Journal of Computational Mathematics
Vol.44, No.1, 2026, 248–285.
http://www.global-sci.org/jcm
doi:10.4208/jcm.2410-m2024-0048

ANALYSIS OF AN EMBEDDED VARIABLE STEP IMPLICIT-EXPLICIT SCHEME FOR NATURAL CONVECTION PROBLEMS*

Mengru Jiang
College of Mathematics and Systems Science, Xinjiang University, Urumqi 830046, China
Email: jmrmath@163.com
Jilian Wu[1)]
School of Mathematics and Statistics, Henan University of Technology, Zhengzhou 450001, China
Email: wjl@haut.edu.cn
Xinlong Feng
College of Mathematics and System Sciences & Institute of Mathematics and Physics, Xinjiang University, Urumqi 830046, China
Email: fxlmath@xju.edu.cn
Ning Li
College of Information and Management of Science, Henan Agricultural University, Zhengzhou 450002, China
Email: ning@henau.edu.cn

Abstract

This report presents a series of implicit-explicit (IMEX) variable stepsize algorithms for natural convection equations. The presented method requires a minimally intrusive modification to an existing program, does not add to the computational complexity, and is conceptually simple. Here, IMEX means the nonlinear term is treated fully explicitly, while the remaining terms are treated implicitly. Due to the increasing demand for low memory solvers, the addition of time adaptive can improve the accuracy and efficiency of the algorithms. For the first-order algorithm, we prove the stability of the variable stepsize backward Euler scheme combined with Adams-Bashforth 2 (VSS BE-AB2) and analyze convergence. Then, the stability of Constant Timestep Filtered-BE-AB2 (BE-AB2+F) is proved. Moreover, we construct adaptive algorithms by extending the approach to variable stepsize. Finally, numerical tests confirm the convergence rates of our method and validate the theoretical results.

Mathematics subject classification: 65N12, 65N30, 65N50, 35Q79.
Key words: Time filter, Stability, Convergence, Adaptive algorithms, Natural convection.

1. Introduction

In this paper, we shall study stability and analyze convergence of a low complexity fully discrete time-stepping finite element method (FEM), then extend the method to adaptive time-stepping and higher order algorithms for natural convection (NC) problems [47]. We consider

* Received March 5, 2024 / Revised version received July 17, 2024 / Accepted October 14, 2024 / Published online December 9, 2024 /
1) Corresponding author

the numerical schemes for solving the time-dependent nonlinear NC equations

$$\begin{cases} \mathbf{u}_t - Pr\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = PrRa\mathbf{j}T & \text{in } \Omega\times(0,T_1], \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega\times(0,T_1], \\ T_t + (\mathbf{u}\cdot\nabla)T - \Delta T = \gamma & \text{in } \Omega\times(0,T_1], \\ \int_\Omega p\, d\mathbf{x} = 0, \quad \mathbf{u} = 0, \quad T = 0 & \text{on } \partial\Omega, \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}^0, \quad T(\mathbf{x},0) = T^0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded domain in R^d ($d = 2$ or 3) with a sufficiently smooth boundary $\partial\Omega, \mathbf{j} = (0,1)^\top$ or $\mathbf{j} = (0,0,1)^\top, Pr, Ra, T_1$ and γ represent the Prandtl number, the Rayleigh number, the given final time and the forcing function, respectively. The unknown functions are velocity vector $\mathbf{u}$, pressure p and temperature T. We consider a homogeneous Dirichlet boundary conditions in the above for simplicity and to fix the idea.

The NC convection problem (1.1) is an important system with dissipative nonlinear terms in atmospheric dynamics [3, 19]. Since this system not only contains the velocity as well as the pressure but also includes the temperature field, finding the numerical solution to problem (1.1) becomes a difficult task. For the research of problem (1.1), many scholars have proposed several kinds of efficient numerical schemes. For example, the standard Galerkin FEM [40], finite difference method [31], lattice Boltzmann method [15], the projection-based stabilized MFEM [7, 9], discontinuous Galerkin method [2, 5], domain decomposition method [37], time-stepping method [18, 52] and references therein. Moreover, NC phenomena are found in many scientific and engineering applications, and have been intensively studied in the literature, cf. [4, 16, 41, 42, 46, 54]. Hence, it is necessary to research the problem for us nowadays.

Time accuracy is critical for obtaining physically relevant solutions in the field of computational fluid dynamics (CFD). Many flow solvers use constant time steps, but there has been an expanding interest in variable step solvers [3, 19]. These methods allow for larger time steps for intervals of the simulation where the physics are stable, while allowing for smaller time steps for portions that are physically interesting. This allows for a decrease in the computational cost of the solver while simultaneously increasing its accuracy. The objective of this report is to design an efficient novel adaptive time-stepping method for NC problems based on time filters (TFs). TFs have some advantages:

(a) They can remove the overdamping of BE while remaining unconditionally energy stable [55].

(b) They can increase the time accuracy and add negligible additional computational complexity to simple, lower accuracy methods for constant time stepsize [12, 22].

(c) They give a low cost error estimator for adapting the time stepsize to ensure time accuracy [23].

The first proposed time filter was the Robert-Asselin (RA) time filter [1,3,29,50], which was used as a method to suppress the non-physical oscillation of leapfrog patterns in weather models. Williams [48, 49] improved the RA time filter, and a Robert-Asselin-Williams (RAW) time filter with third-order precision was obtained. Li and Trenchea [32] proposed a new RA time filter with third-order precision. The proposed method is easy to implement programmatically. Besides, Guzel and Layton [23] combined the backward Euler scheme and the time filter to obtain a new linear multistep method, which is equivalent to the classical two-step method and is a modular method. Latterly, we find that it has been used to solve the natural convection problems recently [51], however, the difference in our paper is that the entire nonlinear term is treated explicitly while the remaining terms are treated implicitly, and through numerical experiments, we found that our algorithm improves the calculation accuracy of pressure. In addition, the error of each numerical solution can be estimated through the time filter, and the time step adaptive algorithm can be used to solve the Navier-Stokes equation [10–12, 28], and the Stokes-Darcy model [38].

In a past paper [13], we can get a nonstandard BE-AB2 combination, where the constant extrapolation $\mathbf{u}_h^{n+1} = \mathbf{u}_h^n + \mathcal{O}(\Delta t)$ and $T_h^{n+1} = T_h^n + \mathcal{O}(\Delta t)$ in the nonlinearity is replaced with a linear extrapolation. For constant stepsize, this means

$$\mathbf{u}_h^{n+1} = 2\mathbf{u}_h^n - \mathbf{u}_h^{n-1} + \mathcal{O}(\Delta t^2), \quad T_h^{n+1} = 2T_h^n - T_h^{n-1} + \mathcal{O}(\Delta t^2).$$

The scheme has an embedded structure so that no additional Stokes solves or function evaluations are required to compute the second-order approximation once the first-order approximation is computed. This is done with an easy-to-implement and efficient time filter as follows. Let y and y^{n-1} be velocity approximations at $T_1 = \Delta tn$. If y^{n+1} is calculated with implicit Euler, a second-order approximation can be constructed by resetting y^{n+1} with

$$y^{n+1} \Longleftarrow y^{n+1} - \frac{1}{3}(y^{n+1} - 2y^n + y^{n-1}) \text{ (Constant stepsize timefilter)}, \quad y = \mathbf{u}, T.$$

The paper is organized as follows. In Section 2, we present some notations and give mathematical preliminaries that will be needed in the ensuing sections for NC problems. The stability analyses of the VSS BE-AB2 and BE-AB2+F algorithms are given, respectively, in Section 3. Section 4 is devoted to the error estimates of velocity, temperature, and pressure for the fully discretized VSS BE-AB2. In Section 5, we introduce the BE-AB2+F algorithm for variable time stepsize and construct an adaptive algorithm with performing stepsize selections to control time accuracy and computational efficiency. Then, we present numerical tests to illustrate the validity and accuracy of our numerical methods in Section 6. The final conclusions are given in Section 7.

2. Nation and Preliminaries

For the mathematical setting of problem (1.1), we introduce standard Hilbert spaces, some notations, and some necessary assumptions, which will be frequently used in the following sections. Note in passing that vector-valued functions and vector-valued spaces are identified with bold fonts. Besides, we use C to denote a generic positive constant that may depend on $\Omega, \mathbf{u}^0, T^0, \mathbf{j}$ and T_1.

2.1. Preliminaries

We now introduce the following standard Hilbert spaces used throughout this paper:

$$\begin{aligned}
&X = H_0^1(\Omega) = \{\mathbf{v} \in H^1(\Omega),\ \mathbf{v} = 0 \text{ on } \partial\Omega\},\\
&\mathbf{X} = X^d, \quad W = H_0^1(\Omega),\\
&Q = L_0^2(\Omega) = \left\{q \in L^2(\Omega),\ \int_\Omega q dx = 0\right\},
\end{aligned}$$

$$\mathbf{V} = \mathbf{X}_{\rm div} = \{\mathbf{v} \in \mathbf{X}\ (\nabla\cdot\mathbf{v}, q) = 0,\ \forall\, q \in Q\}.$$

The $L^2(\Omega)$ inner product and norm are denoted by $(\cdot\,,\cdot)$ and by $\|\cdot\|$, respectively, and the norm in the Hilbert space $\mathbf{H}^m(\Omega) = (H^m(\Omega))^d$ is denoted by

$$\|\mathbf{v}\|_m = \left(\sum_{|i|\le m} \|D^i\mathbf{v}\|^2\right)^{\frac{1}{2}}.$$

For measurable $\mathbf{v} : [0, T_1] \to \mathbf{X}$, define for, respectively, $1 \le p < \infty$ and $p = \infty$,

$$\|\mathbf{v}\|_{L^p(0,T_1;\mathbf{X})} = \left(\int_0^{T_1} \|\mathbf{v}(t)\|^p_{\mathbf{X}} dt\right)^{\frac{1}{p}}, \quad \|\mathbf{v}\|_{L^\infty(0,T_1;\mathbf{X})} = \text{ess}\sup_{0\le t\le T_1} \|\mathbf{v}(t)\|^p_{\mathbf{X}}. \tag{2.1}$$

Besides, we introduce the following discrete norms:

$$|||\mathbf{v}|||_{2,p} = \left(\sum_{n=0}^{T_1/\tau-1} \tau\|\mathbf{v}^{\mathbf{n}}\|^2_p\right)^{\frac{1}{2}}, \quad |||\mathbf{v}|||_{\infty,p} = \max_{0\le n\le T_1/\tau} \|\mathbf{v}^n\|_p. \tag{2.2}$$

The weak formulation of (1.1) is given by: For any $t \in (0, T_1]$ and for any $(\mathbf{v}, q, s) \in \mathbf{X}\times Q\times W$, find $(\mathbf{u}, p, T) \in \mathbf{X}\times Q\times W$ such that

$$\begin{cases}(\mathbf{u}_t, \mathbf{v}) + Pr(\nabla\mathbf{u}, \nabla\mathbf{v}) + (\mathbf{u}\cdot\nabla\mathbf{u}, \mathbf{v}) - (p, \nabla\cdot\mathbf{v}) + (\nabla\cdot\mathbf{u}, q) = PrRa(\mathbf{j}T, \mathbf{v}),\\ (T_t, s) + (\mathbf{u}\cdot\nabla T, s) + (\nabla T, \nabla s) = (\gamma, s).\end{cases} \tag{2.3}$$

Since the finite elements we consider satisfy the inf-sup condition, we can use the following lemma.

Lemma 2.1 ([21]). *We define the trilinear form*

$$\begin{aligned}&b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = (\mathbf{u}\cdot\nabla\mathbf{v}, \mathbf{w}), \quad \forall\, \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{X},\\ &b(\mathbf{u}, T, s) = (\mathbf{u}\cdot\nabla T, s), \qquad \forall\, \mathbf{u} \in \mathbf{X}, \quad T, s \in W,\end{aligned}$$

and the explicitly skew-symmetric trilinear form given by

$$\begin{aligned}&b^*(\mathbf{u}, \mathbf{v}, \mathbf{w}) := \frac{1}{2}(\mathbf{u}\cdot\nabla\mathbf{v}, \mathbf{w}) - \frac{1}{2}(\mathbf{u}\cdot\nabla\mathbf{w}, \mathbf{v}), \quad \forall\, \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{X},\\ &\overline{b^*}(\mathbf{u}, T, s) := \frac{1}{2}(\mathbf{u}\cdot\nabla T, s) - \frac{1}{2}(\mathbf{u}\cdot\nabla s, T), \qquad \forall\, \mathbf{u} \in \mathbf{X}, \quad T, s \in W.\end{aligned}$$

Herein, we recall some lemmas [12, 39, 53] and identify which will be used in the analyses. For some $C > 0$ and $\overline{C} > 0$, the trilinear terms satisfy

$$\begin{aligned}&b^*(\mathbf{u}, \mathbf{v}, \mathbf{v}) := 0, \quad \forall\, \mathbf{u}, \mathbf{v} \in \mathbf{X},\\ &\overline{b^*}(\mathbf{u}, s, s) = 0, \qquad \forall\, \mathbf{u} \in \mathbf{X}, \quad s \in W,\end{aligned} \tag{2.4}$$

$$b^*(\mathbf{u}, \mathbf{v}, \mathbf{w}) \le \begin{cases} C\|\nabla\mathbf{u}\|\|\nabla\mathbf{v}\|\|\nabla\mathbf{w}\|, & \forall\, \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{X},\\ C(\|\mathbf{u}\|\|\nabla\mathbf{u}\|)^{\frac{1}{2}}\|\nabla\mathbf{v}\|\|\nabla\mathbf{w}\|, & \forall\, \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{X},\\ C\|\nabla\mathbf{u}\|(\|\mathbf{v}\|\|\nabla\mathbf{v}\|)^{\frac{1}{2}}\|\nabla\mathbf{w}\|, & \forall\, \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{X},\\ C\|\nabla\mathbf{u}\|\|\nabla\mathbf{v}\|(\|\mathbf{w}\|\|\nabla\mathbf{w}\|)^{\frac{1}{2}}, & \forall\, \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{X},\end{cases} \tag{2.5}$$

$$\overline{b^*}(\mathbf{u}, T, s) \le \begin{cases} \overline{C}\|\nabla\mathbf{u}\|\|\nabla T\|\|\nabla S\|, & \forall\, \mathbf{u} \in \mathbf{X}, \quad T, s \in W,\\ \overline{C}(\|\mathbf{u}\|\|\nabla\mathbf{u}\|)^{\frac{1}{2}}\|\nabla T\|\|\nabla s\|, & \forall\, \mathbf{u} \in \mathbf{X}, \quad T, s \in W,\\ \overline{C}\|\nabla\mathbf{u}\|(\|T\|\|\nabla T\|)^{\frac{1}{2}}\|\nabla s\|, & \forall\, \mathbf{u} \in \mathbf{X}, \quad T, s \in W,\\ \overline{C}\|\nabla\mathbf{u}\|\|\nabla T\|(\|s\|\|\nabla s\|)^{\frac{1}{2}}, & \forall\, \mathbf{u} \in \mathbf{X}, \quad T, s \in W.\end{cases} \tag{2.6}$$

Lemma 2.2. *For any $a,b,c\in R$, the following identity holds:*

$$\begin{aligned}&\left(\frac{3}{2}a-2b+\frac{1}{2}c\right)\left(\frac{3}{2}a-b+\frac{1}{2}c\right)\\ =&\left[\frac{a^2}{4}+\frac{(2a-b)^2}{4}+\frac{(a-b)^2}{4}\right]\\ &-\left[\frac{b^2}{4}+\frac{(2b-c)^2}{4}+\frac{(b-c)^2}{4}\right]+\frac{3}{4}(a-2b+c)^2.\end{aligned}\tag{2.7}$$

Then, we recall some definitions [12] which will be used in the next sections.

We define the discrete system kinetic energy at time level t^n to be

$$\begin{aligned}\epsilon_{\mathbf{u}}^n&:=\frac{1}{2}\left\|\mathbf{u}_h^n\right\|^2+\frac{1}{4}\left\|\mathbf{u}_h^n-\mathbf{u}_h^{n-1}\right\|^2,\\ \epsilon_T^n&:=\frac{1}{2}\left\|T_h^n\right\|^2+\frac{1}{4}\left\|T_h^n-T_h^{n-1}\right\|^2,\end{aligned}\tag{2.8}$$

and the numerical dissipation,

$$\begin{aligned}D_{\mathbf{u}}^n&:=\frac{1}{8\left(1+\omega_n^2\right)}\left\|\mathbf{u}_h^n-\mathbf{u}_h^{n-1}+\omega_n\left(\mathbf{u}_h^{n-1}-\mathbf{u}_h^{n-2}\right)\right\|^2,\\ D_T^n&:=\frac{1}{8\left(1+\omega_n^2\right)}\left\|T_h^n-T_h^{n-1}+\omega_n\left(T_h^{n-1}-T_h^{n-2}\right)\right\|^2.\end{aligned}\tag{2.9}$$

Importantly, $\epsilon_{\mathbf{u}}^n,\epsilon_T^n$ do not depend on Δt_n.

