \leq i \leq M_x - 1, \quad 1 \leq j \leq M_y - 1, \quad 0 \leq k \leq \lfloor T \rfloor - 1, \quad 0 \leq l \leq m - 1.$$

Multiplying (3.2) by $h_x h_y \delta_t e_{ij}^{km+l+1/2}$ and summing up for i from 1 to $M_x - 1$ and j from 1 to $M_y - 1$, we obtain that

$$\begin{aligned}
& h_x h_y \sum_{i=1}^{M_x-1} \sum_{j=1}^{M_y-1} \left(\mathcal{AB}\delta_t e_{ij}^{km+l+\frac{1}{2}} \right) \delta_t e_{ij}^{km+l+\frac{1}{2}} \\
={}& h_x h_y \sum_{i=1}^{M_x-1} \sum_{j=1}^{M_y-1} \big(\alpha\mathcal{B}\delta_x^2 + \beta\mathcal{A}\delta_y^2\big)\mu_t e_{ij}^{km+l+\frac{1}{2}} \delta_t e_{ij}^{km+l+\frac{1}{2}} \\
& - h_x h_y \sum_{i=1}^{M_x-1} \sum_{j=1}^{M_y-1} \left(\frac{1}{4}\alpha\beta\tau^2 \delta_x^2 \delta_y^2 \delta_t e_{ij}^{km+l+\frac{1}{2}} \right) \delta_t e_{ij}^{km+l+\frac{1}{2}} \\
& + h_x h_y \sum_{i=1}^{M_x-1} \sum_{j=1}^{M_y-1} \frac{\mathcal{AB}}{2}\Big[f\big(x_i, y_j, t_{km+l+\frac{1}{2}}, U_{ij}^{km+l}, U_{ij}^{km}\big)
\end{aligned}$$

$$
\begin{aligned}
& - f\left(x_i, y_j, t_{km+l+\frac{1}{2}}, u_{ij}^{km+l}, u_{ij}^{km}\right) \\
& + f\left(x_i, y_j, t_{km+l+\frac{1}{2}}, U_{ij}^{km+l+1}, U_{ij}^{km}\right) \\
& \left. - f\left(x_i, y_j, t_{km+l+\frac{1}{2}}, u_{ij}^{km+l+1}, u_{ij}^{km}\right)\right] \delta_t e_{ij}^{km+l+\frac{1}{2}}
\end{aligned}
$$
$$
+ h_x h_y \sum_{i=1}^{M_x-1} \sum_{j=1}^{M_y-1} R_{ij}^{km+l+\frac{1}{2}} \delta_t e_{ij}^{km+l+\frac{1}{2}}, \quad 0 \le k \le \lfloor T \rfloor - 1, \quad 0 \le l \le m - 1. \tag{3.3}
$$

Next, we estimate each term in (3.3). It follows from inequality Lemma 3.1(iv) that

$$
h_x h_y \sum_{i=1}^{M_x-1} \sum_{j=1}^{M_y-1} \left(\mathcal{AB} \delta_t e_{ij}^{km+l+\frac{1}{2}}\right) \delta_t e_{ij}^{km+l+\frac{1}{2}}
$$
$$
\ge \frac{4}{9} \left\| \delta_t e^{km+l+\frac{1}{2}} \right\|^2, \quad 0 \le k \le \lfloor T \rfloor - 1, \quad 0 \le l \le m - 1. \tag{3.4}
$$

Applying Lemma 3.2 to the first and second terms on the right-side of (3.3) derives for $0 \le k \le \lfloor T \rfloor - 1$ and $0 \le l \le m - 1$ that

$$
h_x h_y \sum_{i=1}^{M_x-1} \sum_{j=1}^{M_y-1} \left(\alpha \mathcal{B} \delta_x^2 + \beta \mathcal{A} \delta_y^2\right) \left(\mu_t e_{ij}^{km+l+\frac{1}{2}}\right) \delta_t e_{ij}^{km+l+\frac{1}{2}}
$$
$$
= -\frac{\alpha}{2\tau} \left[\left(\left\|\delta_x e^{km+l+1}\right\|^2 - \left\|\delta_x e^{km+l}\right\|^2\right) - \frac{h_y^2}{12}\left(\left\|\delta_x \delta_y e^{km+l+1}\right\|^2 - \left\|\delta_x \delta_y e^{km+l}\right\|^2\right)\right]
$$
$$
- \frac{\beta}{2\tau} \left[\left(\left\|\delta_y e^{km+l+1}\right\|^2 - \left\|\delta_y e^{km+l}\right\|^2\right) - \frac{h_x^2}{12}\left(\left\|\delta_x \delta_y e^{km+l+1}\right\|^2 - \left\|\delta_x \delta_y e^{km+l}\right\|^2\right)\right], \tag{3.5}
$$

$$
-h_x h_y \sum_{i=1}^{M_x-1} \sum_{j=1}^{M_y-1} \left(\frac{1}{4}\alpha\beta\tau^2 \delta_x^2 \delta_y^2 \delta_t e_{ij}^{km+l+\frac{1}{2}}\right) \delta_t e_{ij}^{km+l+\frac{1}{2}} = -\frac{1}{4}\alpha\beta\tau^2 \left\|\delta_x \delta_y \delta_t e^{km+l+\frac{1}{2}}\right\|^2. \tag{3.6}
$$

For the third term on the right-hand side of (3.3), taking use of Lipschitz condition (1.9), inequalities Lemma 3.1(iii), Lemma 3.3(iii)-(iv) and the following common inequality:

$$
pq \le \frac{1}{2\varepsilon}p^2 + \frac{\varepsilon}{2}q^2, \quad \forall\, p, q \in \mathbb{R}, \quad \varepsilon > 0, \tag{3.7}
$$

we obtain that

$$
h_x h_y \sum_{i=1}^{M_x-1} \sum_{j=1}^{M_y-1} \frac{\mathcal{AB}}{2} \Big[f\left(x_i, y_j, t_{km+l+\frac{1}{2}}, U_{ij}^{km+l}, U_{ij}^{km}\right)
$$
$$
- f\left(x_i, y_j, t_{km+l+\frac{1}{2}}, u_{ij}^{km+l}, u_{ij}^{km}\right)
$$
$$
+ f\left(x_i, y_j, t_{km+l+\frac{1}{2}}, U_{ij}^{km+l+1}, U_{ij}^{km}\right)
$$
$$
\left. - f\left(x_i, y_j, t_{km+l+\frac{1}{2}}, u_{ij}^{km+l+1}, u_{ij}^{km}\right)\right] \delta_t e_{ij}^{km+l+\frac{1}{2}}
$$
$$
\le \frac{3\mathcal{L}_1^2 (b-a)^2}{32} \left\|\delta_x e^{km+l}\right\|^2 + \frac{3\mathcal{L}_1^2 (d-c)^2}{32} \left\|\delta_y e^{km+l}\right\|^2
$$
$$
+ \frac{3\mathcal{L}_1^2 (b-a)^2}{32} \left\|\delta_x e^{km+l+1}\right\|^2 + \frac{3\mathcal{L}_1^2 (d-c)^2}{32} \left\|\delta_y e^{km+l+1}\right\|^2
$$
$$
+ \frac{3\mathcal{L}_2^2 (b-a)^2}{16} \left\|\delta_x e^{km}\right\|^2 + \frac{3\mathcal{L}_2^2 (d-c)^2}{16} \left\|\delta_y e^{km}\right\|^2 + \frac{1}{3} \left\|\delta_t e^{km+l+\frac{1}{2}}\right\|^2. \tag{3.8}
$$

Also, it holds by inequality (3.7) for $0 \leq k \leq \lfloor T \rfloor - 1$ and $0 \leq l \leq m-1$ that

$$h_x h_y \sum_{i=1}^{M_x-1} \sum_{j=1}^{M_y-1} R_{ij}^{km+l+\frac{1}{2}} \delta_t e_{ij}^{km+l+\frac{1}{2}}$$

$$\leq \frac{9c_0^2(b-a)(d-c)}{4} \left(\tau^2 + h_x^4 + h_y^4 \right)^2 + \frac{1}{9} \left\| \delta_t e^{km+l+\frac{1}{2}} \right\|^2. \tag{3.9}$$