2.2. Finite element approximation

Let $\tau_h=K$ be a uniformly regular family of triangulation of ω, and define the mesh size $h=\max_{K\in\tau_h}\{\mathrm{diam}(k)\}$. To construct a Galerkin approximation of (1.1), we consider the following four sequences of finite dimensional spaces:

$$\begin{aligned}\mathbf{X}_h&=\left\{\mathbf{v}_h\in\mathbf{X}\cap C^0(\overline{\Omega})^d:\mathbf{v}_h|_K\in P_{s+1}(K)^d,\ \forall K\in\tau_h\right\},\\ Q_h&=\left\{q_h\in Q\cap C^0(\overline{\Omega}):q_h|_K\in P_s(K),\ \forall K\in\tau_h\right\},\\ W_h&=\left\{\phi_h\in W\cap C^0(\overline{\Omega}):\phi_h|_K\in P_{s+1}(K),\ \forall K\in\tau_h\right\},\\ \mathbf{V}_h&=\left\{\mathbf{v}_h\in\mathbf{X}_h:(\nabla\cdot\mathbf{v}_h,q_h)=0,\ \forall q_h\in Q_h\right\},\end{aligned}$$

where $P_s(K)$ is the set of all polynomials on K of degree less than $s\in N$. Obviously, Taylor-Hood elements (P_{s+1},P_s) satisfy the discrete inf-sup condition [12,43,53]: There is a constant $\beta>0$ independent of h such that

$$\inf_{0\neq q_h\in Q_h}\sup_{0\neq\mathbf{v}_h\in\mathbf{X}_h}\frac{(q_h,\nabla\cdot\mathbf{v}_h)}{\|\nabla\mathbf{v}_h\|\|q_h\|}\geq\beta>0.\tag{2.10}$$

We will also assume that the mesh satisfies the following standard inverse inequalities (see [4, Lemma 4.9.2]):

$$\|\nabla\mathbf{v}_h\|\leq Ch^{-1}\|\mathbf{v}_h\|,\quad\forall\,\mathbf{v}_h\in\mathbf{X}_h.\tag{2.11}$$

Moreover, suppose that the finite element spaces satisfy the following approximation properties:

$$\begin{aligned}
&\inf_{\mathbf{v}_h\in\mathbf{X}_h}\|\mathbf{v}-\mathbf{v}_h\|\le Ch^{s+1}, && \forall\,\mathbf{v}\in[H^{s+1}(\Omega)]^d,\\
&\inf_{\mathbf{v}_h\in\mathbf{X}_h}\|\nabla(\mathbf{v}-\mathbf{v}_h)\|\le Ch^{s}, && \forall\,\mathbf{v}\in[H^{s+1}(\Omega)]^d,\\
&\inf_{s_h\in W_h}\|T-s_h\|\le Ch^{s+1}, && \forall\,T\in H^{s+1}(\Omega),\\
&\inf_{s_h\in W_h}\|\nabla(T-s_h)\|\le Ch^{s}, && \forall\,T\in H^{s+1}(\Omega),\\
&\inf_{q_h\in Q_h}\|p-q_h\|\le Ch^{s}, && \forall\,q\in H^{s}(\Omega).
\end{aligned} \tag{2.12}$$

Now, we define the dual norms of $\mathbf{X}_h, \mathbf{V}_h$ and W_h, respectively, by

$$\|\overline{\omega}\|_{\mathbf{X}_h^*}=\sup_{\mathbf{v}_h\in\mathbf{X}_h}\frac{(\overline{\omega},\mathbf{v}_h)}{\|\nabla\mathbf{v}_h\|},\quad \|\overline{\omega}\|_{\mathbf{V}_h^*}=\sup_{\mathbf{v}_h\in\mathbf{V}_h}\frac{(\overline{\omega},\mathbf{v}_h)}{\|\nabla\mathbf{v}_h\|},\quad \|\overline{\omega}\|_{W_h^*}=\sup_{s_h\in W_h}\frac{(\overline{\omega},s_h)}{\|\nabla s_h\|}.$$

Then the following lemma [12, 20, 53], which will be used to derive pressure error estimates with a technique shown in Fiordilino [17], from Galvin [20] estimates the equivalence of these norms on $\mathbf{V}_h$.

Lemma 2.3. *Suppose the discrete inf-sup condition holds, for any $\mathbf{v}_h\in\mathbf{V}_h$, the norms $\|\mathbf{v}_h\|_{\mathbf{X}_h^*}$ and $\|\mathbf{v}_h\|_{\mathbf{V}_h^*}$ are equivalent.*

Assumption 2.1. Assume that the true solution $(\mathbf{u},T,p)$ satisfies the following regularity:

$$\begin{aligned}
&\mathbf{u}\in L^\infty\big(0,T_1;\big(H^{s+1}(\omega)\big)^d\big), && \mathbf{u}_t\in L^2\big(0,T_1;\big(H^{s+1}(\omega)\big)^d\big), && u_{tt}\in L^2\big(0,T_1;\big(H^{1}(\omega)\big)^d\big),\\
&\mathbf{u}_{ttt}\in L^2\big(0,T_1;\big(L^{2}(\omega)\big)^d\big), && T\in L^\infty\big(0,T_1;H^{s+1}(\omega)\big), && T_t\in L^2\big(0,T_1;H^{s+1}(\omega)\big),\\
&T_{tt}\in L^2\big(0,T_1;H^{1}(\omega)\big), && T_{ttt}\in L^2\big(0,T_1;L^{2}(\omega)\big), && p\in L^2\big(0,T_1;H^{s+1}(\omega)\big).
\end{aligned}$$

3. Numerical Algorithms and Their Stabilities

In this section, we first develop the constant stepsize fully discrete backward Euler plus time filter algorithms, then derive the stabilities of the algorithms.

3.1. Numerical algorithms

Let $\Delta t_n=t^{n+1}-t^n$. The stepsize ratios are $\omega_n=\Delta t_n/\Delta t_{n-1}$. The second-order extrapolation of $\mathbf{u}_h^{n+1}$ becomes

$$E^{n+1}(\mathbf{u}_h):=(1+\omega_n)\mathbf{u}_h^n-\omega_n\mathbf{u}_h^{n-1},$$

T_h^{n+1} becomes

$$E^{n+1}(T_h):=(1+\omega_n)T_h^n-\omega_nT_h^{n-1}.$$

We then have the variable stepsize BE-AB2 (VSS BE-AB2) method.

This is a second-order perturbation of implicit backward Euler, and applying the time filter results in a second-order method. In the next two subsections, we rigorously show that this new method is variable stepsize stable and is globally convergent. In order to simplify the ensuing analysis, we assume that $\mathbf{u}_h^0,\mathbf{u}_h^1$, and T_h^0,T_h^1 are the L^2 projections of the exact solution at t^0 and t^1 onto the finite element spaces $\mathbf{X}_h$ and W_h. In our numerical tests, when an analytic solution is unavailable, we perform an L^2 projection on the initial condition to obtain $\mathbf{u}_h^0,T_h^0$ and then take a very small time step using BE-FE to obtain $\mathbf{u}_h^1,T_h^1$. The ensuing stepsizes can quickly increase to the largest size allowed by some tolerance.

Algorithm 3.1: Variable Stepsize BE-AB2 (VSS BE-AB2).

Given $\mathbf{u}_h^n, T_h^n$ and $\mathbf{u}_h^{n-1}, T_h^{n-1}$.
Find $(\mathbf{u}_h^{n+1}, T_h^{n+1}, p_h^{n+1})$ satisfying for all $(\mathbf{v}_h, q_h, s_h) \in (\mathbf{X}_h, Q_h, W_h)$,

$$\begin{aligned}
&\left(\frac{\mathbf{u}_h^{n+1}-\mathbf{u}_h^n}{\Delta t_n}, \mathbf{v}_h\right) + b^*\big(E^{n+1}(\mathbf{u}_h), E^{n+1}(\mathbf{u}_h), \mathbf{v}_h\big) \\
&\quad - \big(p_h^{n+1}, \nabla\cdot\mathbf{v}_h\big) + Pr\big(\nabla\mathbf{u}_h^{n+1}, \nabla\mathbf{v}_h\big) = PrRa\big(\mathbf{j}T_h^{n+1}, \mathbf{v}_h\big), \\
&\big(\nabla\cdot\mathbf{u}_h^{n+1}, q_h\big) = 0, \\
&\left(\frac{T_h^{n+1}-T_h^n}{\Delta t_n}, s_h\right) + \overline{b^*}\big(E^{n+1}(\mathbf{u}_h), E^{n+1}(T_h), s_h\big) + \big(\nabla T_h^{n+1}, \nabla s_h\big) = (\gamma, s_h).
\end{aligned} \tag{3.1}$$

Algorithm 3.2: Constant Time-step Filtered-BE-AB2 (BE-AB2+F).

Given $\Delta t, \mathbf{u}_h^n, \mathbf{u}_h^{n-1}$ and T_h^n, T_h^{n-1}.
Find $(\widehat{\mathbf{u}}_h^{n+1}, \widehat{T}_h^{n+1}, p_h^{n+1})$ satisfying for all $(\mathbf{v}_h, q_h, s_h) \in (\mathbf{X}_h, Q_h, W_h)$,

$$\begin{aligned}
&\left(\frac{\widehat{\mathbf{u}}_h^{n+1}-\mathbf{u}_h^n}{\Delta t}, \mathbf{v}_h\right) + b^*\big(2\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, 2\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \mathbf{v}_h\big) \\
&\quad + Pr\big(\nabla\widehat{\mathbf{u}}_h^{n+1}, \nabla\mathbf{v}_h\big) - \big(p_h^{n+1}, \nabla\cdot\mathbf{v}_h\big) = PrRa\big(\mathbf{j}\widehat{T}_h^{n+1}, \mathbf{v}_h\big), \\
&\big(\nabla\cdot\widehat{\mathbf{u}}_h^{n+1}, q_h\big) = 0, \\
&\left(\frac{\widehat{T}_h^{n+1}-T_h^n}{\Delta t}, s_h\right) + \overline{b^*}\big(2\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, 2T_h^n - T_h^{n-1}, s_h\big) \\
&\quad + \big(\nabla\widehat{T}_h^{n+1}, \nabla s_h\big) = (\gamma, s_h).
\end{aligned} \tag{3.2}$$

Then, compute

$$\begin{aligned}
\mathbf{u}_h^{n+1} &= \widehat{\mathbf{u}}_h^{n+1} - \frac{1}{3}\big(\widehat{\mathbf{u}}_h^{n+1} - 2\mathbf{u}_h^n + \mathbf{u}_h^{n-1}\big), \\
T_h^{n+1} &= \widehat{T}_h^{n+1} - \frac{1}{3}\big(\widehat{T}_h^{n+1} - 2T_h^n + T_h^{n-1}\big).
\end{aligned} \tag{3.3}$$

Equivalently, this can be written as

$$\begin{aligned}
&\left(\frac{3\mathbf{u}_h^{n+1}/2 - 2\mathbf{u}_h^n + \mathbf{u}_h^{n-1}/2}{\Delta t}, \mathbf{v}_h\right) + Pr\left(\nabla\left(\frac{3}{2}\mathbf{u}_h^{n+1} - \mathbf{u}_h^n + \frac{1}{2}\mathbf{u}_h^{n-1}\right), \nabla\mathbf{v}_h\right) \\
&\quad - \big(p_h^{n+1}, \nabla\cdot\mathbf{v}_h\big) + b^*\big(2\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, 2\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \mathbf{v}_h\big) \\
&= PrRa\left(\mathbf{j}\left(\frac{3}{2}T_h^{n+1} - T_h^n + \frac{1}{2}T_h^{n-1}\right), \mathbf{v}_h\right), \\
&\left(\nabla\cdot\left(\frac{3}{2}\mathbf{u}_h^{n+1} - 2\mathbf{u}_h^n + \frac{1}{2}\mathbf{u}_h^{n-1}\right), q_h\right) = 0, \\
&\left(\frac{3T_h^{n+1}/2 - 2T_h^n + T_h^{n-1}/2}{\Delta t}, s_h\right) + \overline{b^*}\big(2\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, 2T_h^n - T_h^{n-1}, s_h\big) \\
&\quad + \left(\nabla\left(\frac{3}{2}T_h^{n+1} - T_h^n + \frac{1}{2}T_h^{n-1}\right), \nabla s_h\right) = (\gamma, s_h).
\end{aligned} \tag{3.4}$$

3.2. Energy stability for VSS BE-AB2

In this section, we prove nonlinear, conditional stability of (3.1). We begin with a general stability result. We then show that the timestep condition can be improved in some special cases.

Theorem 3.1. *Consider the method* (3.1)*, let* $\Omega \subset R^d, d = 2, 3$*, and* $C_{stab} > 0$ *be a constant independent of* $h, \Delta t_n, \omega_n, Pr, \mathbf{u}$ *and* T*. Suppose that*

$$\begin{aligned} M_1 &= 1 - \frac{C_{stab}\Delta t_n\left(1+\omega_n^2\right)}{Prh}\left\|\nabla E^{n+1}(\mathbf{u}_h)\right\|^2 \geq 0, \\ M_2 &= 1 - \frac{C_{stab}\Delta t_n\left(1+\omega_n^2\right)}{h}\left\|\nabla E^{n+1}(\mathbf{u}_h)\right\|^2 \geq 0. \end{aligned} \tag{3.5}$$

Then, for any $N > 1$ *the velocity and temperature approximations satisfy*

$$\epsilon_{\mathbf{u}}^N + \frac{Pr}{4}\sum_{n=1}^{N-1}\Delta t_n\left\|\nabla \mathbf{u}_h^{n+1}\right\|^2 + \sum_{n=1}^{N-1} D_u^{n+1} \leq CPrRa^2\left(|||\gamma^{n+1}|||_{2,-1}^2 + \epsilon_T^1\right) + \epsilon_{\mathbf{u}}^1, \tag{3.6}$$

$$\epsilon_T^N + \frac{1}{4}\sum_{n=1}^{N-1}\Delta t_n\left\|\nabla T_h^{n+1}\right\|^2 + \sum_{n=1}^{N-1} D_T^{n+1} \leq |||\gamma^{n+1}|||_{2,-1}^2 + \epsilon_T^1. \tag{3.7}$$

Proof. Setting $\mathbf{v}_h = \mathbf{u}_h^{n+1}, s_h = T_h^{n+1}$ and multiplying by Δt_n, then applying Young's inequality to the right-hand side, we have

$$\begin{aligned} &\frac{1}{2}\left\|\mathbf{u}_h^{n+1}\right\|^2 - \frac{1}{2}\left\|\mathbf{u}_h^{n}\right\|^2 + \frac{1}{2}\left\|\mathbf{u}_h^{n+1} - \mathbf{u}_h^n\right\|^2 \\ &\qquad + \Delta t_n b^*\left(E^{n+1}(\mathbf{u}_h), E^{n+1}(\mathbf{u}_h), \mathbf{u}_h^{n+1}\right) + \Delta t_n Pr\left\|\nabla \mathbf{u}_h^{n+1}\right\|^2 \\ &\leq \frac{\Delta t_n Pr}{4}\left\|\nabla \mathbf{u}_h^{n+1}\right\|^2 + \Delta t_n PrRa^2\left\|T_h^{n+1}\right\|_{-1}^2, \end{aligned} \tag{3.8}$$

$$\begin{aligned} &\frac{1}{2}\left\|T_h^{n+1}\right\|^2 - \frac{1}{2}\left\|T_h^{n}\right\|^2 + \frac{1}{2}\left\|T_h^{n+1} - T_h^n\right\|^2 \\ &\qquad + \Delta t_n \overline{b^*}\left(E^{n+1}(\mathbf{u}_h), E^{n+1}(T_h), T_h^{n+1}\right) + \Delta t_n\left\|\nabla T_h^{n+1}\right\|^2 \\ &\leq \frac{\Delta t_n}{4}\Delta t_n\left\|\nabla T_h^{n+1}\right\|^2 + \Delta t_n\|\gamma^{n+1}\|_{-1}^2. \end{aligned} \tag{3.9}$$

Next, we deal with the nonlinearity. Applying (2.5), (2.6), using the skew symmetry of the nonlinearity, applying the Cauchy-Schwarz-Young inequality, Poincare-Friedrichs, and inverse inequalities we have

$$\begin{aligned} &\Delta t_n b^*\left(E^{n+1}(\mathbf{u}_h), E^{n+1}(\mathbf{u}_h), \mathbf{u}_h^{n+1}\right) \\ &= \Delta t_n b^*\left(E^{n+1}(\mathbf{u}_h), \mathbf{u}_h^{n+1}, \mathbf{u}_h^{n+1} - \mathbf{u}_h^n - \omega_n\left(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\right)\right) \\ &\leq C\Delta t_n h^{-\frac{1}{2}}\left\|\nabla E^{n+1}(\mathbf{u}_h)\right\|\left\|\mathbf{u}_h^{n+1} - \mathbf{u}_h^n - \omega_n\left(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\right)\right\|\left\|\nabla \mathbf{u}_h^{n+1}\right\| \\ &\leq \frac{C{\Delta t_n}^2\left(1+\omega_n^2\right)}{h}\left\|\nabla E^{n+1}(\mathbf{u}_h)\right\|^2\left\|\nabla \mathbf{u}_h^{n+1}\right\|^2 \\ &\qquad + \frac{1}{8\left(1+\omega_n^2\right)}\left\|\mathbf{u}_h^{n+1} - \mathbf{u}_h^n - \omega_n\left(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\right)\right\|^2, \end{aligned} \tag{3.10}$$

$$\begin{aligned}&\Delta t_n \overline{b^*}\big(E^{n+1}(\mathbf{u}_h), E^{n+1}(T_h), T_h^{n+1}\big)\\ \le\ & \frac{C{\Delta t_n}^2\big(1+\omega_n^2\big)}{h}\big\|\nabla E^{n+1}(\mathbf{u}_h)\big\|^2\big\|\nabla T_h^{n+1}\big\|^2\\ &+\frac{1}{8\big(1+\omega_n^2\big)}\big\|T_h^{n+1}-T_h^n-\omega_n\big(T_h^n-T_h^{n-1}\big)\big\|^2. \end{aligned} \tag{3.11}$$

Combining above inequalities, using the parallelogram law and (2.8), (2.9), we then have

$$\begin{aligned}&\epsilon_{\mathbf{u}}^{n+1}-\epsilon_{\mathbf{u}}^n+D_{\mathbf{u}}^{n+1}+\frac{Pr\Delta t_n}{4}\big\|\nabla \mathbf{u}_h^{n+1}\big\|^2+\frac{Pr\Delta t_n}{2}M_1\big\|\nabla \mathbf{u}_h^{n+1}\big\|^2\\ \le\ & \Delta t_n Pr Ra^2\big\|T_h^{n+1}\big\|_{-1}^2, \end{aligned} \tag{3.12}$$

$$\begin{aligned}&\epsilon_T^{n+1}-\epsilon_T^n+D_T^{n+1}+\frac{\Delta t_n}{4}\big\|\nabla T_h^{n+1}\big\|^2+\frac{\Delta t_n}{2}M_2\big\|\nabla T_h^{n+1}\big\|^2\\ \le\ & \Delta t_n\|\gamma^{n+1}\|_{-1}^2. \end{aligned} \tag{3.13}$$

Finally, using condition (3.5), letting $C=C_{stab}$, and summing from $n=1$ to $N-1$, then we can get the result. □

Remark 3.1. In this proof, the zero Dirichlet boundary condition is selected for temperature to reduce the analysis time. Nevertheless, it is notable that the consideration of the non-zero boundary of temperature is of vital significance in practical applications.

Remark 3.2. In the proposed scheme, nonlinear terms are dealt with explicitly, so the Courant-Friedrichs-Lewy condition (CFL) conditions need to be imposed, but we can adopt adaptive algorithms to take a small enough timestep to ensure the CFL conditions.

Theorem 3.2. *Suppose Theorem* 3.1 *holds, then the pressure approximation satisfies*

$$\begin{aligned}\beta\Delta t_n\sum_{n=1}^{N-1}\big\|p_h^{n+1}\big\| \le \big(1+C_*^{-1}\big)\bigg\{&\frac{C}{Pr}\big[CPrRa^2\big(\|\!|\gamma^{n+1}|\!\|_{2,-1}^2+\epsilon_T^1\big)+\epsilon_{\mathbf{u}}^1\big]\\ &+C\sqrt{PrT_1}\big[CPrRa^2\big(\|\!|\gamma^{n+1}|\!\|_{2,-1}^2+\epsilon_T^1\big)+\epsilon_{\mathbf{u}}^1\big]^{\frac12}\\ &+CPrRa\sqrt{T_1}\big(\|\!|\gamma^{n+1}|\!\|_{2,-1}^2+\epsilon_T^1\big)^{\frac12}\bigg\}. \end{aligned} \tag{3.14}$$

Proof. Let $(\mathbf{v}_h,q_h)\in(\mathbf{V}_h,Q_h)$ and isolate the discrete time derivative in (3.1). Then we can get

$$\begin{aligned}\bigg(\frac{\mathbf{u}_h^{n+1}-\mathbf{u}_h^n}{\Delta t_n},\mathbf{v}_h\bigg) =& -b^*\big(E^{n+1}(\mathbf{u}_h),E^{n+1}(\mathbf{u}_h),\mathbf{v}_h\big)\\ &-Pr\big(\nabla\mathbf{u}_h^{n+1},\nabla\mathbf{v}_h\big)+PrRa\big(\mathbf{j}T_h^{n+1},\mathbf{v}_h\big). \end{aligned} \tag{3.15}$$

The terms on the right-hand side of (3.15) can be bounded using Lemma 2.1, the Cauchy-Schwartz inequality, and duality, respectively

$$\begin{aligned}&-b^*\big(E^{n+1}(\mathbf{u}_h),E^{n+1}(\mathbf{u}_h),\mathbf{v}_h\big)\le C\big\|\nabla E^{n+1}(\mathbf{u}_h)\big\|\big\|\nabla E^{n+1}(\mathbf{u}_h)\big\|\|\nabla\mathbf{v}_h\|,\\ &-Pr\big(\nabla\mathbf{u}_h^{n+1},\nabla\mathbf{v}_h\big)\le Pr\big\|\nabla\mathbf{u}_h^{n+1}\big\|\|\nabla\mathbf{v}_h\|,\\ &\ \ PrRa\big(\mathbf{j}T_h^{n+1},\mathbf{v}_h\big)\le CPrRa\big\|\nabla T_h^{n+1}\big\|\|\nabla\mathbf{v}_h\|. \end{aligned} \tag{3.16}$$

Using the above estimates in Eq. (3.15), dividing both sides by $\|\nabla \mathbf{v}_h\|$, taking the supremum over $\mathbf{v}_h \in \mathbf{V}_h$, and using Theorem 3.1 yields

$$\begin{aligned}\left\|\frac{\mathbf{u}_h^{n+1}-\mathbf{u}_h^n}{\Delta t_n}\right\|_{\mathbf{V}_h^*} \leq C\big\|\nabla E^{n+1}(\mathbf{u}_h)\big\|\big\|\nabla E^{n+1}(\mathbf{u}_h)\big\| \\ + Pr\big\|\nabla \mathbf{u}_h^{n+1}\big\| + CPrRa\big\|\nabla T_h^{n+1}\big\|. \end{aligned} \tag{3.17}$$

Lemma 2.3 then implies

$$\begin{aligned}\left\|\frac{\mathbf{u}_h^{n+1}-\mathbf{u}_h^n}{\Delta t_n}\right\|_{\mathbf{X}_h^*} \leq C_*^{-1}\Big[C\big\|\nabla E^{n+1}(\mathbf{u}_h)\big\|\big\|\nabla E^{n+1}(\mathbf{u}_h)\big\| \\ + Pr\big\|\nabla \mathbf{u}_h^{n+1}\big\| + CPrRa\big\|\nabla T_h^{n+1}\big\|\Big]. \end{aligned} \tag{3.18}$$

Now consider that $\mathbf{v}_h \in \mathbf{X}_h$. Isolating the pressure term in (3.1) and using the estimates from (3.16) yields

$$\begin{aligned}\big(p_h^{n+1}, \nabla\cdot \mathbf{v}_h\big) \leq \left(\frac{\mathbf{u}_h^{n+1}-\mathbf{u}_h^n}{\Delta t_n}, \mathbf{v}_h\right) + C\big\|\nabla E^{n+1}(\mathbf{u}_h)\big\|\big\|\nabla E^{n+1}(\mathbf{u}_h)\big\|\|\nabla \mathbf{v}_h\| \\ + Pr\big\|\nabla \mathbf{u}_h^{n+1}\big\|\|\nabla \mathbf{v}_h\| + CPrRa\big\|\nabla T_h^{n+1}\big\|\|\nabla \mathbf{v}_h\|. \end{aligned} \tag{3.19}$$

Divide both sides by $\|\nabla \mathbf{v}_h\|$, take the supremum over $\mathbf{v}_h \in \mathbf{X}_h$, and use both the discrete inf-sup condition (2.10) and estimate (3.16). Then

$$\begin{aligned}\beta\big\|p_h^{n+1}\big\| \leq \big(1+C_*^{-1}\big)\Big[C\big\|\nabla E^{n+1}(\mathbf{u}_h)\big\|\big\|\nabla E^{n+1}(\mathbf{u}_h)\big\| \\ + Pr\big\|\nabla \mathbf{u}_h^{n+1}\big\| + CPrRa\big\|\nabla T_h^{n+1}\big\|\Big]. \end{aligned} \tag{3.20}$$

Multiplying by Δt_n, summing from $n=0$ to $n=N-1$, using the Cauchy-Schwartz inequality and Theorem 3.1, then the proof is complete. □

3.3. Energy stability for BE-AB2+F

Theorem 3.3. *Consider the method* (3.4)*, let* $\Omega \subset R^d, d=2,3$*, and* $C_{stab} > 0$ *be a constant independent of* $h, \Delta t, Pr$ *and* $\mathbf{u}, T$*. Suppose that*

$$\begin{aligned} M_3 &= 1 - \frac{C_{stab}\Delta t}{Prh}\big\|\nabla\big(2\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\big)\big\|^2 \geq 0, \\ M_4 &= 1 - \frac{C_{stab}\Delta t}{h}\big\|\nabla\big(2\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\big)\big\|^2 \geq 0. \end{aligned} \tag{3.21}$$

Then, for any $N > 1$ *the velocity and temperature approximations satisfy*

$$\begin{aligned} &\frac14\big\|\mathbf{u}_h^N\big\|^2 + \frac14\big\|2\mathbf{u}_h^N - \mathbf{u}_h^{N-1}\big\|^2 + \frac14\big\|\mathbf{u}_h^N - \mathbf{u}_h^{N-1}\big\|^2 \\ &\quad + \frac{\Delta t Pr}{4}\sum_{n=1}^{N-1}\left\|\nabla\left(\frac32\mathbf{u}_h^{n+1} - \mathbf{u}_h^n + \frac12\mathbf{u}_h^{n-1}\right)\right\|^2 \\ &\leq CPrRa^2\left(|||\gamma^{n+1}|||_{2,-1}^2 + \frac14\big\|T_h^1\big\|^2 + \frac14\big\|2T_h^1 - T_h^0\big\|^2 + \frac14\big\|T_h^1 - T_h^0\big\|^2\right) \\ &\quad + \frac14\big\|\mathbf{u}_h^1\big\|^2 + \frac14\big\|2\mathbf{u}_h^1 - \mathbf{u}_h^0\big\|^2 + \frac14\big\|\mathbf{u}_h^1 - \mathbf{u}_h^0\big\|^2, \end{aligned} \tag{3.22}$$

$$
\begin{aligned}
&\frac{1}{4}\big\|T_h^N\big\|^2+\frac{1}{4}\big\|2T_h^N-T_h^{N-1}\big\|^2+\frac{1}{4}\big\|T_h^N-T_h^{N-1}\big\|^2\\
&\quad+\frac{\Delta t}{4}\sum_{n=1}^{N-1}\left\|\nabla\left(\frac{3}{2}T_h^{n+1}-T_h^n+\frac{1}{2}T_h^{n-1}\right)\right\|^2\\
&\le \|\!|\gamma^{n+1}|\!\|_{2,-1}^2+\frac{1}{4}\big\|T_h^1\big\|^2+\frac{1}{4}\big\|2T_h^1-T_h^0\big\|^2+\frac{1}{4}\big\|T_h^1-T_h^0\big\|^2.
\end{aligned}\tag{3.23}
$$

The proof is given in Appendix A.1.

Theorem 3.4. *Suppose Theorem* 3.3 *holds, then the pressure approximation satisfies*

$$
\begin{aligned}
&\beta\sum_{n=1}^{N-1}\Delta t\|p_h^{n+1}\| \qquad (3.24)\\
&\le (1+C_*^{-1})\Bigg[CPrh+\sqrt{PrT_1}\\
&\quad\times\Bigg(CPrRa^2\Big(\|\!|\gamma^{n+1}|\!\|_{2,-1}^2+\frac{1}{4}\big\|T_h^1\big\|^2+\frac{1}{4}\big\|2T_h^1-T_h^0\big\|^2+\frac{1}{4}\big\|T_h^1-T_h^0\big\|^2\Big)\\
&\quad+\frac{1}{4}\big\|\mathbf{u}_h^1\big\|^2+\frac{1}{4}\big\|2\mathbf{u}_h^1-\mathbf{u}_h^0\big\|^2+\frac{1}{4}\big\|\mathbf{u}_h^1-\mathbf{u}_h^0\big\|^2\Bigg)^{\frac{1}{2}}\\
&\quad+CPrRa\sqrt{T_1}\Big(\|\!|\gamma^{n+1}|\!\|_{2,-1}^2+\frac{1}{4}\big\|T_h^1\big\|^2+\frac{1}{4}\big\|2T_h^1-T_h^0\big\|^2+\frac{1}{4}\big\|T_h^1-T_h^0\big\|^2\Big)^{\frac{1}{2}}\Bigg].
\end{aligned}
$$

The proof is given in Appendix A.2.

4. Error Analysis

We will give and prove error analyses of Algorithm 3.1 for velocity and temperature in Theorem 4.1, and for pressure in Theorem 4.2, respectively, in this section.

We consider the fully-discrete variable step scheme (3.1). For all $(\mathbf{v}_h,q_h,s_h)\in(\mathbf{X}_h,Q_h,W_h)$, assuming that the discrete inf-sup condition is satisfied, Algorithm 3.1 is equivalent to

$$
\begin{aligned}
&\left(\frac{\mathbf{u}_h^{n+1}-\mathbf{u}_h^n}{\Delta t_n},\mathbf{v}_h\right)+b^*\big(E^{n+1}(\mathbf{u}_h),E^{n+1}(\mathbf{u}_h),\mathbf{v}_h\big)+Pr\big(\nabla\mathbf{u}_h^{n+1},\nabla\mathbf{v}_h\big)=PrRa\big(\mathbf{j}T_h^{n+1},\mathbf{v}_h\big),\\
&\big(\nabla\cdot\mathbf{u}_h^{n+1},q_h\big)=0,\\
&\left(\frac{T_h^{n+1}-T_h^n}{\Delta t_n},s_h\right)+\overline{b^*}\big(E^{n+1}(\mathbf{u}_h),E^{n+1}(T_h),s_h\big)+\big(\nabla T_h^{n+1},\nabla s_h\big)=(\gamma,s_h).
\end{aligned}\tag{4.1}
$$

For the error analysis we will split the velocity, temperature and pressure as follows:

$$
\begin{aligned}
&e_{\mathbf{u}}^{n+1}=\mathbf{u}^{n+1}-\mathbf{u}_h^{n+1}=\big(\mathbf{u}^{n+1}-I_h(\mathbf{u}^{n+1})\big)-\big(\mathbf{u}_h^{n+1}-I_h(\mathbf{u}^{n+1})\big)=\eta_{\mathbf{u}}^{n+1}-\phi_{\mathbf{u}}^{n+1},\\
&e_T^{n+1}=T^{n+1}-T_h^{n+1}=\big(T^{n+1}-I_h(T^{n+1})\big)-\big(T_h^{n+1}-I_h(T^{n+1})\big)=\eta_T^{n+1}-\phi_T^{n+1},\\
&e_p^{n+1}=p^{n+1}-p_h^{n+1}=\big(p^{n+1}-\lambda_h^{n+1}\big)-\big(p_h^{n+1}-\lambda_h^{n+1}\big),
\end{aligned}\tag{4.2}
$$

where I_h is the L^2 projection into the discretely divergence-free space $\mathbf{V}_h$ and W_h,λ_h^{n+1} is the L^2 projection into the discretely divergence-free space Q_h.