Inserting (3.4)-(3.6) and (3.8)-(3.9) into (3.3) follows that

$$\frac{\alpha}{2\tau} \left[\left(\left\| \delta_x e^{km+l+1} \right\|^2 - \left\| \delta_x e^{km+l} \right\|^2 \right) - \frac{h_y^2}{12} \left(\left\| \delta_x \delta_y e^{km+l+1} \right\|^2 - \left\| \delta_x \delta_y e^{km+l} \right\|^2 \right) \right]$$

$$+ \frac{\beta}{2\tau} \left[\left(\left\| \delta_y e^{km+l+1} \right\|^2 - \left\| \delta_y e^{km+l} \right\|^2 \right) - \frac{h_x^2}{12} \left(\left\| \delta_x \delta_y e^{km+l+1} \right\|^2 - \left\| \delta_x \delta_y e^{km+l} \right\|^2 \right) \right]$$

$$\leq \frac{3\mathcal{L}_1^2(b-a)^2}{32} \left\| \delta_x e^{km+l} \right\|^2 + \frac{3\mathcal{L}_1^2(d-c)^2}{32} \left\| \delta_y e^{km+l} \right\|^2 + \frac{3\mathcal{L}_1^2(b-a)^2}{32} \left\| \delta_x e^{km+l+1} \right\|^2$$

$$+ \frac{3\mathcal{L}_1^2(d-c)^2}{32} \left\| \delta_y e^{km+l+1} \right\|^2 + \frac{3\mathcal{L}_2^2(b-a)^2}{16} \left\| \delta_x e^{km} \right\|^2 + \frac{3\mathcal{L}_2^2(d-c)^2}{16} \left\| \delta_y e^{km} \right\|^2$$

$$+ \frac{9c_0^2(b-a)(d-c)}{4} \left(\tau^2 + h_x^4 + h_y^4 \right)^2, \quad 0 \leq k \leq \lfloor T \rfloor - 1, \quad 0 \leq l \leq m-1. \tag{3.10}$$

On the both sides of (3.10), taking sum of l from 0 to $\tilde{m}$ ($\leq m-1$) generates that

$$\frac{\alpha}{2\tau} \left[\left(\left\| \delta_x e^{km+\tilde{m}+1} \right\|^2 - \left\| \delta_x e^{km} \right\|^2 \right) - \frac{h_y^2}{12} \left(\left\| \delta_x \delta_y e^{km+\tilde{m}+1} \right\|^2 - \left\| \delta_x \delta_y e^{km} \right\|^2 \right) \right]$$

$$+ \frac{\beta}{2\tau} \left[\left(\left\| \delta_y e^{km+\tilde{m}+1} \right\|^2 - \left\| \delta_y e^{km} \right\|^2 \right) - \frac{h_x^2}{12} \left(\left\| \delta_x \delta_y e^{km+\tilde{m}+1} \right\|^2 - \left\| \delta_x \delta_y e^{km} \right\|^2 \right) \right]$$

$$\leq 3 \sum_{l=0}^{\tilde{m}} \left[\frac{\mathcal{L}_1^2(b-a)^2}{32} \left\| \delta_x e^{km+l} \right\|^2 + \frac{\mathcal{L}_1^2(d-c)^2}{32} \left\| \delta_y e^{km+l} \right\|^2 \right.$$

$$\left. + \frac{\mathcal{L}_1^2(b-a)^2}{32} \left\| \delta_x e^{km+l+1} \right\|^2 + \frac{\mathcal{L}_1^2(d-c)^2}{32} \left\| \delta_y e^{km+l+1} \right\|^2 \right]$$

$$+ \frac{3\mathcal{L}_2^2(\tilde{m}+1)}{16} \left[(b-a)^2 \left\| \delta_x e^{km} \right\|^2 + (d-c)^2 \left\| \delta_y e^{km} \right\|^2 \right]$$

$$+ \frac{9c_0^2(b-a)(d-c)(\tilde{m}+1)}{4} \left(\tau^2 + h_x^4 + h_y^4 \right)^2, \quad 0 \leq k \leq \lfloor T \rfloor - 1, \quad 0 \leq \tilde{m} \leq m-1. \tag{3.11}$$

With (3.11) and inequalities Lemma 3.3(i)-(ii), it can be deduced that

$$\frac{\alpha}{2\tau} \left(\left\| \delta_x e^{km+\tilde{m}+1} \right\|^2 - \left\| \delta_x e^{km} \right\|^2 - \frac{1}{3} \left\| \delta_x e^{km+\tilde{m}+1} \right\|^2 \right)$$

$$+ \frac{\beta}{2\tau} \left(\left\| \delta_y e^{km+\tilde{m}+1} \right\|^2 - \left\| \delta_y e^{km} \right\|^2 - \frac{1}{3} \left\| \delta_y e^{km+\tilde{m}+1} \right\|^2 \right)$$

$$\leq \sum_{l=0}^{\tilde{m}} \left[\frac{3\mathcal{L}_1^2(b-a)^2}{16} \left\| \delta_x e^{km+l} \right\|^2 + \frac{3\mathcal{L}_1^2(d-c)^2}{16} \left\| \delta_y e^{km+l} \right\|^2 \right]$$

$$+ \frac{3\mathcal{L}_1^2(b-a)^2}{32} \left\| \delta_x e^{km+\tilde{m}+1} \right\|^2 + \frac{3\mathcal{L}_1^2(d-c)^2}{32} \left\| \delta_y e^{km+\tilde{m}+1} \right\|^2$$

$$+ \frac{3\mathcal{L}_2^2(b-a)^2(\tilde{m}+1)}{16} \left\| \delta_x e^{km} \right\|^2 + \frac{3\mathcal{L}_2^2(d-c)^2(\tilde{m}+1)}{16} \left\| \delta_y e^{km} \right\|^2$$

$$+ \frac{9c_0^2(b-a)(d-c)(\tilde{m}+1)}{4} \left(\tau^2 + h_x^4 + h_y^4 \right)^2, \quad 0 \leq k \leq \lfloor T \rfloor - 1, \quad 0 \leq \tilde{m} \leq m-1, \tag{3.12}$$

which can be rearranged as

$$
\left[\alpha - \frac{9\mathcal{L}_1^2(b-a)^2\tau}{32}\right]\left\|\delta_x e^{km+\tilde{m}+1}\right\|^2 + \left[\beta - \frac{9\mathcal{L}_1^2(d-c)^2\tau}{32}\right]\left\|\delta_y e^{km+\tilde{m}+1}\right\|^2
$$

$$
\leq \sum_{l=0}^{\tilde{m}}\left[\frac{9\mathcal{L}_1^2(b-a)^2\tau}{16}\left\|\delta_x e^{km+l}\right\|^2 + \frac{9\mathcal{L}_1^2(d-c)^2\tau}{16}\left\|\delta_y e^{km+l}\right\|^2\right]
$$

$$
+ \left[\frac{3\alpha}{2} + \frac{9\mathcal{L}_2^2(b-a)^2}{16}\right]\left\|\delta_x e^{km}\right\|^2 + \left[\frac{3\beta}{2} + \frac{9\mathcal{L}_2^2(d-c)^2}{16}\right]\left\|\delta_y e^{km}\right\|^2
$$

$$
+ \frac{27c_0^2(b-a)(d-c)}{4}\left(\tau^2 + h_x^4 + h_y^4\right)^2, \quad 0 \leq k \leq \lfloor T \rfloor - 1, \quad 0 \leq \tilde{m} \leq m - 1. \tag{3.13}
$$

Introduce $\lambda_1 = \min\{\alpha, \beta\}$, $\lambda_2 = \max\{\alpha, \beta\}$ and $\lambda_3 = \max\{(b-a)^2, (d-c)^2\}$. Then (3.13) can be written as

$$
\left(\lambda_1 - \frac{9\mathcal{L}_1^2\lambda_3\tau}{32}\right)\left(\left\|\delta_x e^{km+\tilde{m}+1}\right\|^2 + \left\|\delta_y e^{km+\tilde{m}+1}\right\|^2\right)
$$

$$
\leq \frac{9\mathcal{L}_1^2\lambda_3\tau}{16}\sum_{l=0}^{\tilde{m}}\left(\left\|\delta_x e^{km+l}\right\|^2 + \left\|\delta_y e^{km+l}\right\|^2\right)
$$

$$
+ \left(\frac{3\lambda_2}{2} + \frac{9\mathcal{L}_2^2\lambda_3}{16}\right)\left(\left\|\delta_x e^{km}\right\|^2 + \left\|\delta_y e^{km}\right\|^2\right) + \frac{27c_0^2\lambda_3}{4}\left(\tau^2 + h_x^4 + h_y^4\right)^2. \tag{3.14}
$$