Lemma 4.1 (Consistency Error). *For* $\mathbf{u}$ *and* T *satisfying the regularity assumptions in Assumption* 2.1 *the following inequalities hold:*

$$\begin{aligned} &\Big\|\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t_n}-\mathbf{u}_t^{n+1}\Big\|^2 \le C\Delta t_n\|\mathbf{u}_{tt}\|^2_{L^2(t^n,t^{n+1};L^2(\Omega))},\\ &\big\|\nabla\big(\mathbf{u}^{n+1}-E^{n+1}(\mathbf{u})\big)\big\|^2 \le C(\Delta t_{n-1}+\Delta t_n)^3\|\nabla\mathbf{u}_{tt}\|^2_{L^2(t^{n-1},t^{n+1};L^2(\Omega))},\\ &\Big\|\frac{T^{n+1}-T^n}{\Delta t_n}-T_t^{n+1}\Big\|^2 \le C\Delta {t_n}^2\|T_{tt}\|^2_{L^2(t^n,t^{n+1};L^2(\Omega))},\\ &\big\|\nabla\big(T^{n+1}-E^{n+1}(T)\big)\big\|^2 \le C(\Delta t_{n-1}+\Delta t_n)^3\|\nabla T_{tt}\|^2_{L^2(t^{n-1},t^{n+1};L^2(\Omega))}. \end{aligned} \tag{4.3}$$

We now prove an estimate for the nonlinear terms which will appear in the error analysis.

Lemma 4.2 (Estimate on the Nonlinear Term). *For* T *and* $\mathbf{u}$ *satisfying the regularity assumptions in Assumption* 2.1 *the following inequality holds for the nonlinear term:*

$$\begin{aligned} &\overline{b^*}\big(\mathbf{u}^{n+1},T^{n+1},\phi_T^{n+1}\big)-\overline{b^*}\big(E^{n+1}(\mathbf{u}_h),E^{n+1}(T_h),\phi_T^{n+1}\big)\\ \le{}& \frac{5}{64}Pr\big\|\nabla\phi_T^{n+1}\big\|^2+\frac{C}{Pr}\big\|\nabla E^{n+1}(\eta_{\mathbf{u}})\big\|^2+\frac{C(\Delta t_{n-1}+\Delta t_n)^3}{Pr}\|\nabla\mathbf{u}_{tt}\|^2_{L^2(t^{n-1},t^{n+1};L^2(\Omega))}\\ &+CPr^{-3}\big\|\phi_{\mathbf{u}}^n\big\|^2+CPr^{-3}\big\|\phi_{\mathbf{u}}^{n-1}\big\|^2+\frac{1}{16}Pr\big\|\nabla\phi_{\mathbf{u}}^n\big\|^2+\frac{1}{16}Pr\big\|\nabla\phi_{\mathbf{u}}^{n-1}\big\|^2\\ &+\frac{C}{Pr}\big\|E^{n+1}(\mathbf{u}_h)\big\|\big\|\nabla E^{n+1}(\mathbf{u}_h)\big\|\big\|\nabla\eta_T^{n+1}\big\|^2+\frac{1}{4\Delta t_n}\big\|\phi_T^{n+1}-\phi_T^n\big\|^2\\ &+\frac{1}{4\Delta t_n}\big\|\phi_T^n-\phi_T^{n-1}\big\|^2+\frac{5C_{stab}\Delta t_n\big(1+{\omega_n}^2\big)}{8h}\big\|\nabla E^{n+1}(\mathbf{u}_h)\big\|^2\big\|\nabla\phi_T^{n+1}\big\|^2\\ &+\frac{Ch\big(\Delta t_{n-1}+\Delta t_n\big)^3}{C_{stab}\Delta t_n\big(1+{\omega_n}^2\big)}\big(\|\nabla T_{tt}\|^2_{L^2(t^{n-1},t^{n+1};L^2(\Omega))}+\|\nabla(\eta_T)_{tt}\|^2_{L^2(t^{n-1},t^{n+1};L^2(\Omega))}\big), \end{aligned} \tag{4.4}$$

$$\begin{aligned} &b^*\big(\mathbf{u}^{n+1},\mathbf{u}^{n+1},\phi_{\mathbf{u}}^{n+1}\big)-b^*\big(E^{n+1}(\mathbf{u}_h),E^{n+1}(\mathbf{u}_h),\phi_{\mathbf{u}}^{n+1}\big)\\ \le{}& \frac{5}{64}Pr\big\|\nabla\phi_{\mathbf{u}}^{n+1}\big\|^2+\frac{C}{Pr}\big\|\nabla E^{n+1}(\eta_{\mathbf{u}})\big\|^2+\frac{C(\Delta t_{n-1}+\Delta t_n)^3}{Pr}\|\nabla\mathbf{u}_{tt}\|^2_{L^2(t^{n-1},t^{n+1};L^2(\Omega))}\\ &+CPr^{-3}\big\|\phi_{\mathbf{u}}^n\big\|^2+CPr^{-3}\big\|\phi_{\mathbf{u}}^{n-1}\big\|^2+\frac{1}{16}Pr\big\|\nabla\phi_{\mathbf{u}}^n\big\|^2+\frac{1}{16}Pr\big\|\nabla\phi_{\mathbf{u}}^{n-1}\big\|^2\\ &+\frac{C}{Pr}\big\|E^{n+1}(\mathbf{u}_h)\big\|\big\|\nabla E^{n+1}(\mathbf{u}_h)\big\|\big\|\nabla\eta_{\mathbf{u}}^{n+1}\big\|^2+\frac{1}{4\Delta t_n}\big\|\phi_{\mathbf{u}}^{n+1}-\phi_{\mathbf{u}}^n\big\|^2\\ &+\frac{1}{4\Delta t_n}\big\|\phi_{\mathbf{u}}^n-\phi_{\mathbf{u}}^{n-1}\big\|^2+\frac{5C_{stab}\Delta t_n\big(1+{\omega_n}^2\big)}{8h}\big\|\nabla E^{n+1}(\mathbf{u}_h)\big\|^2\big\|\nabla\phi_{\mathbf{u}}^{n+1}\big\|^2\\ &+\frac{Ch(\Delta t_{n-1}+\Delta t_n)^3}{C_{stab}\Delta t_n\big(1+{\omega_n}^2\big)}\big(\|\nabla\mathbf{u}_{tt}\|^2_{L^2(t^{n-1},t^{n+1};L^2(\Omega))}+\|\nabla(\eta_{\mathbf{u}})_{tt}\|^2_{L^2(t^{n-1},t^{n+1};L^2(\Omega))}\big). \end{aligned} \tag{4.5}$$

The proof is given in Appendix B.1.

Lemma 4.3. *Let* C *be a constant number and* a_k, b_k, d_k, t_k *be a positive sequence and*

$$a_n+\sum_{k=j_1}^{n}t_kb_k\le\sum_{k=j_1}^{s}t_kd_ka_k+C,\quad \forall\, n\ge j_1. \tag{4.6}$$

If $s=n$*, then under the assumption* $t_kd_k<1$*, we have*

$$a_n+\sum_{k=j_1}^{n}t_kb_k\le\exp\left(\sum_{k=j_1}^{n}\frac{t_kd_k}{1-t_kd_k}\right)C,\quad \forall\, n\ge j_1. \tag{4.7}$$

Theorem 4.1 (Error Analysis for the Velocity and Temperature). *Consider the VSS BE-AB2 Algorithm* 3.1. *Suppose for any* $1 \le n \le N-1$, *the stability conditions from Theorem* 3.1 *and the regularity of the solution given in Assumption* 2.1 *holds. Define the maximum stepsize ratio for* $1 \le n \le N-1$ *as*

$$\omega_{N^*} = \max_{n=1,\dots,N-1} \omega_n. \tag{4.8}$$

We then have the following error estimate:

$$\begin{aligned}
&\|e_{\mathbf{u}}^N\|^2 + \frac{Pr}{4}\sum_{n=1}^{N-1}\Delta t_n\|\nabla e_{\mathbf{u}}^{n+1}\|^2 + \|e_T^N\|^2 + \frac{11}{16}\sum_{n=1}^{N-1}\Delta t_n\|\nabla e_T^{n+1}\|^2 \\
&\le C\Bigg[h^{2s+2} + PrT_1h^{2s} + T_1h^{2s} + \exp\left(\sum_{n=1}^{N-1}\Delta t_n d_n\right) \\
&\qquad \times\Bigg\{\|\phi_{\mathbf{u}}^1\|^2 + \frac{1}{2}\|\phi_{\mathbf{u}}^1 - \phi_{\mathbf{u}}^0\|^2 + \frac{\Delta t_1 Pr}{4}\|\phi_{\mathbf{u}}^1\|^2 + \frac{\Delta t_1 Pr}{8}\|\phi_{\mathbf{u}}^0\|^2 + \|\phi_T^1\|^2 + \frac{1}{2}\|\phi_T^1 - \phi_T^0\|^2 \\
&\qquad\quad + \sum_{n=1}^{N-1}\Bigg(Ch^{2s}\Delta t_n Pr\|\nabla\mathbf{u}^{n+1}\|^2 + \frac{Ch^{2s}\Delta t_n\omega_n}{Pr}(\|\nabla\mathbf{u}^n\|^2 + \|\nabla\mathbf{u}^{n-1}\|^2) \\
&\qquad\qquad + \frac{C\Delta t_n(\Delta t_{n-1}+\Delta t_n)^3}{Pr}\|\nabla\mathbf{u}_{tt}\|^2_{L^2(t^n,t^{n+1};L^2(\Omega))} \\
&\qquad\qquad + \frac{C\Delta t_n^2}{Pr}\|\mathbf{u}_{tt}\|^2_{L^2(t^n,t^{n+1};L^2(\Omega))} \\
&\qquad\qquad + \frac{Ch^{2s}\Delta t_n}{Pr}\|p^{n+1}\|^2 + Ch^{2s}\Delta t_n PrRa\|\nabla T^{n+1}\|^2 \\
&\qquad\qquad + Ch^{2s}\Delta t_n\|\nabla T^{n+1}\|^2 + C\Delta t_n^2\|T_{tt}\|^2_{L^2(t^n,t^{n+1};L^2(\Omega))} \\
&\qquad\qquad + \frac{Ch^{2s}\omega_{N^*}}{Pr^{\frac{3}{2}}}\left[PrRa^2\big(|||\gamma^{n+1}|||^2_{2,-1} + \epsilon_T^1\big) + \epsilon_{\mathbf{u}}^1\right] \\
&\qquad\qquad \times\left[\left(\sum_{n=1}^{N-1}\Delta t_n\|\nabla\mathbf{u}^{n+1}\|^4\right)^{\frac{1}{2}} + \left(\sum_{n=1}^{N-1}\Delta t_n\|\nabla T^{n+1}\|^4\right)^{\frac{1}{2}}\right] \\
&\qquad\qquad + \frac{Ch\Delta t_n(\Delta t_{n-1}+\Delta t_n)^3(1+h^{2s})}{C_{stab}\Delta t_n(1+\omega_n{}^2)} \\
&\qquad\qquad \times\Big(\|\nabla\mathbf{u}_{tt}\|^2_{L^2(t^{n-1},t^{n+1};L^2(\Omega))} + \|\nabla T_{tt}\|^2_{L^2(t^{n-1},t^{n+1};L^2(\Omega))}\Big)\Bigg)\Bigg\}\Bigg].
\end{aligned} \tag{4.9}$$

Proof. The true solutions of the NC satisfy, for all $n = 1,\dots,N-1$,

$$\begin{aligned}
&\left(\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t_n}, \mathbf{v}_h\right) + b^*\big(\mathbf{u}^{n+1},\mathbf{u}^{n+1},\mathbf{v}_h\big) \\
&\quad - \big(p^{n+1},\nabla\cdot\mathbf{v}_h\big) + Pr\big(\nabla\mathbf{u}^{n+1},\nabla\mathbf{v}_h\big) \\
&= PrRa\big(\mathbf{j}T^{n+1},\mathbf{v}_h\big) + \tau_{\mathbf{u}}\big(\mathbf{u}^{n+1};\mathbf{v}_h\big), \\
&\big(\nabla\cdot\mathbf{u}_h^{n+1}, q_h\big) = 0, \\
&\left(\frac{T^{n+1}-T^n}{\Delta t_n}, s_h\right) + \overline{b^*}\big(\mathbf{u}^{n+1},T^{n+1},s_h\big) + \big(\nabla T^{n+1},\nabla s_h\big) \\
&= (\gamma, s_h) + \tau_T\big(T^{n+1};s_h\big),
\end{aligned} \tag{4.10}$$

where $\tau_{\mathbf{u}}(\mathbf{u}^{n+1};\mathbf{v}_h), \tau_T(T^{n+1};s_h)$ are defined as

$$\begin{aligned}\tau_{\mathbf{u}}\big(\mathbf{u}^{n+1};\mathbf{v}_h\big) &= \left(\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t_n} - \mathbf{u}_t(t_n), \mathbf{v}_h\right),\\ \tau_T\big(T^{n+1};s_h\big) &= \left(\frac{T^{n+1}-T^n}{\Delta t_n} - T_t(t_n), s_h\right).\end{aligned} \tag{4.11}$$

Subtracting (4.1) from (4.10) yields the error equation

$$\begin{aligned}&\left(\frac{e_{\mathbf{u}}^{n+1}-e_{\mathbf{u}}^n}{\Delta t_n}, \mathbf{v}_h\right) + b^*\big(\mathbf{u}^{n+1},\mathbf{u}^{n+1},\mathbf{v}_h\big)\\ &\quad - b^*\big(E^{n+1}(\mathbf{u}_h),E^{n+1}(\mathbf{u}_h),\mathbf{v}_h\big)\\ &\quad + Pr\big(\nabla e_{\mathbf{u}}^{n+1},\nabla\mathbf{v}_h\big) - \big(e_p^{n+1},\nabla\cdot\mathbf{v}_h\big)\\ &= PrRa\big(\mathbf{j}e_T^{n+1},\mathbf{v}_h\big) + \tau_{\mathbf{u}}\big(\mathbf{u}^{n+1};\mathbf{v}_h\big),\\ &\left(\frac{e_T^{n+1}-e_T^n}{\Delta t_n}, s_h\right) + \overline{b^*}\big(\mathbf{u}^{n+1},T^{n+1},s_h\big)\\ &\quad - \overline{b^*}\big(E^{n+1}(\mathbf{u}_h),E^{n+1}(T_h),s_h\big) + \big(\nabla e_T^{n+1},\nabla s_h\big)\\ &= \tau_T\big(T^{n+1};s_h\big).\end{aligned} \tag{4.12}$$

This can be equivalently written as

$$\begin{aligned}&\left(\frac{\phi_{\mathbf{u}}^{n+1}-\phi_{\mathbf{u}}^n}{\Delta t_n}, \mathbf{v}_h\right) + Pr\big(\nabla\phi_{\mathbf{u}}^{n+1},\nabla\mathbf{v}_h\big)\\ &= \left(\frac{\eta_{\mathbf{u}}^{n+1}-\eta_{\mathbf{u}}^n}{\Delta t_n}, \mathbf{v}_h\right) + Pr\big(\nabla\eta_{\mathbf{u}}^{n+1},\nabla\mathbf{v}_h\big) - \big(e_p^{n+1},\nabla\cdot\mathbf{v}_h\big)\\ &\quad + b^*\big(\mathbf{u}^{n+1},\mathbf{u}^{n+1},\mathbf{v}_h\big) - b^*\big(E^{n+1}(\mathbf{u}_h),E^{n+1}(\mathbf{u}_h),\mathbf{v}_h\big)\\ &\quad - \tau_{\mathbf{u}}\big(\mathbf{u}^{n+1};\mathbf{v}_h\big) - PrRa\big(\mathbf{j}e_T^{n+1},\mathbf{v}_h\big),\end{aligned} \tag{4.13}$$

$$\begin{aligned}&\left(\frac{\phi_T^{n+1}-\phi_T^n}{\Delta t_n}, s_h\right) + \big(\nabla\phi_T^{n+1},\nabla s_h\big)\\ &= \left(\frac{\eta_T^{n+1}-\eta_T^n}{\Delta t_n}, s_h\right) + \big(\nabla\eta_T^{n+1},\nabla s_h\big) + \overline{b^*}\big(\mathbf{u}^{n+1},T^{n+1},s_h\big)\\ &\quad - \overline{b^*}\big(E^{n+1}(\mathbf{u}_h),E^{n+1}(T_h),s_h\big) - \tau_T\big(T^{n+1};s_h\big).\end{aligned} \tag{4.14}$$