Since there always exist constants $\bar{\tau}, \gamma > 0$ such that

$$
32\lambda_1 - 9\mathcal{L}_1^2\lambda_3\tau \geq \gamma > 0, \quad \forall\, \tau \in (0, \bar{\tau}], \tag{3.15}
$$

we have by (3.14) and the fact $\tau = 1/m$ ($m \in \mathbb{N}$) that

$$
\left\|\delta_x e^{km+\tilde{m}+1}\right\|^2 + \left\|\delta_y e^{km+\tilde{m}+1}\right\|^2
$$

$$
\leq \frac{18\mathcal{L}_1^2\lambda_3\tau}{32\lambda_1 - 9\mathcal{L}_1^2\lambda_3\tau}\sum_{l=0}^{\tilde{m}}\left(\left\|\delta_x e^{km+l}\right\|^2 + \left\|\delta_y e^{km+l}\right\|^2\right)
$$

$$
+ \frac{48\lambda_2 + 18\mathcal{L}_2^2\lambda_3}{32\lambda_1 - 9\mathcal{L}_1^2\lambda_3\tau}\left(\left\|\delta_x e^{km}\right\|^2 + \left\|\delta_y e^{km}\right\|^2\right) + \frac{216c_0^2\lambda_3}{32\lambda_1 - 9\mathcal{L}_1^2\lambda_3\tau}\left(\tau^2 + h_x^4 + h_y^4\right)^2
$$

$$
\leq \gamma_1\tau\sum_{l=0}^{\tilde{m}}\left(\left\|\delta_x e^{km+l}\right\|^2 + \left\|\delta_y e^{km+l}\right\|^2\right) + \gamma_2\left(\left\|\delta_x e^{km}\right\|^2 + \left\|\delta_y e^{km}\right\|^2\right)
$$

$$
+ \gamma_3\left(\tau^2 + h_x^4 + h_y^4\right)^2, \quad 0 \leq k \leq \lfloor T \rfloor - 1, \quad 0 \leq \tilde{m} \leq m - 1, \quad m \geq \left\lfloor\frac{1}{\bar{\tau}}\right\rfloor, \tag{3.16}
$$

where

$$
\gamma_1 = \frac{18\mathcal{L}_1^2\lambda_3}{\gamma}, \quad \gamma_2 = \frac{48\lambda_2 + 18\mathcal{L}_2^2\lambda_3}{\gamma}, \quad \gamma_3 = \frac{216c_0^2\lambda_3}{\gamma}.
$$

Inequality (3.16) means that

$$
|e^{km+\tilde{m}+1}|_1^2 \leq \gamma_1\tau\sum_{l=0}^{\tilde{m}}|e^{km+l}|_1^2 + \gamma_2|e^{km}|_1^2 + \gamma_3\left(\tau^2 + h_x^4 + h_y^4\right)^2,
$$

$$
0 \leq k \leq \lfloor T \rfloor - 1, \quad 0 \leq \tilde{m} \leq m - 1, \quad m \geq \left\lfloor\frac{1}{\bar{\tau}}\right\rfloor. \tag{3.17}
$$

Next, on the basis of (3.17), we use the mathematical induction to prove the following inequality:

$$|e^{km}|_1^2 \leq \frac{\gamma_3 \exp(\gamma_1) \left\{ 1 - [\gamma_2 \exp(\gamma_1)]^k \right\}}{1 - \gamma_2 \exp(\gamma_1)} \left(\tau^2 + h_x^4 + h_y^4 \right)^2,$$

$$0 \leq k \leq \lfloor T \rfloor - 1, \quad m \geq \left\lfloor \frac{1}{\bar{\tau}} \right\rfloor, \tag{3.18}$$

where it is remarkable that $1 - [\gamma_2 \exp(\gamma_1)]^k < 0$ for $0 \leq k \leq \lfloor T \rfloor - 1$ by the following fact:

$$\gamma_2 \exp(\gamma_1) = \frac{48\lambda_2 + 18\mathcal{L}_2^2\lambda_3}{\gamma} \exp\left(\frac{18\mathcal{L}_1^2\lambda_3}{\gamma} \right)$$

$$> \frac{48\lambda_2 + 18\mathcal{L}_2^2\lambda_3}{\gamma} \geq \frac{48\lambda_2 + 18\mathcal{L}_2^2\lambda_3}{32\lambda_1 - 9\mathcal{L}_1^2\lambda_3\tau} > 1,$$

in which the last inequality is obtained by (3.15) and the definitions of λ_i, $i = 1, 2, 3$. When $k = 0$, (3.18) holds since under this case the left and right sides of (3.18) are equal to zero. Assume that (3.18) holds for $0 < k \leq \tilde{k} \, (< \lfloor T \rfloor - 1)$, we go to prove that (3.18) is also true for $k = \tilde{k} + 1$. Setting $\tilde{m} = m - 1$ in (3.17) and using the inductive assumption, we have that

$$|e^{\tilde{k}m+m}|_1^2 \leq \gamma_1\tau \sum_{l=0}^{m-1} |e^{\tilde{k}m+l}|_1^2 + \frac{\gamma_3 \left\{ 1 - [\gamma_2 \exp(\gamma_1)]^{\tilde{k}+1} \right\}}{1 - \gamma_2 \exp(\gamma_1)} \left(\tau^2 + h_x^4 + h_y^4 \right)^2. \tag{3.19}$$

An application of Lemma 3.4 to (3.19) gives

$$|e^{(\tilde{k}+1)m}|_1^2 \leq \frac{\gamma_3 \exp(\gamma_1) \left\{ 1 - [\gamma_2 \exp(\gamma_1)]^{\tilde{k}+1} \right\}}{1 - \gamma_2 \exp(\gamma_1)} (\tau^2 + h_x^4 + h_y^4)^2,$$

$$0 \leq k \leq \lfloor T \rfloor - 1, \quad m \geq \left\lfloor \frac{1}{\bar{\tau}} \right\rfloor, \tag{3.20}$$

which shows that (3.18) holds for $k = \tilde{k} + 1$ and thus (3.18) is proved for $0 \leq k \leq \lfloor T \rfloor - 1$ and $m \geq \lfloor 1/\bar{\tau} \rfloor$. Combining (3.17) and (3.18) follows for $0 \leq k \leq \lfloor T \rfloor - 1, 0 \leq \tilde{m} \leq m - 1$ and $m \geq \lfloor 1/\bar{\tau} \rfloor$ that

$$|e^{km+\tilde{m}+1}|_1^2 \leq \gamma_1\tau \sum_{l=0}^{\tilde{m}} |e^{km+l}|_1^2 + \frac{\gamma_3 \left\{ 1 - [\gamma_2 \exp(\gamma_1)]^{k+1} \right\}}{1 - \gamma_2 \exp(\gamma_1)} \left(\tau^2 + h_x^4 + h_y^4 \right)^2. \tag{3.21}$$

Applying Lemma 3.4 to (3.21) gives

$$|e^{km+\tilde{m}+1}|_1 \leq \sqrt{\frac{\gamma_3 \exp(\gamma_1) \left\{ 1 - [\gamma_2 \exp(\gamma_1)]^{k+1} \right\}}{1 - \gamma_2 \exp(\gamma_1)}} \left(\tau^2 + h_x^4 + h_y^4 \right)$$

$$\leq \sqrt{\frac{\gamma_3 \exp(\gamma_1) \left\{ 1 - [\gamma_2 \exp(\gamma_1)]^{\lfloor T \rfloor} \right\}}{1 - \gamma_2 \exp(\gamma_1)}} \left(\tau^2 + h_x^4 + h_y^4 \right), \tag{3.22}$$

$$0 \leq k \leq \lfloor T \rfloor - 1, \quad 0 \leq \tilde{m} \leq m - 1, \quad m \geq \left\lfloor \frac{1}{\bar{\tau}} \right\rfloor.$$