Letting $\mathbf{v}_h = 2\Delta t_n\phi_{\mathbf{u}}^{n+1}$, using the fact that $2(\eta_{\mathbf{u}}^{n+1}-\eta_{\mathbf{u}}^n,\mathbf{v}_h)=0$ by the definition of L^2 projection, and the polarization identity yields

$$\begin{aligned}&\big\|\phi_{\mathbf{u}}^{n+1}\big\|^2 - \big\|\phi_{\mathbf{u}}^n\big\|^2 + \big\|\phi_{\mathbf{u}}^{n+1}-\phi_{\mathbf{u}}^n\big\|^2 + 2\Delta t_n Pr\big\|\nabla\phi_{\mathbf{u}}^{n+1}\big\|^2\\ &\equiv \sum_{i=1}^6 A_i = 2\Delta t_n Pr\big(\nabla\eta_{\mathbf{u}}^{n+1},\nabla\phi_{\mathbf{u}}^{n+1}\big) - 2\Delta t_n\big(e_p^{n+1},\nabla\cdot\phi_{\mathbf{u}}^{n+1}\big)\\ &\quad - 2\Delta t_n\tau_{\mathbf{u}}\big(\mathbf{u}^{n+1};\phi_{\mathbf{u}}^{n+1}\big) - 2\Delta t_n PrRa\big(\mathbf{j}e_T^{n+1},\phi_{\mathbf{u}}^{n+1}\big)\\ &\quad + 2\Delta t_n b^*\big(\mathbf{u}^{n+1},\mathbf{u}^{n+1},\phi_{\mathbf{u}}^{n+1}\big) - 2\Delta t_n b^*\big(E^{n+1}(\mathbf{u}_h),E^{n+1}(\mathbf{u}_h),\phi_{\mathbf{u}}^{n+1}\big).\end{aligned} \tag{4.15}$$

Letting $s_h = 2\Delta t_n\phi_T^{n+1}$, using the fact that $2(\eta_T^{n+1}-\eta_T^n,s_h)=0$ by the definition of L^2 projection, and the polarization identity yields

$$\big\|\phi_T^{n+1}\big\|^2 - \big\|\phi_T^n\big\|^2 + \big\|\phi_T^{n+1}-\phi_T^n\big\|^2 + 2\Delta t_n\big\|\nabla\phi_T^{n+1}\big\|^2$$

$$\equiv \sum_{i=1}^{4} B_i = 2\Delta t_n\big(\nabla\eta_T^{n+1}, \nabla\phi_T^{n+1}\big) - 2\Delta t_n\tau_T\big(T^{n+1}; \phi_T^{n+1}\big) + 2\Delta t_n\overline{b^*}\big(\mathbf{u}^{n+1}, T^{n+1}, \phi_T^{n+1}\big) - 2\Delta t_n\overline{b^*}\big(E^{n+1}(\mathbf{u}_h), E^{n+1}(T_h), \phi_T^{n+1}\big). \tag{4.16}$$

By the Cauchy-Schwarz-Young and Poincare-Friedrichs inequalities, we bound the first term on the right-hand side

$$A_1 \le \frac{Pr\Delta t_n}{\delta_1}\big\|\nabla\eta_{\mathbf{u}}^{n+1}\big\|^2 + \Delta t_n\delta_1 Pr\big\|\nabla\phi_{\mathbf{u}}^{n+1}\big\|^2. \tag{4.17}$$

Next, we consider the pressure term. Since $\phi_{\mathbf{u}}^{n+1} \in \mathbf{V}_h$, applying $(\lambda_h, \nabla\cdot\phi_{\mathbf{u}}^{n+1}) = 0$ for $\forall\lambda_h \in Q_h$, we have

$$\begin{aligned} A_2 &= 2\Delta t_n\big[\big(p^{n+1} - \lambda_h^{n+1}\big) - \big(p_h^{n+1} - \lambda_h^{n+1}\big), \nabla\cdot\phi_{\mathbf{u}}^{n+1}\big] \\ &= 2\Delta t_n\big(p^{n+1} - \lambda_h^{n+1}, \nabla\cdot\phi_{\mathbf{u}}^{n+1}\big) \\ &\le \frac{\Delta t_n}{\delta_2 Pr}\big\|p^{n+1} - \lambda_h^{n+1}\big\|^2 + \Delta t_n\delta_2 Pr\big\|\nabla\phi_{\mathbf{u}}^{n+1}\big\|^2. \end{aligned} \tag{4.18}$$

Using Lemma 4.1 and Cauchy-Schwarz-Young inequality the consistency term is bounded as

$$A_3 \le \frac{C\Delta t_n^2}{\delta_3 Pr}\|\mathbf{u}_{tt}\|^2_{L^2(t^n,t^{n+1};L^2(\Omega))} + \Delta t_n Pr\delta_3\big\|\nabla\phi_{\mathbf{u}}^{n+1}\big\|^2. \tag{4.19}$$

Applying Cauchy-Schwarz-Young inequality yields

$$\begin{aligned} A_4 &\le C\Delta t_n PrRa\big\|\eta_T^{n+1}\big\|\big\|\phi_{\mathbf{u}}^{n+1}\big\| + C\Delta t_n PrRa\big\|\phi_T^{n+1}\big\|\big\|\phi_{\mathbf{u}}^{n+1}\big\| \\ &\le \frac{C\Delta t_n PrRa}{2\delta_4}\big\|\eta_T^{n+1}\big\|^2 + C\delta_4\Delta t_n PrRa\big\|\phi_{\mathbf{u}}^{n+1}\big\|^2 + \frac{C\Delta t_n PrRa}{2\delta_4}\big\|\phi_T^{n+1}\big\|^2. \end{aligned} \tag{4.20}$$

Lastly, the nonlinear terms are bounded using Lemma 4.2

$$\begin{aligned} A_5 + A_6 \le 2\Delta t_n\bigg\{ &\frac{5}{64}Pr\big\|\nabla\phi_{\mathbf{u}}^{n+1}\big\|^2 + \frac{C}{Pr}\big\|\nabla E^{n+1}(\eta_{\mathbf{u}})\big\|^2 + CPr^{-3}\big(\big\|\phi_{\mathbf{u}}^{n}\big\|^2 + \big\|\phi_{\mathbf{u}}^{n-1}\big\|^2\big) \\ &+ \frac{C(\Delta t_{n-1} + \Delta t_n)^3}{Pr}\|\nabla\mathbf{u}_{tt}\|^2_{L^2(t^{n-1},t^{n+1};L^2(\Omega))} + \frac{1}{16}Pr\big\|\nabla\phi_{\mathbf{u}}^{n}\big\|^2 \\ &+ \frac{1}{16}Pr\big\|\nabla\phi_{\mathbf{u}}^{n-1}\big\|^2 + \frac{1}{4\Delta t_n}\big\|\phi_{\mathbf{u}}^{n} - \phi_{\mathbf{u}}^{n-1}\big\|^2 \\ &+ \frac{C}{Pr}\big\|E^{n+1}(\mathbf{u}_h)\big\|\big\|\nabla E^{n+1}(\mathbf{u}_h)\big\|\big\|\nabla\eta_{\mathbf{u}}^{n+1}\big\|^2 \\ &+ \frac{5C_{stab}\Delta t_n\big(1+{\omega_n}^2\big)}{8h}\big\|\nabla E^{n+1}(\mathbf{u}_h)\big\|^2\big\|\nabla\phi_{\mathbf{u}}^{n+1}\big\|^2 \\ &+ \frac{Ch(\Delta t_{n-1} + \Delta t_n)^3}{C_{stab}\Delta t_n\big(1+{\omega_n}^2\big)} \\ &\times\big(\|\nabla\mathbf{u}_{tt}\|^2_{L^2(t^{n-1},t^{n+1};L^2(\Omega))} + \|\nabla(\eta_{\mathbf{u}})_{tt}\|^2_{L^2(t^{n-1},t^{n+1};L^2(\Omega))}\big) \\ &+ \frac{1}{4\Delta t_n}\big\|\phi_{\mathbf{u}}^{n+1} - \phi_{\mathbf{u}}^{n}\big\|^2\bigg\}, \end{aligned} \tag{4.21}$$

$$B_1 \le \frac{\Delta t_n}{\widetilde{\delta_1}}\big\|\nabla\eta_T^{n+1}\big\|^2 + \Delta t_n\widetilde{\delta_1}\big\|\nabla\phi_T^{n+1}\big\|^2, \tag{4.22}$$

$$B_2 \le \frac{C{\Delta t_n}^2}{\widetilde{\delta_2}}\|T_{tt}\|^2_{L^2(t^n,t^{n+1};L^2(\Omega))} + \Delta t_n\widetilde{\delta_2}\big\|\nabla\phi_T^{n+1}\big\|^2, \tag{4.23}$$

$$
\begin{aligned}
B_3+B_4 \le 2\Delta t_n\bigg\{ & \frac{5}{64}Pr\big\|\nabla\phi_T^{n+1}\big\|^2+\frac{C}{Pr}\big\|\nabla E^{n+1}(\eta_{\mathbf{u}})\big\|^2+CPr^{-3}\big(\big\|\phi_{\mathbf{u}}^n\big\|^2+\big\|\phi_{\mathbf{u}}^{n-1}\big\|^2\big)\\
&+\frac{C(\Delta t_{n-1}+\Delta t_n)^3}{Pr}\|\nabla\mathbf{u}_{tt}\|^2_{L^2(t^{n-1},t^{n+1};L^2(\Omega))}+\frac{1}{16}Pr\big\|\nabla\phi_{\mathbf{u}}^n\big\|^2\\
&+\frac{1}{16}Pr\big\|\nabla\phi_{\mathbf{u}}^{n-1}\big\|^2+\frac{1}{4\Delta t_n}\big\|\phi_T^{n+1}-\phi_T^n\big\|^2\\
&+\frac{C}{Pr}\big\|E^{n+1}(\mathbf{u}_h)\big\|\big\|\nabla E^{n+1}(\mathbf{u}_h)\big\|\big\|\nabla\eta_T^{n+1}\big\|^2\\
&+\frac{5C_{stab}\Delta t_n\big(1+\omega_n{}^2\big)}{8h}\big\|\nabla E^{n+1}(\mathbf{u}_h)\big\|^2\big\|\nabla\phi_T^{n+1}\big\|^2\\
&+\frac{1}{4\Delta t_n}\big\|\phi_T^n-\phi_T^{n-1}\big\|^2+\frac{Ch(\Delta t_{n-1}+\Delta t_n)^3}{C_{stab}\Delta t_n(1+\omega_n{}^2)}\\
&\times\big(\|\nabla T_{tt}\|^2_{L^2(t^{n-1},t^{n+1};L^2(\Omega))}+\|\nabla(\eta_T)_{tt}\|^2_{L^2(t^{n-1},t^{n+1};L^2(\Omega))}\big)\bigg\}.
\end{aligned}
\tag{4.24}
$$

Then, we can get

$$
\begin{aligned}
&\big\|\phi_{\mathbf{u}}^{n+1}\big\|^2-\big\|\phi_{\mathbf{u}}^n\big\|^2+\big\|\phi_{\mathbf{u}}^{n+1}-\phi_{\mathbf{u}}^n\big\|^2+2\Delta t_nPr\big\|\nabla\phi_{\mathbf{u}}^{n+1}\big\|^2\\
\le{}&\frac{Pr\Delta t_n}{\delta_1}\big\|\nabla\eta_{\mathbf{u}}^{n+1}\big\|^2+\Delta t_n\delta_1Pr\big\|\nabla\phi_{\mathbf{u}}^{n+1}\big\|^2+\frac{\Delta t_n}{\delta_2Pr}\big\|p^{n+1}-\lambda_h^{n+1}\big\|^2\\
&+\Delta t_n\delta_2Pr\big\|\nabla\phi_{\mathbf{u}}^{n+1}\big\|^2+\frac{C\Delta t_n^2}{\delta_3Pr}\|\mathbf{u}_{tt}\|^2_{L^2(t^n,t^{n+1};L^2(\Omega))}+\Delta t_nPr\delta_3\big\|\nabla\phi_{\mathbf{u}}^{n+1}\big\|^2\\
&+\frac{C\Delta t_nPrRa}{2\delta_4}\big\|\eta_T^{n+1}\big\|^2+C\delta_4\Delta t_nPrRa\big\|\phi_{\mathbf{u}}^{n+1}\big\|^2+\frac{C\Delta t_nPrRa}{2\delta_4}\big\|\phi_T^{n+1}\big\|^2\\
&+2\Delta t_n\bigg\{\frac{5}{64}Pr\big\|\nabla\phi_{\mathbf{u}}^{n+1}\big\|^2+\frac{C}{Pr}\big\|\nabla E^{n+1}(\eta_{\mathbf{u}})\big\|^2\\
&\qquad+\frac{C(\Delta t_{n-1}+\Delta t_n)^3}{Pr}\|\nabla\mathbf{u}_{tt}\|^2_{L^2(t^{n-1},t^{n+1};L^2(\Omega))}\\
&\qquad+CPr^{-3}\big(\big\|\phi_{\mathbf{u}}^n\big\|^2+\big\|\phi_{\mathbf{u}}^{n-1}\big\|^2\big)+\frac{1}{16}Pr\big(\big\|\nabla\phi_{\mathbf{u}}^n\big\|^2+\big\|\nabla\phi_{\mathbf{u}}^{n-1}\big\|^2\big)\\
&\qquad+\frac{1}{4\Delta t_n}\big(\big\|\phi_{\mathbf{u}}^{n+1}-\phi_{\mathbf{u}}^n\big\|^2+\big\|\phi_{\mathbf{u}}^n-\phi_{\mathbf{u}}^{n-1}\big\|^2\big)\\
&\qquad+\frac{C}{Pr}\big\|E^{n+1}(\mathbf{u}_h)\big\|\big\|\nabla E^{n+1}(\mathbf{u}_h)\big\|\big\|\nabla\eta_{\mathbf{u}}^{n+1}\big\|^2\\
&\qquad+\frac{5C_{stab}\Delta t_n\big(1+\omega_n{}^2\big)}{8h}\big\|\nabla E^{n+1}(\mathbf{u}_h)\big\|^2\big\|\nabla\phi_{\mathbf{u}}^{n+1}\big\|^2\\
&\qquad+\frac{Ch(\Delta t_{n-1}+\Delta t_n)^3}{C_{stab}\Delta t_n\big(1+\omega_n{}^2\big)}\\
&\qquad\times\big(\|\nabla\mathbf{u}_{tt}\|^2_{L^2(t^{n-1},t^{n+1};L^2(\Omega))}+\|\nabla(\eta_{\mathbf{u}})_{tt}\|^2_{L^2(t^{n-1},t^{n+1};L^2(\Omega))}\big)\bigg\},
\end{aligned}
\tag{4.25}
$$

$$
\begin{aligned}
&\big\|\phi_T^{n+1}\big\|^2-\big\|\phi_T^n\big\|^2+\big\|\phi_T^{n+1}-\phi_T^n\big\|^2+2\Delta t_n\big\|\nabla\phi_T^{n+1}\big\|^2\\
\le{}&\frac{\Delta t_n}{\widetilde{\delta_1}}\big\|\nabla\eta_T^{n+1}\big\|^2+\Delta t_n\widetilde{\delta_1}\big\|\nabla\phi_T^{n+1}\big\|^2+\frac{C\Delta t_n^2}{\widetilde{\delta_2}}\|T_{tt}\|^2_{L^2(t^n,t^{n+1};L^2(\Omega))}+\Delta t_n\widetilde{\delta_2}\big\|\nabla\phi_T^{n+1}\big\|^2\\
&+2\Delta t_n\bigg\{\frac{5}{64}Pr\big\|\nabla\phi_T^{n+1}\big\|^2+\frac{C}{Pr}\big\|\nabla E^{n+1}(\eta_{\mathbf{u}})\big\|^2\\
&\qquad+\frac{C(\Delta t_{n-1}+\Delta t_n)^3}{Pr}\|\nabla\mathbf{u}_{tt}\|^2_{L^2(t^{n-1},t^{n+1};L^2(\Omega))}+CPr^{-3}\big(\big\|\phi_{\mathbf{u}}^n\big\|^2+\big\|\phi_{\mathbf{u}}^{n-1}\big\|^2\big)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{16}Pr\big(\big\|\nabla\phi_{\mathbf{u}}^{n}\big\|^2+\big\|\nabla\phi_{\mathbf{u}}^{n-1}\big\|^2\big)\\
&+\frac{1}{4\Delta t_n}\big(\big\|\phi_T^{n+1}-\phi_T^{n}\big\|^2+\big\|\phi_T^{n}-\phi_T^{n-1}\big\|^2\big)\\
&+\frac{C}{Pr}\big\|E^{n+1}(\mathbf{u}_h)\big\|\big\|\nabla E^{n+1}(\mathbf{u}_h)\big\|\big\|\nabla\eta_T^{n+1}\big\|^2\\
&+\frac{5C_{stab}\Delta t_n\big(1+{\omega_n}^2\big)}{8h}\big\|\nabla E^{n+1}(\mathbf{u}_h)\big\|^2\big\|\nabla\phi_T^{n+1}\big\|^2\\
&+\frac{Ch(\Delta t_{n-1}+\Delta t_n)^3}{C_{stab}\Delta t_n\big(1+{\omega_n}^2\big)}\\
&\times\big(\|\nabla T_{tt}\|^2_{L^2(t^{n-1},t^{n+1};L^2(\Omega))}+\|\nabla(\eta_T)_{tt}\|^2_{L^2(t^{n-1},t^{n+1};L^2(\Omega))}\big)\Big\}. \qquad (4.26)
\end{aligned}
$$