By (3.22) and inequalities Lemma 3.3(iii)-(iv), it holds that

$$\|e^{km+\tilde{m}+1}\| \leq \sqrt{\frac{(b-a)^2}{8} \left\| \delta_x e^{km+\tilde{m}+1} \right\|^2 + \frac{(d-c)^2}{8} \left\| \delta_y e^{km+\tilde{m}+1} \right\|^2}$$

$$= \frac{\sqrt{2\lambda_3}}{4} |e^{km+\tilde{m}+1}|_1 \leq c_1 \left(\tau^2 + h_x^4 + h_y^4 \right), \tag{3.23}$$

$$0 \leq k \leq \lfloor T \rfloor - 1, \quad 0 \leq \tilde{m} \leq m - 1, \quad m \geq \left\lfloor \frac{1}{\bar{\tau}} \right\rfloor,$$

where

$$c_1 = \sqrt{\frac{\gamma_3^2 \exp(\gamma_1)\big\{1 - [\gamma_2 \exp(\gamma_1)]^{\lfloor T \rfloor}\big\}}{8[1 - \gamma_2 \exp(\gamma_1)]}}.$$

Combining (3.22) with (3.23), we have for $0 \le k \le \lfloor T \rfloor - 1$, $0 \le \tilde{m} \le m - 1$, $m \ge \lfloor 1/\bar{\tau} \rfloor$ that

$$\|e^{km+\tilde{m}+1}\|_1 = \sqrt{\|e^{km+\tilde{m}+1}\|^2 + |e^{km+\tilde{m}+1}|_1^2} \le c_2(\tau^2 + h_x^4 + h_y^4), \qquad (3.24)$$

where

$$c_2 = \sqrt{\frac{[\gamma_3 \exp(\gamma_1)(\gamma_3 + 8)\big\{1 - [\gamma_2 \exp(\gamma_1)]^{\lfloor T \rfloor}\big\}}{8[1 - \gamma_2 \exp(\gamma_1)]}}.$$

A combination of (3.23) and (3.24) gives the error estimates in (3.1).   □

Theorem 3.1 shows that method (2.11)-(2.12) is convergent of order two (respectively four) in time (respectively space) under the given conditions. As to the stability of method (2.11)-(2.12), we can obtain it by the similar analytical techniques and arguments to these for Theorem 3.1. Due to the limitation in the length of the paper and the similarity to the proof of Theorem 3.1, in the following, we will only provide the stability theorem and omit its proof. For state the stability theorem, we introduce the following perturbation problem with initial perturbation function $\hat{\varphi}(x, y)$:

$$\begin{cases} \hat{u}_t(x, y, t) = \alpha \hat{u}_{xx}(x, y, t) + \beta \hat{u}_{yy}(x, y, t) \\ \qquad\qquad + f\big(x, y, t, \hat{u}(x, y, t), \hat{u}(x, y, \lfloor t \rfloor)\big), & (x, y) \in \Omega, \quad t \in (0, T], \\ \hat{u}(x, y, 0) = \hat{\varphi}(x, y), & (x, y) \in \bar{\Omega}, \\ \hat{u}(x, y, t) = \psi(x, y, t), & (x, y) \in \partial\Omega, \quad t \in (0, T]. \end{cases} \qquad (3.25)$$

Let

$$\hat{u}^n = \big\{\hat{u}_{ij}^n \,|\, 0 \le i \le M_x - 1, \, 0 \le j \le M_y - 1\big\}, \quad 1 \le n \le N$$

be the numerical solution on the n-th temporal layer generated by applying method (2.11)-(2.12) to problem (3.25). Then a stability theorem of method (2.11)-(2.12) can be presented as follows.

Theorem 3.2. *Assume that Lipschitz condition* (1.9) *holds. Then method* (2.11)-(2.12) *for problem* (1.8) *satisfies the following stability properties for* $1 \le n \le N$:

$$\|u^n - \hat{u}^n\| \le c_3 \max_{(x,y)\in\bar{\Omega}} \big[|\varphi(x, y) - \hat{\varphi}(x, y)| + |\varphi_x(x, y) - \hat{\varphi}_x(x, y)| + |\varphi_y(x, y) - \hat{\varphi}_y(x, y)|\big],$$

$$\|u^n - \hat{u}^n\|_1 \le c_4 \max_{(x,y)\in\bar{\Omega}} \big[|\varphi(x, y) - \hat{\varphi}(x, y)| + |\varphi_x(x, y) - \hat{\varphi}_x(x, y)| + |\varphi_y(x, y) - \hat{\varphi}_y(x, y)|\big],$$

where $c_3, c_4 > 0$ *are two constants independent of stepsizes* τ, h_x *and* h_y.

4. Two Types of ADI Schemes

In order to improve the computational efficiency of compact difference method (2.11)-(2.12), in the following, we will propose two types of ADI schemes.

Let

$$u_{ij}^* = \left(\mathcal{B} - \frac{\beta\tau}{2}\delta_y^2\right) u_{ij}^{km+l+1}.$$

Then, based on this symbol and method (2.13), a high-order compact ADI scheme can be obtained as follows:

$$
\begin{aligned}
\left(\mathcal{A} - \frac{\alpha\tau}{2}\delta_x^2\right)u_{ij}^* &= \left(\mathcal{A} + \frac{\alpha\tau}{2}\delta_x^2\right)\left(\mathcal{B} + \frac{\beta\tau}{2}\delta_y^2\right)u_{ij}^{km+l} \\
&\quad + \frac{\tau\mathcal{A}\mathcal{B}}{2}\Big[f\big(x_i, y_j, t_{km+l+\frac{1}{2}}, u_{ij}^{km+l}, u_{ij}^{km}\big) \\
&\qquad\qquad + f\big(x_i, y_j, t_{km+l+\frac{1}{2}}, u_{ij}^{km+l+1}, u_{ij}^{km}\big)\Big],
\end{aligned}
\tag{4.1}
$$

$$
1 \leq i \leq M_x - 1, \quad 1 \leq j \leq M_y - 1, \quad 0 \leq k \leq \lfloor T \rfloor - 1, \quad 0 \leq l \leq m - 1,
$$

$$
\left(\mathcal{B} - \frac{\beta\tau}{2}\delta_y^2\right)u_{ij}^{km+l+1} = u_{ij}^*,
\tag{4.2}
$$

$$
1 \leq i \leq M_x - 1, \quad 1 \leq j \leq M_y - 1, \quad 0 \leq k \leq \lfloor T \rfloor - 1, \quad 0 \leq l \leq m - 1,
$$

whose initial and boundary values are given by

$$
\begin{aligned}
u_{ij}^0 &= \varphi(x_i, y_j), & 0 \leq i \leq M_x, \quad 0 \leq j \leq M_y, \\
u_{ij}^n &= \psi(x_i, y_j, t_n), & (i,j) \in \Gamma, \quad 1 \leq n \leq N,
\end{aligned}
\tag{4.3}
$$

$$
u_{ij}^* = \left(\mathcal{B} - \frac{\beta\tau}{2}\delta_y^2\right)\psi_{ij}^{km+l+1},
\tag{4.4}
$$

$$
i = 0, M_x, \quad 0 \leq j \leq M_y, \quad 0 \leq k \leq \lfloor T \rfloor - 1, \quad 0 \leq l \leq m - 1.
$$

In what follows, scheme (4.1)-(4.4) will be called I-type ADI scheme, which can be implemented efficiently by combining Newton iteration algorithm and Thomas algorithm (see, e.g. [12]). Moreover, since I-type ADI scheme is obtained by an identical deformation of method (2.11)-(2.12), the both methods are actually equivalent and thus Theorems 3.1 and 3.2 can also cover I-type ADI scheme, namely, under the same conditions of the above theorems, I-type ADI scheme is convergent of order two (respectively four) in time (respectively space) and globally stable.