Taking $\delta_1,\delta_2,\delta_3,\delta_4=1/32$, adding and subtracting $Pr\|\nabla\phi_{\mathbf{u}}^{n}\|/8$, and rearranging terms, we have

$$
\begin{aligned}
&\big\|\phi_{\mathbf{u}}^{n+1}\big\|^2-\big\|\phi_{\mathbf{u}}^{n}\big\|^2+\frac{1}{2}\big\|\phi_{\mathbf{u}}^{n+1}-\phi_{\mathbf{u}}^{n}\big\|^2-\frac{1}{2}\big\|\phi_{\mathbf{u}}^{n}-\phi_{\mathbf{u}}^{n-1}\big\|^2\\
&\quad+\frac{\Delta t_nPr}{4}\big\|\nabla\phi_{\mathbf{u}}^{n+1}\big\|^2+\frac{5\Delta t_nPr}{4}\big\|\nabla\phi_{\mathbf{u}}^{n+1}\big\|^2M_2\\
&\quad+\frac{\Delta t_nPr}{4}\big(\big\|\nabla\phi_{\mathbf{u}}^{n+1}\big\|^2-\big\|\nabla\phi_{\mathbf{u}}^{n}\big\|^2\big)+\frac{\Delta t_nPr}{8}\big(\big\|\nabla\phi_{\mathbf{u}}^{n}\big\|^2-\big\|\nabla\phi_{\mathbf{u}}^{n-1}\big\|^2\big)\\
&\le\frac{C\Delta t_n}{Pr}\big\|E^{n+1}(\mathbf{u}_h)\big\|\big\|\nabla E^{n+1}(\mathbf{u}_h)\big\|\big\|\nabla\eta_{\mathbf{u}}^{n+1}\big\|^2+\frac{C\Delta t_n}{Pr}\big\|\nabla E^{n+1}(\eta_{\mathbf{u}})\big\|^2\\
&\quad+C\Delta t_nPr^{-3}\big(\big\|\phi_{\mathbf{u}}^{n}\big\|^2+\big\|\phi_{\mathbf{u}}^{n-1}\big\|^2\big)\\
&\quad+\frac{C\Delta t_n(\Delta t_{n-1}+\Delta t_n)^3}{Pr}\|\nabla\mathbf{u}_{tt}\|^2_{L^2(t^{n-1},t^{n+1};L^2(\Omega))}+CPr\Delta t_n\big\|\nabla\eta_{\mathbf{u}}^{n+1}\big\|^2\\
&\quad+\frac{C\Delta t_n^2}{Pr}\|\mathbf{u}_{tt}\|^2_{L^2(t^{n},t^{n+1};L^2(\Omega))}+\frac{Ch\Delta t_n(\Delta t_{n-1}+\Delta t_n)^3}{C_{stab}\Delta t_n\big(1+{\omega_n}^2\big)}\\
&\quad\times\big(\|\nabla\mathbf{u}_{tt}\|^2_{L^2(t^{n-1},t^{n+1};L^2(\Omega))}+\|\nabla(\eta_{\mathbf{u}})_{tt}\|^2_{L^2(t^{n-1},t^{n+1};L^2(\Omega))}\big)\\
&\quad+\frac{C\Delta t_n}{Pr}\big\|p^{n+1}-\lambda_h^{n+1}\big\|^2+C\Delta t_nPrRa\big(\big\|\eta_T^{n+1}\big\|^2+\big\|\phi_{\mathbf{u}}^{n+1}\big\|^2+\big\|\phi_T^{n+1}\big\|^2\big). \qquad (4.27)
\end{aligned}
$$

Next taking $\widetilde{\delta_1},\widetilde{\delta_2}=1/32$, we can get

$$
\begin{aligned}
&\big\|\phi_T^{n+1}\big\|^2-\big\|\phi_T^{n}\big\|^2+\frac{1}{2}\big\|\phi_T^{n+1}-\phi_T^{n}\big\|^2-\frac{1}{2}\big\|\phi_T^{n}-\phi_T^{n-1}\big\|^2\\
&\quad+\frac{11\Delta t_n}{16}\big\|\nabla\phi_T^{n+1}\big\|^2+\frac{5\Delta t_nPr}{4}\big\|\nabla\phi_T^{n+1}\big\|^2M_2\\
&\le\frac{C\Delta t_n}{Pr}\big\|E^{n+1}(\mathbf{u}_h)\big\|\big\|\nabla E^{n+1}(\mathbf{u}_h)\big\|\big\|\nabla\eta_T^{n+1}\big\|^2+\frac{C\Delta t_n}{Pr}\big\|\nabla E^{n+1}(\eta_{\mathbf{u}})\big\|^2\\
&\quad+\frac{5}{32}\Delta t_nPr\big\|\nabla\phi_{\mathbf{u}}^{n+1}\big\|^2+\frac{C\Delta t_n(\Delta t_{n-1}+\Delta t_n)^3}{Pr}\|\nabla\mathbf{u}_{tt}\|^2_{L^2(t^{n-1},t^{n+1};L^2(\Omega))}\\
&\quad+C\Delta t_n\big\|\nabla\eta_T^{n+1}\big\|^2+\frac{1}{8}\Delta t_nPr\big(\big\|\nabla\phi_{\mathbf{u}}^{n}\big\|^2+\big\|\nabla\phi_{\mathbf{u}}^{n-1}\big\|^2\big)\\
&\quad+\frac{Ch\Delta t_n(\Delta t_{n-1}+\Delta t_n)^3}{C_{stab}\Delta t_n\big(1+{\omega_n}^2\big)}\\
&\quad\times\big(\|\nabla T_{tt}\|^2_{L^2(t^{n-1},t^{n+1};L^2(\Omega))}+\|\nabla(\eta_T)_{tt}\|^2_{L^2(t^{n-1},t^{n+1};L^2(\Omega))}\big)\\
&\quad+C\Delta t_nPr^{-3}\big(\big\|\phi_{\mathbf{u}}^{n}\big\|^2+\big\|\phi_{\mathbf{u}}^{n-1}\big\|^2\big)+C\Delta t_n^2\|T_{tt}\|^2_{L^2(t^{n},t^{n+1};L^2(\Omega))}. \qquad (4.28)
\end{aligned}
$$

We note that using Cauchy-Schwarz-Young inequality and the stability estimate from Theorem 3.1 we have that

$$
\begin{aligned}
&\sum_{n=1}^{N-1} \frac{C\Delta t_n}{Pr} \|E^{n+1}(\mathbf{u}_h)\| \|\nabla E^{n+1}(\mathbf{u}_h)\| \|\nabla \eta_{\mathbf{u}}^{n+1}\|^2 \\
&\leq \frac{C}{Pr} \Big(\max_{n=1,\dots,N-1} \|E^{n+1}(\mathbf{u}_h)\| \Big) \sum_{n=1}^{N-1} \Delta t_n \|\nabla E^{n+1}(\mathbf{u}_h)\| \|\nabla \eta_{\mathbf{u}}^{n+1}\|^2 \\
&\leq \frac{C\omega_{N^*}^{\frac{1}{2}}}{Pr} \big[PrRa^2 \big(|||\gamma^{n+1}|||_{2,-1}^2 + \epsilon_T^1 \big) + \epsilon_{\mathbf{u}}^1 \big]^{\frac{1}{2}} \\
&\quad \times \left(\sum_{n=1}^{N-1} \Delta t_n \|\nabla E^{n+1}(\mathbf{u}_h)\|^2 \right)^{\frac{1}{2}} \left(\sum_{n=1}^{N-1} \Delta t_n \|\nabla \eta_{\mathbf{u}}^{n+1}\|^4 \right)^{\frac{1}{2}} \\
&\leq \frac{C\omega_{N^*}}{Pr^{\frac{3}{2}}} \big[PrRa^2 (|||\gamma^{n+1}|||_{2,-1}^2 + \epsilon_T^1) + \epsilon_{\mathbf{u}}^1 \big] \left(\sum_{n=1}^{N-1} \Delta t_n \|\nabla \eta_{\mathbf{u}}^{n+1}\|^4 \right)^{\frac{1}{2}}, \qquad (4.29)
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{n=1}^{N-1} \frac{C\Delta t_n}{Pr} \|E^{n+1}(\mathbf{u}_h)\| \|\nabla E^{n+1}(\mathbf{u}_h)\| \|\nabla \eta_T^{n+1}\|^2 \\
&\leq \frac{C\omega_{N^*}}{Pr^{\frac{3}{2}}} \big[PrRa^2 \big(|||\gamma^{n+1}|||_{2,-1}^2 + \epsilon_T^1 \big) + \epsilon_{\mathbf{u}}^1 \big] \left(\sum_{n=1}^{N-1} \Delta t_n \|\nabla \eta_T^{n+1}\|^4 \right)^{\frac{1}{2}}. \qquad (4.30)
\end{aligned}
$$

Then, using Theorem 3.1, summing from $n = 1$ to $n = N - 1$, dropping positive terms on the left-hand side, and using the above bound we have

$$
\begin{aligned}
&\|\phi_{\mathbf{u}}^N\|^2 + \frac{Pr}{4} \sum_{n=1}^{N-1} \Delta t_n \|\nabla \phi_{\mathbf{u}}^{n+1}\|^2 \\
&\leq \|\phi_{\mathbf{u}}^1\|^2 + \frac{1}{2} \|\phi_{\mathbf{u}}^1 - \phi_{\mathbf{u}}^0\|^2 + \frac{\Delta t_1 Pr}{4} \|\phi_{\mathbf{u}}^1\|^2 + \frac{\Delta t_1 Pr}{8} \|\nabla \phi_{\mathbf{u}}^0\|^2 + CPr^{-3} \sum_{n=1}^{N-1} \Delta t_n \|\phi_{\mathbf{u}}^n\|^2 \\
&\quad + \sum_{n=1}^{N-1} \Bigg\{ C\Delta t_n Pr \|\nabla \eta_{\mathbf{u}}^{n+1}\|^2 + \frac{C\Delta t_n \omega_n}{Pr} \big(\|\nabla \eta_{\mathbf{u}}^n\|^2 + \|\nabla \eta_{\mathbf{u}}^{n-1}\|^2 \big) \\
&\qquad + \frac{Ch\Delta t_n (\Delta t_{n-1} + \Delta t_n)^3}{C_{stab} \Delta t_n \big(1 + {\omega_n}^2\big)} \\
&\qquad \times \big(\|\nabla \mathbf{u}_{tt}\|_{L^2(t^{n-1},t^{n+1};L^2(\Omega))}^2 + \|\nabla (\eta_{\mathbf{u}})_{tt}\|_{L^2(t^{n-1},t^{n+1};L^2(\Omega))}^2 \big) \\
&\qquad + \frac{C\Delta t_n (\Delta t_{n-1} + \Delta t_n)^3}{Pr} \|\nabla \mathbf{u}_{tt}\|_{L^2(t^{n-1},t^{n+1};L^2(\Omega))}^2 \\
&\qquad + \frac{C\Delta t_n^2}{Pr} \|\mathbf{u}_{tt}\|_{L^2(t^n,t^{n+1};L^2(\Omega))}^2 \\
&\qquad + \frac{C\omega_{N^*}}{Pr^{\frac{3}{2}}} \big[PrRa^2 \big(|||\gamma^{n+1}|||_{2,-1}^2 + \epsilon_T^1 \big) + \epsilon_{\mathbf{u}}^1 \big] \left(\sum_{n=1}^{N-1} \Delta t_n \|\nabla \eta_{\mathbf{u}}^{n+1}\|^4 \right)^{\frac{1}{2}} \\
&\qquad + \frac{C\Delta t_n}{Pr} \|p^{n+1} - \lambda_h^{n+1}\|^2 \\
&\qquad + C\Delta t_n PrRa \big(\|\eta_T^{n+1}\|^2 + \|\phi_{\mathbf{u}}^{n+1}\|^2 + \|\phi_T^{n+1}\|^2 \big) \Bigg\}, \qquad (4.31)
\end{aligned}
$$

$$
\begin{aligned}
&\|\phi_T^N\|^2 + \frac{11}{16}\sum_{n=1}^{N-1}\Delta t_n \|\nabla\phi_T^{n+1}\|^2 \\
\leq\ & \|\phi_T^1\|^2 + \frac{1}{2}\|\phi_T^1 - \phi_T^0\|^2 + CPr^{-3}\sum_{n=1}^{N-1}\Delta t_n\|\phi_{\mathbf{u}}^n\|^2 \\
&+ \sum_{n=1}^{N-1}\Bigg\{ C\Delta t_n\|\nabla\eta_T^{n+1}\|^2 + \frac{C\Delta t_n\omega_n}{Pr}\big(\|\nabla\eta_{\mathbf{u}}^n\|^2 + \|\nabla\eta_{\mathbf{u}}^{n-1}\|^2\big) \\
&\qquad + \frac{1}{8}\Delta t_n Pr\big(\|\nabla\phi_{\mathbf{u}}^n\|^2 + \|\nabla\phi_{\mathbf{u}}^{n-1}\|^2\big) + \frac{Ch\Delta t_n(\Delta t_{n-1}+\Delta t_n)^3}{C_{stab}\Delta t_n(1+\omega_n{}^2)} \\
&\qquad \times\big(\|\nabla T_{tt}\|^2_{L^2(t^{n-1},t^{n+1};L^2(\Omega))} + \|\nabla(\eta_T)_{tt}\|^2_{L^2(t^{n-1},t^{n+1};L^2(\Omega))}\big) \\
&\qquad + \frac{C\Delta t_n(\Delta t_{n-1}+\Delta t_n)^3}{Pr}\|\nabla\mathbf{u}_{tt}\|^2_{L^2(t^{n-1},t^{n+1};L^2(\Omega))} \\
&\qquad + C\Delta t_n^2\|T_{tt}\|^2_{L^2(t^n,t^{n+1};L^2(\Omega))} + \frac{C\omega_{N^*}}{Pr^{\frac{3}{2}}}\big[PrRa^2\big(|||\gamma^{n+1}|||^2_{2,-1} + \epsilon_T^1\big) + \epsilon_{\mathbf{u}}^1\big] \\
&\qquad \times\left(\sum_{n=1}^{N-1}\Delta t_n\|\nabla\eta_T^{n+1}\|^4\right)^{\frac{1}{2}} + \frac{5}{32}\Delta t_n Pr\|\nabla\phi_{\mathbf{u}}^{n+1}\|^2\Bigg\}. \qquad (4.32)
\end{aligned}
$$