As an alternative approximation technique for dealing with nonlinear problems of differential equations, Newton linearized method is frequently used, whose advantage is that it can transform nonlinear computational problems into linear ones and thus derives some efficient numerical methods. In the following, we consider adopting this technique to construct a new ADI method for solving IBVPs (1.8). Applying Taylor expansion to each term of equality (2.1) derives for $1 \leq i \leq M_x - 1, 1 \leq j \leq M_y - 1, 0 \leq n \leq N - 1$ that

$$
\begin{aligned}
\delta_t U_{ij}^{n+\frac{1}{2}} &= \frac{\alpha}{2}\left[\frac{\partial^2 u}{\partial x^2}(x_i, y_j, t_n) + \frac{\partial^2 u}{\partial x^2}(x_i, y_j, t_{n+1})\right] \\
&\quad + \frac{\beta}{2}\left[\frac{\partial^2 u}{\partial y^2}(x_i, y_j, t_n) + \frac{\partial^2 u}{\partial y^2}(x_i, y_j, t_{n+1})\right] \\
&\quad + f\big(x_i, y_j, t_{n+\frac{1}{2}}, u(x_i, y_j, t_n), u\big(x_i, y_j, \lfloor t_{n+\frac{1}{2}} \rfloor\big)\big) \\
&\quad + \frac{\tau}{2}\left[f_u\big(x_i, y_j, t_{n+\frac{1}{2}}, u(x_i, y_j, t_n), u\big(x_i, y_j, \lfloor t_{n+\frac{1}{2}} \rfloor\big)\big)\delta_t U_{ij}^{n+\frac{1}{2}}\right] + \bar{r}_{ij}^{n+\frac{1}{2}},
\end{aligned}
\tag{4.5}
$$

where the following expansion of $f(x_i, y_j, t_{n+1/2}, u(x_i, y_j, t_{n+1/2}), u(x_i, y_j, \lfloor t_{n+1/2} \rfloor))$ is used:

$$
\begin{aligned}
&f\Big(x_i, y_j, t_{n+\frac{1}{2}}, u\big(x_i, y_j, t_{n+\frac{1}{2}}\big), u\big(x_i, y_j, \lfloor t_{n+\frac{1}{2}} \rfloor\big)\Big) \\
&= f\Big(x_i, y_j, t_{n+\frac{1}{2}}, u(x_i, y_j, t_n), u\big(x_i, y_j, \lfloor t_{n+\frac{1}{2}} \rfloor\big)\Big)
\end{aligned}
$$

$$+ \frac{\tau}{2} f_u\Big(x_i, y_j, t_{n+\frac{1}{2}}, u(x_i, y_j, t_n), u\big(x_i, y_j, \lfloor t_{n+\frac{1}{2}}\rfloor\big)\Big) \delta_t u\big(x_i, y_j, t_{n+\frac{1}{2}}\big)$$

$$+ \frac{\tau^2}{8} \Big\{ f_u\Big(x_i, y_j, t_{n+\frac{1}{2}}, u(x_i, y_j, \kappa_n), u\big(x_i, y_j, \lfloor t_{n+\frac{1}{2}}\rfloor\big)\Big) \frac{\partial^2 u}{\partial t^2}(x_i, y_j, \kappa_n)$$

$$+ g_u\Big(x_i, y_j, t_{n+\frac{1}{2}}, u(x_i, y_j, \kappa_n), u\big(x_i, y_j, \lfloor t_{n+\frac{1}{2}}\rfloor\big)\Big) \Big[\frac{\partial u}{\partial t}(x_i, y_j, \kappa_n)\Big]^2 \Big\}$$

$$- \frac{\tau^2}{4} f_u\Big(x_i, y_j, t_{n+\frac{1}{2}}, u(x_i, y_j, \vartheta_n), u\big(x_i, y_j, \lfloor t_{n+\frac{1}{2}}\rfloor\big)\Big) \frac{\partial^2 u}{\partial t^2}(x_i, y_j, \vartheta_n),$$

$$\kappa_n \in \big(t_n, t_{n+\frac{1}{2}}\big), \quad \vartheta_n \in (t_n, t_{n+1}), \tag{4.6}$$

which is different from that in (2.2), and

$$\bar{r}_{ij}^{n+\frac{1}{2}} = \frac{\tau^2}{24} \frac{\partial^3 u}{\partial t^3}(x_i, y_j, \varsigma_n) - \frac{\alpha \tau^2}{8} \frac{\partial^4 u}{\partial x^2 \partial t^2}(x_i, y_j, \varpi_n) - \frac{\beta \tau^2}{8} \frac{\partial^4 u}{\partial y^2 \partial t^2}(x_i, y_j, \zeta_n)$$

$$+ \frac{\tau^2}{8} \Big\{ f_u\Big(x_i, y_j, t_{n+\frac{1}{2}}, u(x_i, y_j, \kappa_n), u\big(x_i, y_j, \lfloor t_{n+\frac{1}{2}}\rfloor\big)\Big) \frac{\partial^2 u}{\partial t^2}(x_i, y_j, \kappa_n)$$

$$+ g_u\Big(x_i, y_j, t_{n+\frac{1}{2}}, u(x_i, y_j, \kappa_n), u\big(x_i, y_j, \lfloor t_{n+\frac{1}{2}}\rfloor\big)\Big) \Big[\frac{\partial u}{\partial t}(x_i, y_j, \kappa_n)\Big]^2 \Big\} \tag{4.7}$$

$$- \frac{\tau^2}{4} f_u\Big(x_i, y_j, t_{n+\frac{1}{2}}, u(x_i, y_j, \vartheta_n), u\big(x_i, y_j, \lfloor t_{n+\frac{1}{2}}\rfloor\big)\Big) \frac{\partial^2 u}{\partial t^2}(x_i, y_j, \vartheta_n),$$

$$\varsigma_n \in (t_n, t_{n+1}), \quad \varpi_n \in (t_n, t_{n+1}), \quad \zeta_n \in (t_n, t_{n+1}), \quad \kappa_n \in (t_n, t_{n+\frac{1}{2}}), \quad \vartheta_n \in (t_n, t_{n+1}).$$

Since $\lfloor t_{n+1/2} \rfloor = t_{km}$ when setting $n = km + l$, where k and l are two nonnegative integers with $k \leq \lfloor T \rfloor - 1$ and $l \leq m - 1$, scheme (4.5) can be rewritten as

$$\delta_t U_{ij}^{km+l+\frac{1}{2}} = \frac{\alpha}{2} \left[\frac{\partial^2 u}{\partial x^2}(x_i, y_j, t_{km+l}) + \frac{\partial^2 u}{\partial x^2}(x_i, y_j, t_{km+l+1}) \right]$$

$$+ \frac{\beta}{2} \left[\frac{\partial^2 u}{\partial y^2}(x_i, y_j, t_{km+l}) + \frac{\partial^2 u}{\partial y^2}(x_i, y_j, t_{km+l+1}) \right]$$

$$+ f\big(x_i, y_j, t_{km+l+\frac{1}{2}}, U_{ij}^{km+l}, U_{ij}^{km}\big) \tag{4.8}$$

$$+ \frac{\tau}{2} f_u\big(x_i, y_j, t_{km+l+\frac{1}{2}}, U_{ij}^{km+l}, U_{ij}^{km}\big) \delta_t U_{ij}^{km+l+\frac{1}{2}} + \bar{r}_{ij}^{km+l+\frac{1}{2}},$$

$$1 \leq i \leq M_x - 1, \quad 1 \leq j \leq M_y - 1, \quad 0 \leq k \leq \lfloor T \rfloor - 1, \quad 0 \leq l \leq m - 1.$$

Applying the operator $\mathcal{AB}$ to the both sides of (4.8), gives for $1 \leq i \leq M_x - 1, 1 \leq j \leq M_y - 1$, $0 \leq k \leq \lfloor T \rfloor - 1$ and $0 \leq l \leq m - 1$ that

$$\mathcal{AB} \delta_t U_{ij}^{km+l+\frac{1}{2}} = \frac{\alpha \mathcal{AB}}{2} \left[\frac{\partial^2 u}{\partial x^2}(x_i, y_j, t_{km+l}) + \frac{\partial^2 u}{\partial x^2}(x_i, y_j, t_{km+l+1}) \right]$$

$$+ \frac{\beta \mathcal{AB}}{2} \left[\frac{\partial^2 u}{\partial y^2}(x_i, y_j, t_{km+l}) + \frac{\partial^2 u}{\partial y^2}(x_i, y_j, t_{km+l+1}) \right]$$

$$+ \mathcal{AB} f\big(x_i, y_j, t_{km+l+\frac{1}{2}}, U_{ij}^{km+l}, U_{ij}^{km}\big)$$

$$+ \frac{\tau \mathcal{AB}}{2} f_u\big(x_i, y_j, t_{km+l+\frac{1}{2}}, U_{ij}^{km+l}, U_{ij}^{km}\big) \delta_t U_{ij}^{km+l+\frac{1}{2}} + \mathcal{AB} \bar{r}_{ij}^{km+l+\frac{1}{2}}. \tag{4.9}$$