Combining (4.31) and (4.32), we can get

$$
\begin{aligned}
&\|\phi_{\mathbf{u}}^N\|^2 + \frac{Pr}{4}\sum_{n=1}^{N-1}\Delta t_n\|\nabla\phi_{\mathbf{u}}^{n+1}\|^2 + \|\phi_T^N\|^2 + \frac{11}{16}\sum_{n=1}^{N-1}\Delta t_n\|\nabla\phi_T^{n+1}\|^2 \\
\leq\ & \|\phi_{\mathbf{u}}^1\|^2 + \frac{1}{2}\|\phi_{\mathbf{u}}^1 - \phi_{\mathbf{u}}^0\|^2 + \frac{\Delta t_1 Pr}{4}\|\phi_{\mathbf{u}}^1\|^2 + \frac{\Delta t_1 Pr}{8}\|\nabla\phi_{\mathbf{u}}^0\|^2 + \|\phi_T^1\|^2 + \frac{1}{2}\|\phi_T^1 - \phi_T^0\|^2 \\
&+ \sum_{n=1}^{N-1}\Big(\widehat{C_1}Pr^{-3}\Delta t_n\|\phi_{\mathbf{u}}^n\|^2 + \frac{5}{32}\Delta t_n Pr\|\nabla\phi_{\mathbf{u}}^{n+1}\|^2 \\
&\qquad + \frac{1}{8}\Delta t_n Pr\big(\|\nabla\phi_{\mathbf{u}}^n\|^2 + \|\nabla\phi_{\mathbf{u}}^{n-1}\|^2\big) \\
&\qquad + \widehat{C_2}\Delta t_n PrRa\|\phi_{\mathbf{u}}^{n+1}\|^2 + \widehat{C_3}\Delta t_n PrRa\|\phi_T^{n+1}\|^2\Big) \\
&+ \sum_{n=1}^{N-1}\Bigg\{ C\Delta t_n Pr\|\nabla\eta_{\mathbf{u}}^{n+1}\|^2 + \frac{C\Delta t_n\omega_n}{Pr}\big(\|\nabla\eta_{\mathbf{u}}^n\|^2 + \|\nabla\eta_{\mathbf{u}}^{n-1}\|^2\big) \\
&\qquad + \frac{Ch\Delta t_n(\Delta t_{n-1}+\Delta t_n)^3}{C_{stab}\Delta t_n\big(1+\omega_n{}^2\big)} \\
&\qquad \times\big(\|\nabla\mathbf{u}_{tt}\|^2_{L^2(t^{n-1},t^{n+1};L^2(\Omega))} + \|\nabla(\eta_{\mathbf{u}})_{tt}\|^2_{L^2(t^{n-1},t^{n+1};L^2(\Omega))} \\
&\qquad\quad + \|\nabla T_{tt}\|^2_{L^2(t^{n-1},t^{n+1};L^2(\Omega))} + \|\nabla(\eta_T)_{tt}\|^2_{L^2(t^{n-1},t^{n+1};L^2(\Omega))}\big) \\
&\qquad + \frac{C\Delta t_n(\Delta t_{n-1}+\Delta t_n)^3}{Pr}\|\nabla\mathbf{u}_{tt}\|^2_{L^2(t^{n-1},t^{n+1};L^2(\Omega))} \\
&\qquad + \frac{C\Delta t_n^2}{Pr}\|\mathbf{u}_{tt}\|^2_{L^2(t^n,t^{n+1};L^2(\Omega))} \\
&\qquad + \frac{C\omega_{N^*}}{Pr^{\frac{3}{2}}}\big[PrRa^2\big(|||\gamma^{n+1}|||^2_{2,-1} + \epsilon_T^1\big) + \epsilon_{\mathbf{u}}^1\big] \\
&\qquad \times\left[\left(\sum_{n=1}^{N-1}\Delta t_n\|\nabla\eta_{\mathbf{u}}^{n+1}\|^4\right)^{\frac{1}{2}} + \left(\sum_{n=1}^{N-1}\Delta t_n\|\nabla\eta_T^{n+1}\|^4\right)^{\frac{1}{2}}\right]
\end{aligned}
$$

$$+\frac{C\Delta t_n}{Pr}\left\|p^{n+1}-\lambda_h^{n+1}\right\|^2+C\Delta t_n PrRa\left\|\eta_T^{n+1}\right\|^2$$
$$\left.+C\Delta t_n\left\|\nabla\eta_T^{n+1}\right\|^2+C\Delta t_n^2\|T_{tt}\|^2_{L^2(t^n,t^{n+1};L^2(\Omega))}\right\}. \tag{4.33}$$

Define

$$d_n=\max\left\{\widehat{C_1}Pr^{-3},\frac{5Pr}{32},\frac{Pr}{8},\widehat{C_2}PrRa,\widehat{C_3}PrRa\right\},$$

and assume $\Delta t_n d_n<1$. Then using invoking the discrete Gronwall inequality gives and interpolation inequalities we can get

$$\begin{aligned}
&\left\|\phi_{\mathbf{u}}^N\right\|^2+\frac{Pr}{4}\sum_{n=1}^{N-1}\Delta t_n\left\|\nabla\phi_{\mathbf{u}}^{n+1}\right\|^2+\left\|\phi_T^N\right\|^2+\frac{11}{16}\sum_{n=1}^{N-1}\Delta t_n\left\|\nabla\phi_T^{n+1}\right\|^2\\
\le&\exp\left(\sum_{n=1}^{N-1}\Delta t_n d_n\right)\left\{\left\|\phi_{\mathbf{u}}^1\right\|^2+\frac{1}{2}\left\|\phi_{\mathbf{u}}^1-\phi_{\mathbf{u}}^0\right\|^2+\frac{\Delta t_1 Pr}{4}\left\|\phi_{\mathbf{u}}^1\right\|^2\right.\\
&+\frac{\Delta t_1 Pr}{8}\left\|\nabla\phi_{\mathbf{u}}^0\right\|^2+\left\|\phi_T^1\right\|^2+\frac{1}{2}\left\|\phi_T^1-\phi_T^0\right\|^2\\
&+\sum_{n=1}^{N-1}\left(Ch^{2s}\Delta t_n Pr\|\nabla\mathbf{u}^{n+1}\|^2\right.\\
&+\frac{Ch^{2s}\Delta t_n\omega_n}{Pr}(\|\nabla\mathbf{u}^n\|^2+\|\nabla\mathbf{u}^{n-1}\|^2)\\
&+\frac{Ch\Delta t_n(\Delta t_{n-1}+\Delta t_n)^3(1+h^{2s})}{C_{stab}\Delta t_n\big(1+{\omega_n}^2\big)}\\
&\times\left(\|\nabla\mathbf{u}_{tt}\|^2_{L^2(t^{n-1},t^{n+1};L^2(\Omega))}+\|\nabla T_{tt}\|^2_{L^2(t^{n-1},t^{n+1};L^2(\Omega))}\right)\\
&+\frac{C\Delta t_n(\Delta t_{n-1}+\Delta t_n)^3}{Pr}\|\nabla\mathbf{u}_{tt}\|^2_{L^2(t^{n-1},t^{n+1};L^2(\Omega))}\\
&+\frac{C\Delta t_n^2}{Pr}\|\mathbf{u}_{tt}\|^2_{L^2(t^n,t^{n+1};L^2(\Omega))}\\
&+\frac{Ch^{2s}\omega_{N^*}}{Pr^{\frac{3}{2}}}\left[PrRa^2\big(\|\gamma^{n+1}\|^2_{2,-1}+\epsilon_T^1\big)+\epsilon_{\mathbf{u}}^1\right]\\
&\times\left[\left(\sum_{n=1}^{N-1}\Delta t_n\|\nabla\mathbf{u}^{n+1}\|^4\right)^{\frac{1}{2}}+\left(\sum_{n=1}^{N-1}\Delta t_n\|\nabla T^{n+1}\|^4\right)^{\frac{1}{2}}\right]\\
&+\frac{Ch^{2s}\Delta t_n}{Pr}\|p^{n+1}\|^2+Ch^{2s}\Delta t_n PrRa\|\nabla T^{n+1}\|^2\\
&\left.\left.+Ch^{2s}\Delta t_n\|\nabla T^{n+1}\|^2+C\Delta t_n^2\|T_{tt}\|^2_{L^2(t^n,t^{n+1};L^2(\Omega))}\right)\right\}.
\end{aligned}\tag{4.34}$$

Finally, by the triangle inequality, we have

$$e_{\mathbf{u}}^n\le 2\big(\phi_{\mathbf{u}}^n+\eta_{\mathbf{u}}^n\big),\quad e_T^n\le 2\big(\phi_T^n+\eta_T^n\big).$$

Applying inequality (4.34), interpolation inequalities, and absorbing constants, the result follows. The proof is complete. □

Theorem 4.2 (Error Analysis for the Pressure). *Let* $\mathbf{u}, p, T$ *satisfy Theorem* 3.2. *Let the regularity of the solution given in Assumption* 2.1 *be satisfied. Then there exists a constant* $C > 0$ *such that*

$$\beta\Delta t_n \sum_{n=1}^{N-1} \|e_p^{n+1}\| \le C(\Delta t + h^s + h^{s+1}). \tag{4.35}$$

Proof. Using the Eq. (4.13), we have for any $\mathbf{v}_h \in \mathbf{V}_h$ that

$$\begin{aligned}
\Big(\frac{\phi_{\mathbf{u}}^{n+1} - \phi_{\mathbf{u}}^{n}}{\Delta t_n}, \mathbf{v}_h\Big) &= \sum_{i=1}^{7} C_i \\
= \Big(\frac{\eta_{\mathbf{u}}^{n+1} - \eta_{\mathbf{u}}^{n}}{\Delta t_n}, \mathbf{v}_h\Big) &+ Pr\big(\nabla e_{\mathbf{u}}^{n+1}, \nabla \mathbf{v}_h\big) \\
&- \big(p^{n+1} - \lambda_h^{n+1}, \nabla\cdot\mathbf{v}_h\big) + b^*\big(\mathbf{u}^{n+1}, \mathbf{u}^{n+1}, \mathbf{v}_h\big) \\
&- b^*\big(E^{n+1}(\mathbf{u}_h), E^{n+1}(\mathbf{u}_h), \mathbf{v}_h\big) \\
&- \tau_{\mathbf{u}}\big(\mathbf{u}^{n+1}; \mathbf{v}_h\big) - PrRa\big(\mathbf{j}e_T^{n+1}, \mathbf{v}_h\big),
\end{aligned} \tag{4.36}$$

where $\lambda_h^{n+1} \in Q_h$ is an approximation to $p(t^{n+1})$. By the Lemma 4.1, Cauchy-Schwarz-Young and Poincare-Friedrichs inequalities, we bound the seven individual terms on the right hands as follows:

$$C_1 \le C\Delta t_n^{-\frac{1}{2}} \|(\eta_{\mathbf{u}})_t\|_{L^2(t^{n-1},t^{n+1};L^2(\Omega))} \|\nabla \mathbf{v}_h\|, \quad C_2 \le Pr\big\|\nabla e_{\mathbf{u}}^{n+1}\big\| \|\nabla \mathbf{v}_h\|, \tag{4.37}$$

$$C_3 \le \big\|p^{n+1} - \lambda_h^{n+1}\big\| \|\nabla \mathbf{v}_h\|, \qquad C_7 \le PrRa\big\|\nabla e_T^{n+1}\big\| \|\nabla \mathbf{v}_h\|, \tag{4.38}$$

$$C_6 \le C \left\| \frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t_n} - \mathbf{u}_t^{n+1} \right\| \|\nabla \mathbf{v}_h\| \le C\Delta t_n \|\mathbf{u}_{tt}\|_{L^2(t^{n-1},t^{n+1};L^2(\Omega))} \|\nabla \mathbf{v}_h\|, \tag{4.39}$$

$$\begin{aligned}
C_4 + C_5 &= b^*\big(\mathbf{u}^{n+1} - E^{n+1}(\mathbf{u}), \mathbf{u}^{n+1}, \mathbf{v}_h\big) + b^*\big(E^{n+1}(\mathbf{u}), \mathbf{u}^{n+1} - E^{n+1}(\mathbf{u}), \mathbf{v}_h\big) \\
&\quad + b^*\big(E^{n+1}(e_{\mathbf{u}}), E^{n+1}(\mathbf{u}), \mathbf{v}_h\big) + b^*\big(E^{n+1}(\mathbf{u}_h), E^{n+1}(e_{\mathbf{u}}), \mathbf{v}_h\big) \\
&\le C(\Delta t_{n-1} + \Delta t_n)^{\frac{3}{2}} \|\nabla \mathbf{u}_{tt}\|_{L^2(t^{n-1},t^{n+1};L^2(\Omega))} \|\nabla \mathbf{u}^{n+1}\| \|\nabla \mathbf{v}_h\| \\
&\quad + C\big\|\nabla E^{n+1}(e_{\mathbf{u}})\big\| \|\nabla E^{n+1}(\mathbf{u})\| \|\nabla \mathbf{v}_h\| \\
&\quad + C\big\|\nabla E^{n+1}(\mathbf{u}_h)\big\| \big\|\nabla E^{n+1}(e_{\mathbf{u}})\big\| \|\nabla \mathbf{v}_h\|.
\end{aligned} \tag{4.40}$$

Consider (4.36) and Lemma 2.3, using $C_1 - C_7$, dividing both sides by $\|\nabla \mathbf{v}_h\|$ and taking a supremum over $\mathbf{V}_h$, gives

$$\begin{aligned}
\left\| \frac{\phi_{\mathbf{u}}^{n+1} - \phi_{\mathbf{u}}^{n}}{\Delta t_n} \right\|_{X_h^*} \le C\Big[&\Delta t_n^{-\frac{1}{2}} \|(\eta_{\mathbf{u}})_t\|_{L^2(t^{n-1},t^{n+1};L^2(\Omega))} + Pr\big\|\nabla e_{\mathbf{u}}^{n+1}\big\| + \big\|p^{n+1} - \lambda_h^{n+1}\big\| \\
&+ C(\Delta t_{n-1} + \Delta t_n)^{\frac{3}{2}} \|\nabla \mathbf{u}_{tt}\|_{L^2(t^{n-1},t^{n+1};L^2(\Omega))} \|\nabla \mathbf{u}^{n+1}\| \\
&+ C\big\|\nabla E^{n+1}(e_{\mathbf{u}})\big\| \|\nabla E^{n+1}(\mathbf{u})\| \\
&+ C\big\|\nabla E^{n+1}(\mathbf{u}_h)\big\| \big\|\nabla E^{n+1}(e_{\mathbf{u}})\big\| \\
&+ \Delta t_n \|\mathbf{u}_{tt}\|_{L^2(t^{n-1},t^{n+1};L^2(\Omega))} + PrRa\big\|\nabla e_T^{n+1}\big\|\Big].
\end{aligned} \tag{4.41}$$

Separating the pressure error term for $\forall \mathbf{v}_h \in \mathbf{X}_h$ and rearranging implies

$$\big(p_h^{n+1} - \lambda_h^{n+1}, \nabla\cdot\mathbf{v}_h\big) = \Big(\frac{\phi_{\mathbf{u}}^{n+1} - \phi_{\mathbf{u}}^{n}}{\Delta t_n}, \mathbf{v}_h\Big) - \Big(\frac{\eta_{\mathbf{u}}^{n+1} - \eta_{\mathbf{u}}^{n}}{\Delta t_n}, \mathbf{v}_h\Big)$$

$$
\begin{aligned}
&- Pr\big(\nabla e_{\mathbf{u}}^{n+1}, \nabla \mathbf{v}_h\big) + PrRa\big(\mathbf{j}e_T^{n+1}, \mathbf{v}_h\big) \\
&+ \big(p^{n+1} - \lambda_h^{n+1}, \nabla \cdot \mathbf{v}_h\big) - b^*\big(\mathbf{u}^{n+1}, \mathbf{u}^{n+1}, \mathbf{v}_h\big) \\
&+ b^*\big(E^{n+1}(\mathbf{u}_h), E^{n+1}(\mathbf{u}_h), \mathbf{v}_h\big) + \tau_{\mathbf{u}}\big(\mathbf{u}^{n+1}; \mathbf{v}_h\big). \qquad (4.42)
\end{aligned}
$$

Consider the estimates in $C_1 - C_7$ and (4.41). Divide by $\|\nabla \mathbf{v}_h\|$, and take supremum over $\mathbf{v}_h \in \mathbf{X}_h$ and use discrete inf-sup condition to obtain

$$
\begin{aligned}
\beta\big\|p_h^{n+1} - \lambda_h^{n+1}\big\| \le C\Big[&\Delta t_n{}^{-\frac{1}{2}}\|(\eta_{\mathbf{u}})_t\|_{L^2(t^{n-1},t^{n+1};L^2(\Omega))} \\
&+ Pr\big\|\nabla e_{\mathbf{u}}^{n+1}\big\| + \big\|p^{n+1} - \lambda_h^{n+1}\big\| + PrRa\big\|\nabla e_T^{n+1}\big\| \\
&+ (\Delta t_{n-1} + \Delta t_n)^{\frac{3}{2}}\|\nabla \mathbf{u}_{tt}\|_{L^2(t^{n-1},t^{n+1};L^2(\Omega))}\|\nabla \mathbf{u}^{n+1}\|\|\nabla \mathbf{v}_h\| \\
&+ \big\|\nabla E^{n+1}(e_{\mathbf{u}})\big\|\|\nabla E^{n+1}(\mathbf{u})\|\|\nabla \mathbf{v}_h\| \\
&+ \big\|\nabla E^{n+1}(\mathbf{u}_h)\big\|\big\|\nabla E^{n+1}(e_{\mathbf{u}})\big\|\|\nabla \mathbf{v}_h\| \\
&+ \Delta t_n\|\mathbf{u}_{tt}\|_{L^2(t^{n-1},t^{n+1};L^2(\Omega))} + \Delta t_n\|\mathbf{u}_{tt}\|_{L^2(t^{n-1},t^{n+1};L^2(\Omega))}\Big]. \qquad (4.43)
\end{aligned}
$$

Define $\Delta t = \max_{n=1,\ldots,N-1}\Delta t_n$, and multiplying both sides of (4.43) by Δt_n, summing n from 1 to $N-1$, using Theorem 3.1 and triangle inequality, we obtain that

$$
\begin{aligned}
\beta\Delta t_n \sum_{n=1}^{N-1}\big\|e_p^{n+1}\big\| \le C\Big[&\Delta t^{\frac{1}{2}}\|(\eta_{\mathbf{u}})_t\|_{L^2(0,T_1;L^2(\Omega))} + Pr\sqrt{T_1}\big|\big\|\nabla e_{\mathbf{u}}^{n+1}\big\|\big|_{2,0} \\
&+ \sqrt{T_1}\big|\big\|p^{n+1} - \lambda_h^{n+1}\big\|\big|_{2,0} + \Delta t^{\frac{5}{2}}\|\nabla \mathbf{u}_{tt}\|_{L^2(0,T_1;L^2(\Omega))} \\
&+ \big|\big\|\nabla E^{n+1}(e_{\mathbf{u}})\big\|\big|_{2,0} + \Delta t^2\|\mathbf{u}_{tt}\|_{L^2(0,T_1;L^2(\Omega))} \\
&+ PrRa\sqrt{T_1}\big|\big\|\nabla e_T^{n+1}\big\|\big|_{2,0}\Big]. \qquad (4.44)
\end{aligned}
$$

Finally, combining Theorem 4.1 and interpolation inequalities, we complete the proof. □

5. Adaptive Algorithms

In this section, we combine BE-AB2 algorithm and BE-AB2+F algorithm with general adaptive method to construct adaptive BE-AB2 algorithm and adaptive BE-AB2+F algorithm, respectively. Given $\Delta t_n = t^{n+1} - t^n$, $\omega_n = \Delta t_n/\Delta t_{n-1}$, $TOL_{\mathbf{u}}$ and TOL_T are tolerable errors for $\mathbf{u}_h$ and T_h, respectively. The adaptive algorithm we show is as follows.