Inserting (2.6)-(2.7) into (4.9) and then carrying out an identical transformation to the right-

side of the generated equality by adding and subtracting the following small term:

$$-\frac{1}{4}\alpha\beta\tau^2\delta_x^2\delta_y^2\delta_t U_{ij}^{km+l+\frac{1}{2}} - \frac{\alpha\tau^2\mathcal{B}}{4}\delta_x^2\left[f_u\left(x_i,y_j,t_{km+l+\frac{1}{2}},U_{ij}^{km+l},U_{ij}^{km}\right)\delta_t U_{ij}^{km+l+\frac{1}{2}}\right],$$

we obtain for $1 \le i \le M_x - 1, 1 \le j \le M_y - 1, 0 \le k \le \lfloor T \rfloor - 1$ and $0 \le l \le m - 1$ that

$$
\begin{aligned}
\mathcal{AB}\delta_t U_{ij}^{km+l+\frac{1}{2}} &= \left(\alpha\mathcal{B}\delta_x^2 + \beta\mathcal{A}\delta_y^2\right)\mu_t U_{ij}^{km+l+\frac{1}{2}} - \frac{1}{4}\alpha\beta\tau^2\delta_x^2\delta_y^2\delta_t U_{ij}^{km+l+\frac{1}{2}} \\
&\quad + \mathcal{AB}f\left(x_i,y_j,t_{km+l+\frac{1}{2}},U_{ij}^{km+l},U_{ij}^{km}\right) \\
&\quad + \frac{\tau\mathcal{AB}}{2}f_u\left(x_i,y_j,t_{km+l+\frac{1}{2}},U_{ij}^{km+l},U_{ij}^{km}\right)\delta_t U_{ij}^{km+l+\frac{1}{2}} \\
&\quad - \frac{\alpha\tau^2\mathcal{B}}{4}\delta_x^2\left[f_u\left(x_i,y_j,t_{km+l+\frac{1}{2}},U_{ij}^{km+l},U_{ij}^{km}\right)\delta_t U_{ij}^{km+l+\frac{1}{2}}\right] + \bar{R}_{ij}^{km+l+\frac{1}{2}}, \quad (4.10)
\end{aligned}
$$

where

$$
\begin{aligned}
\bar{R}_{ij}^{km+l+\frac{1}{2}} &= \mathcal{AB}\bar{r}_{ij}^{km+l+\frac{1}{2}} + \frac{\alpha h_x^4}{480}\mathcal{B}\left[\frac{\partial^6 u}{\partial x^6}(\theta_i,y_j,t_{km+l}) + \frac{\partial^6 u}{\partial x^6}(\theta_i,y_j,t_{km+l+1})\right] \\
&\quad + \frac{\beta h_y^4}{480}\mathcal{A}\left[\frac{\partial^6 u}{\partial y^6}(x_i,\nu_j,t_{km+l}) + \frac{\partial^6 u}{\partial y^6}(x_i,\nu_j,t_{km+l+1})\right] \\
&\quad + \frac{\alpha\tau^2\mathcal{B}}{4}\delta_x^2\left(f_u\left(x_i,y_j,t_{km+l+\frac{1}{2}},U_{ij}^{km+l},U_{ij}^{km}\right)\delta_t U_{ij}^{km+l+\frac{1}{2}}\right) \\
&\quad + \frac{1}{4}\alpha\beta\tau^2\delta_x^2\delta_y^2 U_{ij}^{km+l+\frac{1}{2}}.
\end{aligned}
\quad (4.11)
$$

When $u(x,y,t) \in C^{6,3}(\bar{\Omega} \times \mathcal{D}_n)$ $(0 \le n \le N - 1)$, it follows from (4.7), (4.11) and Taylor expansion that there exists a constant $c_3 > 0$ independent of τ, h_x, h_y such that

$$
\begin{aligned}
&\left|\bar{R}_{ij}^{km+l+\frac{1}{2}}\right| \le c_3\left(\tau^2 + h_x^4 + h_y^4\right), \\
&1 \le i \le M_x - 1, \quad 1 \le j \le M_y - 1, \quad 0 \le k \le \lfloor T \rfloor - 1, \quad 0 \le l \le m - 1.
\end{aligned}
\quad (4.12)
$$

In equality (4.10), dropping local truncation error $\bar{R}_{ij}^{km+l+1/2}$ and replacing $U_{ij}^{km+\hat{l}}$ with its corresponding approximation $\bar{u}_{ij}^{km+\hat{l}}$, $\hat{l} = 0, l, l + 1/2$, a Newton linearized high-order compact difference method can be followed, namely

$$
\begin{aligned}
\mathcal{AB}\delta_t \bar{u}_{ij}^{km+l+\frac{1}{2}} &= \left(\alpha\mathcal{B}\delta_x^2 + \beta\mathcal{A}\delta_y^2\right)\mu_t \bar{u}_{ij}^{km+l+\frac{1}{2}} - \frac{1}{4}\alpha\beta\tau^2\delta_x^2\delta_y^2\delta_t \bar{u}_{ij}^{km+l+\frac{1}{2}} \\
&\quad + \mathcal{AB}f\left(x_i,y_j,t_{km+l+\frac{1}{2}},\bar{u}_{ij}^{km+l},\bar{u}_{ij}^{km}\right) \\
&\quad + \frac{\tau\mathcal{AB}}{2}f_u\left(x_i,y_j,t_{km+l+\frac{1}{2}},\bar{u}_{ij}^{km+l},\bar{u}_{ij}^{km}\right)\delta_t \bar{u}_{ij}^{km+l+\frac{1}{2}} \\
&\quad - \frac{\alpha\tau^2\mathcal{B}}{4}\delta_x^2\left[f_u\left(x_i,y_j,t_{km+l+\frac{1}{2}},\bar{u}_{ij}^{km+l},\bar{u}_{ij}^{km}\right)\delta_t \bar{u}_{ij}^{km+l+\frac{1}{2}}\right], \\
1 \le i \le M_x - 1, &\quad 1 \le j \le M_y - 1, \quad 0 \le k \le \lfloor T \rfloor - 1, \quad 0 \le l \le m - 1.
\end{aligned}
\quad (4.13)
$$

Method (4.13) also can be integrated into

$$
\begin{aligned}
&\left(\mathcal{A} - \frac{\alpha\tau}{2}\delta_x^2\right)\left\{\mathcal{B}\left[1 - \frac{\tau}{2}f_u\left(x_i,y_j,t_{km+l+\frac{1}{2}},\bar{u}_{ij}^{km+l},\bar{u}_{ij}^{km}\right)\right] - \frac{\beta\tau}{2}\delta_y^2\right\}\bar{u}_{ij}^{km+l+1} \\
&= \left[\mathcal{AB} - \frac{\tau}{2}\left(\mathcal{AB} - \frac{\alpha\tau}{2}\mathcal{B}\delta_x^2\right)f_u\left(x_i,y_j,t_{km+l+\frac{1}{2}},\bar{u}_{ij}^{km+l},\bar{u}_{ij}^{km}\right)\right]
\end{aligned}
$$

$$+ \frac{\tau}{2}\left(\alpha\mathcal{B}\delta_x^2 + \beta\mathcal{A}\delta_y^2 + \frac{\alpha\beta\tau}{2}\delta_x^2\delta_y^2\right)\Big]\bar{u}_{ij}^{km+l} + \tau\mathcal{A}\mathcal{B}f\big(x_i, y_j, t_{km+l+\frac{1}{2}}, \bar{u}_{ij}^{km+l}, \bar{u}_{ij}^{km}\big),$$

$$1 \le i \le M_x - 1, \quad 1 \le j \le M_y - 1, \quad 0 \le k \le \lfloor T \rfloor - 1, \quad 0 \le l \le m - 1, \qquad (4.14)$$

where the initial and boundary values are given by

$$\begin{aligned}
\bar{u}_{ij}^0 &= \varphi(x_i, y_j), & 0 \le i \le M_x, \quad 0 \le j \le M_y, \\
\bar{u}_{ij}^n &= \psi(x_i, y_j, t_n), & (i,j) \in \Gamma, \qquad 1 \le n \le N.
\end{aligned} \qquad (4.15)$$

Write

$$\bar{u}_{ij}^* = \left\{ \mathcal{B}\left[1 - \frac{\tau}{2}f_u\big(x_i, y_j, t_{km+l+\frac{1}{2}}, \bar{u}_{ij}^{km+l}, \bar{u}_{ij}^{km}\big)\right] - \frac{\beta\tau}{2}\delta_y^2 \right\} \bar{u}_{ij}^{km+l+1}.$$

With this symbol and (4.14)-(4.15), a Newton linearized high-order compact ADI scheme for $1 \le i \le M_x - 1, 1 \le j \le M_y - 1, 0 \le k \le \lfloor T \rfloor - 1, 0 \le l \le m - 1$ can be presented as follows:

$$\begin{aligned}
\left(\mathcal{A} - \frac{\alpha\tau}{2}\delta_x^2\right)\bar{u}_{ij}^* &= \Big[\mathcal{A}\mathcal{B} - \frac{\tau}{2}\left(\mathcal{A}\mathcal{B} - \frac{\alpha\tau}{2}\mathcal{B}\delta_x^2\right)f_u\big(x_i, y_j, t_{km+l+\frac{1}{2}}, \bar{u}_{ij}^{km+l}, \bar{u}_{ij}^{km}\big) \\
&\quad + \frac{\tau}{2}\left(\alpha\mathcal{B}\delta_x^2 + \beta\mathcal{A}\delta_y^2 + \frac{\alpha\beta\tau}{2}\delta_x^2\delta_y^2\right)\Big]\bar{u}_{ij}^{km+l} \\
&\quad + \tau\mathcal{A}\mathcal{B}f\big(x_i, y_j, t_{km+l+\frac{1}{2}}, \bar{u}_{ij}^{km+l}, \bar{u}_{ij}^{km}\big),
\end{aligned} \qquad (4.16)$$

$$\left\{ \mathcal{B}\left[1 - \frac{\tau}{2}f_u\big(x_i, y_j, t_{km+l+\frac{1}{2}}, \bar{u}_{ij}^{km+l}, \bar{u}_{ij}^{km}\big)\right] - \frac{\beta\tau}{2}\delta_y^2 \right\} \bar{u}_{ij}^{km+l+1} = \bar{u}_{ij}^*, \qquad (4.17)$$

whose initial and boundary values are given by

$$\begin{aligned}
\bar{u}_{ij}^0 &= \varphi(x_i, y_j), & 0 \le i \le M_x, \quad 0 \le j \le M_y, \\
\bar{u}_{ij}^n &= \psi(x_i, y_j, t_n), & (i,j) \in \Gamma, \qquad 1 \le n \le N, \\
\bar{u}_{ij}^* &= \left\{ \mathcal{B}\left[1 - \frac{\tau}{2}f_u\big(x_i, y_j, t_{km+l+\frac{1}{2}}, \bar{u}_{ij}^{km+l}, \bar{u}_{ij}^{km}\big)\right] - \frac{\beta\tau}{2}\delta_y^2 \right\} \psi_{ij}^{km+l+1}, \\
i &= 0, M_x, \quad 0 \le j \le M_y, \quad 0 \le k \le \lfloor T \rfloor - 1, \quad 0 \le l \le m - 1.
\end{aligned}$$

$$(4.18)$$
$$(4.19)$$

Scheme (4.16)-(4.19) can be implemented directly by Thomas algorithm (cf. [12]). To distinguish from the previous I-type ADI scheme, we call scheme (4.16)-(4.19) as II-type ADI scheme.

5. Numerical Experiments

In order to illustrate the computational effectiveness and numerical accuracy of the methods presented in previous sections, we consider the following IBVPs of 2D SLRDEs with PCA:

$$\begin{cases}
u_t(x, y, t) = u_{xx}(x, y, t) + u_{yy}(x, y, t), \\
\qquad + f\big(x, y, t, u(x, y, t), u(x, y, \lfloor t \rfloor)\big), & (x, y, t) \in (0, 1) \times (0, 1) \times (0, 5], \\
u(x, y, 0) = \tanh(x + y), & (x, y) \in [0, 1] \times [0, 1], \\
u(0, y, t) = \exp(t)\tanh(y), \quad u(1, y, t) = \exp(t)\tanh(1 + y), & y \in [0, 1], \quad t \in (0, 5], \\
u(x, 0, t) = \exp(t)\tanh(x), \quad u(x, 1, t) = \exp(t)\tanh(1 + x), & x \in [0, 1], \quad t \in (0, 5],
\end{cases} \qquad (5.1)$$

where

$$f\big(x, y, t, u(x,y,t), u(x,y,\lfloor t \rfloor)\big) = \sin[u(x,y,t)] + \frac{u(x, y, \lfloor t \rfloor)}{\sqrt{1 + [u(x, y, \lfloor t \rfloor)]^2}} + g(x, y, t),$$

$$g(x, y, t) = \exp(t) \tanh(x+y)\big[1 - 4\big(\tanh(x+y)\big)^2 - 4\big] - \sin[\exp(t) \tanh(x+y)]$$
$$- \frac{\exp(\lfloor t \rfloor) \tanh(x+y)}{\sqrt{1 + [\exp(\lfloor t \rfloor) \tanh(x+y)]^2}},$$

and $\tanh(\cdot)$ denotes the hyperbolic tangent function defined by

$$\tanh(\hat{x}) = \frac{\exp(\hat{x}) - \exp(-\hat{x})}{\exp(\hat{x}) + \exp(-\hat{x})}, \quad \hat{x} \in \mathbb{R}.$$

It can be checked that problem (5.1) has a sufficiently smooth exact solutions $u(x, y, t) = \exp(t) \tanh(x+y)$ and its right-function f satisfies Lipschitz conditions (1.9) with $\mathcal{L}_1 = \mathcal{L}_2 = 1$. Hence, in terms of Theorems 3.1 and 3.2, method (2.11)-(2.12) for problem (5.1) is stable and convergent of order two (respectively four) in time (respectively space).

For describing the global errors and convergence orders of the presented numerical methods, we introduce the following formulas:

$$E_1(h, \tau) = \max_{1 \le n \le N} \|U^n - u^n\|_1, \quad E_2(h, \tau) = \max_{1 \le n \le N} \|U^n - u^n\|,$$

$$\bar{E}_1(h, \tau) = \max_{1 \le n \le N} \|U^n - \bar{u}^n\|_1, \quad \bar{E}_2(h, \tau) = \max_{1 \le n \le N} \|U^n - \bar{u}^n\|,$$

$$p_1 = \frac{\log[E_1(h_1, \tau_1)/E_1(h_2, \tau_2)]}{\log(h_1/h_2)}, \quad p_2 = \frac{\log[E_2(h_1, \tau_1)/E_2(h_2, \tau_2)]}{\log(h_1/h_2)},$$

$$\bar{p}_1 = \frac{\log[\bar{E}_1(h_1, \tau_1)/\bar{E}_1(h_2, \tau_2)]}{\log(h_1/h_2)}, \quad \bar{p}_2 = \frac{\log[\bar{E}_2(h_1, \tau_1)/\bar{E}_2(h_2, \tau_2)]}{\log(h_1/h_2)},$$

where $E_1(h, \tau), E_2(h, \tau), \bar{E}_1(h, \tau), \bar{E}_2(h, \tau)$ (respectively $p_1, p_2, \bar{p}_1, \bar{p}_2$) denote the global errors (respectively convergence orders) of the corresponding numerical methods with $h = h_x = h_y$, $\tau_1 = h_1^2$ and $\tau_2 = h_2^2$. Moreover, for convenience of the statement, we refer to method (2.11)-(2.12) and method (4.14)-(4.15) as I-type basic scheme and II-type basic scheme, respectively.

Applying I-type basic scheme, II-type basic scheme, I-type ADI scheme and II-type ADI scheme with $\tau = h^2$, $h = 1/(4j)$, $j = 5, \ldots, 9$, to solve problem (5.1), the global errors, convergence orders and CPU times of four schemes are displayed in Tables 5.1 and 5.2, respectively. From the data in Tables 5.1-5.2, it can be found that the four numerical schemes are very effective for problem (5.1) and can reach second-order in time and fourth-order in space. Nevertheless, it is remarkable that the exact orders of II-type basic scheme and II-type ADI scheme

Table 5.1: Global errors, convergence orders and CPU times of I-type basic scheme and I-type ADI scheme with $\tau = h^2$ for problem (5.1).

h	τ	I-type basic scheme					I-type ADI scheme				
		E_1	p_1	E_2	p_2	CPU	E_1	p_1	E_2	p_2	CPU
1/20	1/400	1.8795e-04	–	2.1465e-05	–	1.7210e+02	1.8795e-04	–	2.1465e-05	–	5.8344e+00
1/24	1/576	8.0461e-05	4.65	8.8442e-06	4.86	7.8439e+02	8.0461e-05	4.65	8.8442e-06	4.86	1.1419e+01
1/28	1/784	4.3810e-05	3.94	4.7989e-06	3.97	2.4563e+03	4.3810e-05	3.94	4.7989e-06	3.97	1.9906e+01
1/32	1/1024	2.5790e-05	3.97	2.8123e-06	4.00	6.5372e+03	2.5791e-05	3.97	2.8126e-06	4.00	3.4617e+01
1/36	1/1296	1.6194e-05	3.95	1.7555e-06	4.00	1.5618e+04	1.6196e-05	3.95	1.7562e-06	4.00	5.4335e+01

Table 5.2: Global errors, convergence orders and CPU times of II-type basic scheme and II-type ADI schemes with $\tau = h^2$ for problem (5.1).

h	τ	II-type basic scheme					II-type ADI scheme				
		$\bar{E}_1$	$\bar{p}_1$	$\bar{E}_2$	$\bar{p}_2$	CPU	$\bar{E}_1$	$\bar{p}_1$	$\bar{E}_2$	$\bar{p}_2$	CPU
1/20	1/400	2.9000e-03	–	6.3095e-05	–	6.4959e+01	2.9000e-03	–	6.3095e-05	–	3.8376e+00
1/24	1/576	1.6000e-03	3.09	3.1031e-05	3.89	2.8512e+02	1.6000e-03	3.09	3.1031e-05	3.89	4.4148e+00
1/28	1/784	9.3348e-04	3.65	1.6208e-05	4.21	8.9701e+02	9.3348e-04	3.65	1.6208e-05	4.21	8.1121e+00
1/32	1/1024	6.0512e-04	3.25	9.5880e-06	3.93	2.4977e+03	6.0512e-04	3.25	9.5880e-06	3.93	1.4040e+01
1/36	1/1296	3.7378e-04	4.09	5.7903e-06	4.28	5.9438e+03	3.7378e-04	4.09	5.7903e-06	4.28	2.2511e+01

exhibit the significant fluctuation. This may be related to the stability of the both schemes. Owing to the complexities of II-type schemes and the lack of some useful mathematical tools, in the present paper, we are unable to derive the theoretical results on stability and convergence of II-type basic scheme and II-type ADI scheme and hope to leave this opening issue for our future research. As to the computational efficiency of the four schemes, according to the CPU times in Tables 5.1-5.2, we can know that I-type ADI scheme (respectively II-type ADI scheme) has the better computational efficiency than I-type basic scheme (respectively II-type basic scheme), II-type basic scheme has slightly better computational efficiency than I-type basic scheme, and the both types of ADI schemes have almost same computational efficiency.

In the end, for visually showing the computational accuracy of the presented ADI schemes for problem (5.1), we also plot their error surfaces on time layers $t = 2, 3, 4, 5$ in Figs. 5.1 and 5.2, where $h = \tau = 1/32$. These figures further verify that I-type and II-type ADI schemes are highly effective for solving IBVPs of 2D SLRDEs with PCA. As to the error surfaces of I-type and II-type basic schemes on time layers $t = 2, 3, 4, 5$, here we omit them since they are similar in vision to those of the corresponding I-type and II-type ADI schemes.

(a) $t = 2$

(b) $t = 3$

(c) $t = 4$

(d) $t = 5$
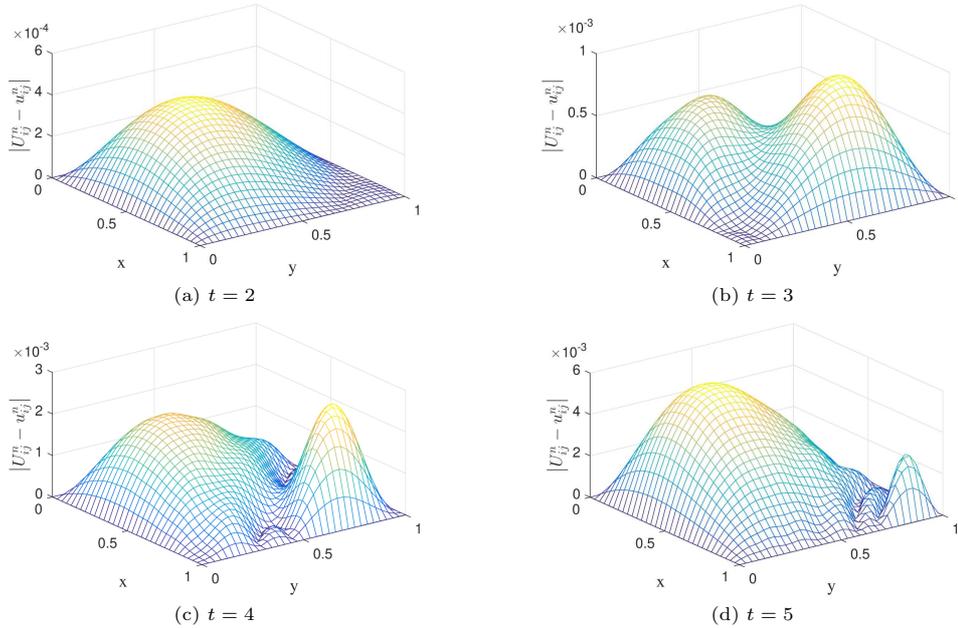
Fig. 5.1. Error surfaces of I-type ADI scheme with $h = \tau = 1/32$ for problem (5.1).
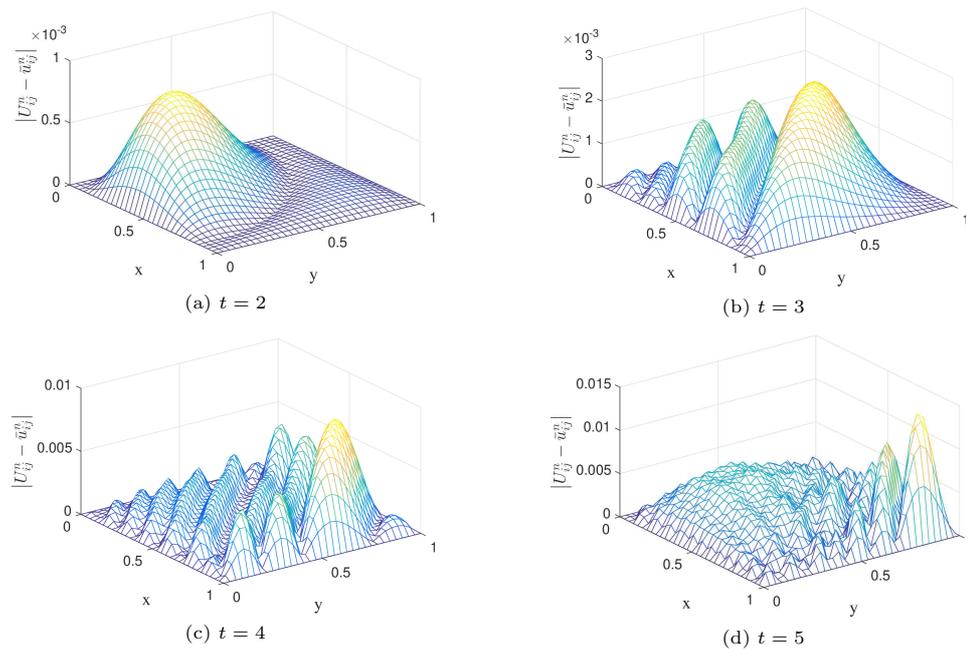
Fig. 5.2. Error surfaces of II-type ADI scheme with $h = \tau = 1/32$ for problem (5.1).

6. Conclusion

In this paper, we have proposed four kinds of numerical schemes for solving 2D SLRDEs with PCA in reaction term, namely, I- and II-type basic schemes and I- and II-type ADI schemes. It is proved under the suitable conditions that I-type basic scheme has the computational accuracy $\mathcal{O}(\tau^2 + h_x^4 + h_y^4)$ and is stable. Since I-type ADI scheme is derived by an identical deformation of I-type basic scheme, the both schemes have the same computational accuracy and I-type ADI scheme is also stable. As to the error estimates and stability of II-type basic scheme and II-type ADI scheme, we fail to derive their theoretical results owing to the extremal complexity of the schemes and the lack of some necessary mathematical arguments. At present, we only can show the computational effectiveness and accuracy of the both schemes by the numerical experiments in Section 5.

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