Algorithm 5.1: Adaptive VSS BE-AB2.

Step 1. Given $(\mathbf{u}_h^{n-1}, p_h^{n-1}, T_h^{n-1}), (\mathbf{u}_h^n, p_h^n, T_h^n)$.
Find $(\mathbf{u}_h^{n+1}, p_h^{n+1}, T_h^{n+1})$ satisfying for all $(\mathbf{v}_h, q_h, s_h) \in (\mathbf{X}_h, Q_h, W_h)$,

$$
\begin{aligned}
&\left(\frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t_n}, \mathbf{v}_h\right) + b^*\big(E^{n+1}(\mathbf{u}_h), E^{n+1}(\mathbf{u}_h), \mathbf{v}_h\big) \\
&\quad - \big(p_h^{n+1}, \nabla \cdot \mathbf{v}_h\big) + Pr\big(\nabla \mathbf{u}_h^{n+1}, \nabla \mathbf{v}_h\big) = PrRa\big(\mathbf{j}T_h^{n+1}, \mathbf{v}_h\big), \\
&\big(\nabla \cdot \mathbf{u}_h^{n+1}, q_h\big) = 0, \qquad (5.1) \\
&\left(\frac{T_h^{n+1} - T_h^n}{\Delta t_n}, s_h\right) + \overline{b^*}\big(E^{n+1}(\mathbf{u}_h), E^{n+1}(T_h), s_h\big) + \big(\nabla T_h^{n+1}, \nabla s_h\big) = (\gamma, s_h).
\end{aligned}
$$

Step 2. Compute the error estimators of $\mathbf{u}_h^{n+1}, T_h^{n+1}$

$$EST_{\mathbf{u},1} = \frac{\omega_n}{1+2\omega_n}\big(\mathbf{u}_h^{n+1} - (1+\omega_n)\mathbf{u}_h^n + \omega_n \mathbf{u}_h^{n-1}\big), \tag{5.2}$$

$$EST_{T,1} = \frac{\omega_n}{1+2\omega_n}\big(T_h^{n+1} - (1+\omega_n)T_h^n + \omega_n T_h^{n-1}\big). \tag{5.3}$$

Step 3. If $\|EST_{\mathbf{u},1}\| > TOL_{\mathbf{u}}$ or $\|EST_{T,1}\| > TOL_T$, this step is rejected and repeated by resetting Δt_n

$$\Delta t_n = \max\bigg\{\min\bigg\{0.7\Delta t_n\Big(\frac{TOL_{\mathbf{u}}}{\|EST_{\mathbf{u},1}\|}\Big)^{\frac{1}{2}}, 0.7\Delta t_n\Big(\frac{TOL_T}{\|EST_{T,1}\|}\Big)^{\frac{1}{2}}\bigg\}, 0.5\Delta t_n\bigg\}, \tag{5.4}$$

otherwise, set the time step Δt_{n+1} as follows, and proceed to the next step

$$\Delta t_{n+1} = \min\bigg\{\min\bigg\{\Delta t_n\Big(\frac{TOL_{\mathbf{u}}}{\|EST_{\mathbf{u},1}\|}\Big)^{\frac{1}{2}}, \Delta t_n\Big(\frac{TOL_T}{\|EST_{T,1}\|}\Big)^{\frac{1}{2}}\bigg\}, 1.5\Delta t_n\bigg\}. \tag{5.5}$$

Algorithm 5.2: Adaptive BE-AB2+F.

Step 1. Given $(\mathbf{u}_h^{n-2}, p_h^{n-2}, T_h^{n-2}), (\mathbf{u}_h^{n-1}, p_h^{n-1}, T_h^{n-1}), (\mathbf{u}_h^{n}, p_h^{n}, T_h^{n})$.
Find $(\mathbf{u}_h^{n+1}, p_h^{n+1}, T_h^{n+1})$ and $\widehat{\mathbf{u}}_h^{n+1}$, $\widehat{T}_h^{n+1}$ satisfying for all $(\mathbf{v}_h, q_h, s_h) \in (\mathbf{X}_h, Q_h, W_h)$,

$$\begin{aligned}
&\Big(\frac{\widehat{\mathbf{u}}_h^{n+1} - \mathbf{u}_h^n}{\Delta t_n}, \mathbf{v}_h\Big) + b^*\big(E^{n+1}(\mathbf{u}_h), E^{n+1}(\mathbf{u}_h), \mathbf{v}_h\big) \\
&\quad - \big(p_h^{n+1}, \nabla \mathbf{v}_h\big) + Pr\big(\nabla \widehat{\mathbf{u}}_h^{n+1}, \nabla \mathbf{v}_h\big) = PrRa\big(\mathbf{j}\widehat{T}_h^{n+1}, \mathbf{v}_h\big), \\
&\big(\nabla \cdot \widehat{\mathbf{u}}_h^{n+1}, q_h\big) = 0, \\
&\Big(\frac{\widehat{T}_h^{n+1} - T_h^n}{\Delta t_n}, s_h\Big) + \overline{b^*}\big(E^{n+1}(\mathbf{u}_h), E^{n+1}(T_h), s_h\big) + \big(\nabla \widehat{T}_h^{n+1}, \nabla s_h\big) = (\gamma, s_h),
\end{aligned} \tag{5.6}$$

$$\mathbf{u}_h^{n+1} = \widehat{\mathbf{u}}_h^{n+1} - \frac{\omega^n}{1+2\omega^n}\big(\widehat{\mathbf{u}}_h^{n+1} - (1+\omega^n)\mathbf{u}_h^n + \omega^n \mathbf{u}_h^{n-1}\big), \tag{5.7}$$

$$T_h^{n+1} = \widehat{T}_h^{n+1} - \frac{\omega^n}{1+2\omega^n}\big(\widehat{T}_h^{n+1} - (1+\omega^n)T_h^n + \omega^n T_h^{n-1}\big). \tag{5.8}$$

Step 2. Compute the error estimators of $\mathbf{u}_h^{n+1}, T_h^{n+1}$

$$\begin{aligned}
EST_{\mathbf{u},2} = &\frac{\omega_{n-1}\omega_n(1+\omega_n)}{1+2\omega_n + \omega_{n-1}\big(1+4\omega_n+3\omega_n^2\big)} \\
&\times \bigg(\mathbf{u}_h^{n+1} - \frac{(1+\omega_n)\big(1+\omega_{n-1}(1+\omega_n)\big)}{1+\omega_{n-1}}\mathbf{u}_h^n \\
&\qquad + \omega_n\big(1+\omega_{n-1}(1+\omega_n)\big)\mathbf{u}_h^{n-1} - \frac{\omega_{n-1}^2\omega_n(1+\omega_n)}{1+\omega_{n-1}}\mathbf{u}_h^{n-2}\bigg),
\end{aligned} \tag{5.9}$$

$$EST_{T,2} = \frac{\omega_{n-1}\omega_n(1+\omega_n)}{1+2\omega_n+\omega_{n-1}\big(1+4\omega_n+3\omega_n^2\big)} \times \Bigg(T_h^{n+1} - \frac{(1+\omega_n)\big(1+\omega_{n-1}(1+\omega_n)\big)}{1+\omega_{n-1}} T_h^n + \omega_n\big(1+\omega_{n-1}(1+\omega_n)\big)T_h^{n-1} - \frac{\omega_{n-1}^2\omega_n(1+\omega_n)}{1+\omega_{n-1}} T_h^{n-2}\Bigg). \qquad (5.10)$$

Step 3. If $\|EST_{\mathbf{u},2}\| > TOL_{\mathbf{u}}$ or $\|EST_{T,2}\| > TOL_T$, this step is rejected and repeated by resetting Δt_n

$$\Delta t_n = \max\left\{ \min\left\{0.7\Delta t_n \left(\frac{TOL_{\mathbf{u}}}{\|EST_{\mathbf{u},2}\|}\right)^{\frac{1}{3}}, 0.7\Delta t_n \left(\frac{TOL_T}{\|EST_{T,2}\|}\right)^{\frac{1}{3}}\right\}, 0.5\Delta t_n\right\}, \qquad (5.11)$$

otherwise, set the time step Δt_{n+1} as follows, and proceed to the next step

$$\Delta t_{n+1} = \min\left\{ \min\left\{\Delta t_n \left(\frac{TOL_{\mathbf{u}}}{\|EST_{\mathbf{u},2}\|}\right)^{\frac{1}{3}}, \Delta t_n \left(\frac{TOL_T}{\|EST_{T,2}\|}\right)^{\frac{1}{3}}\right\}, 1.5\Delta t_n\right\}. \qquad (5.12)$$

6. Numerical Experiments

In this section, we present some numerical experiments to check the numerical theory developed in the previous sections and illustrate the efficiency of the algorithms proposed in Sections 3 and 5. Throughout this section, the computations are performed using the Freefem++ [24]. First, we consider a flow problem with a manufactured analytical solution. The second one is the accuracy of the fully adaptive.

6.1. Examples with analytical solution

In this subsection, we consider a flow problem with a smooth analytic solution in $\Omega = [0,1]\times[0,1]$ and we take the exact solution

$$\begin{aligned}
&p(x_1,x_2,t) = 10(2x_1-1)(2x_2-1)\cos(t),\\
&u_1(x_1,x_2,t) = 10x_1^2(x_1-1)^2x_2(x_2-1)(2x_2-1)\cos(t),\\
&u_2(x_1,x_2,t) = -10x_1(x_1-1)(2x_1-1)x_2^2(x_2-1)^2\cos(t),\\
&T(x_1,x_2,t) = u_1(x_1,x_2,t) + u_2(x_1,x_2,t).
\end{aligned}$$

Two source terms $f_{\mathbf{u}}$ and Γ are determined by original problem (1.1).

We choose the parameters $Pr = 1.0, Ra = 1.0$, and $T_1 = 1.0$. We now recall the naming conventions for the various methods that we test. We use a nonstandard BE-AB2 combination where the constant extrapolation $y_h^{n+1} = y_h^n + \mathcal{O}(\Delta t)$ in the nonlinearity is replaced with a linear extrapolation, where $y = \mathbf{u}, T$. For constant stepsize, this means $y_h^{n+1} = 2y_h^n - y_h^{n-1} + \mathcal{O}(\Delta t^2)$. BE-AB2+F is BE-AB2 post-processed by the time filter.

For all adaptive methods, we imposed a stepsize ratio limiter, which is a common heuristic. The stepsizes are limited to a maximum increase of 1.5 times each timestep, and cannot be less than half of the previous attempted timestep. However, the algorithms may reject several solutions in a row, effectively allowing the timestep to shrink as small as necessary.

We carry out the accuracy tests calculated by the formula $\log(E_i/E_{i+1})/\log(\Delta t_i/\Delta t_{i+1})$ with respect to the timestep, where E_i and E_{i+1} are the relative errors corresponding to the timesteps Δt_i and Δt_{i+1}, respectively. Computations are made on a fixed small enough mesh size with different timesteps so that the spatial discretization error can be negligible compared with the time error. For BE-AB2 and BE-AB2+F, we use the Taylor-Hood elements (P2-P1-P2) for $\mathbf{u}, p, T$, and $h = 1/200$. The results are given in Tables 6.1-6.2. Moreover, Tables 6.3-6.4 show the space convergence rate of L^2 norm and H^1 semi-norm when fixed $\Delta t = 1/1000$. For adaptive BE-AB2 and adaptive BE-AB2+F, we choose (P3-P2-P3) for $\mathbf{u}, p, T$, and $h = 1/160$. The results are given in Tables 6.5-6.6.

Table 6.1: The convergence performance for the BE-AB2 method at time $t_N = 1$ with $h = 1/200$.

Δt	1/15	1/30	1/45	1/60
$\frac{\Vert\nabla(\mathbf{u}(t_N) - \mathbf{u}_h^N)\Vert}{\Vert\nabla\mathbf{u}(t_N)\Vert}$	6.7404E-04	3.3802E-04	2.3087E-04	1.7950E-04
Rate	$*$	0.996	0.940	0.875
$\frac{\Vert\mathbf{u}(t_N) - \mathbf{u}_h^N\Vert}{\Vert\mathbf{u}(t_N)\Vert}$	6.7975E-04	3.3467E-04	2.2195E-04	1.6602E-04
Rate	$*$	1.022	1.013	1.001
$\frac{\Vert p(t_N) - p_h^N\Vert}{\Vert p(t_N)\Vert}$	2.6731E-05	1.3078E-05	8.6625E-06	6.4773E-06
Rate	$*$	1.031	1.016	1.010
$\frac{\Vert\nabla(T(t_N) - T_h^N)\Vert}{\Vert\nabla T(t_N)\Vert}$	6.7945E-04	3.3659E-04	2.2563E-04	1.7129E-04
Rate	$*$	1.013	0.986	0.960
$\frac{\Vert T(t_N) - T_h^N\Vert}{\Vert T(t_N)\Vert}$	7.0625E-04	3.4759E-04	2.3050E-04	1.7241E-04
Rate	$*$	1.023	1.013	1.009

Table 6.2: The convergence performance for the BE-AB2+F method at time $t_N = 1$ with $h = 1/200$.

Δt	1/15	1/30	1/45	1/60
$\frac{\Vert\nabla(\mathbf{u}(t_N) - \mathbf{u}_h^N)\Vert}{\Vert\nabla\mathbf{u}(t_N)\Vert}$	2.5570E-03	6.1500E-04	2.7717E-04	1.6636E-04
Rate	$*$	2.056	1.966	1.775
$\frac{\Vert\mathbf{u}(t_N) - \mathbf{u}_h^N\Vert}{\Vert\mathbf{u}(t_N)\Vert}$	2.5583E-03	6.1115E-04	2.6743E-04	1.4925E-04
Rate	$*$	2.066	2.038	2.027
$\frac{\Vert p(t_N) - p_h^N\Vert}{\Vert p(t_N)\Vert}$	4.4156E-03	1.0525E-03	4.6004E-04	2.5658E-04
Rate	$*$	2.069	2.041	2.030
$\frac{\Vert\nabla(T(t_N) - T_h^N)\Vert}{\Vert\nabla T(t_N)\Vert}$	2.5551E-03	6.1186E-04	2.7066E-04	1.5542E-04
Rate	$*$	2.062	2.012	1.928
$\frac{\Vert T(t_N) - T_h^N\Vert}{\Vert T(t_N)\Vert}$	2.5614E-03	6.1192E-04	2.6778E-04	1.4945E-04
Rate	$*$	2.066	2.038	2.027

Table 6.3: The convergence performance for the BE-AB2 method at time $t_N = 1$ with $\Delta t = 1/1000$.

h	1/4	1/8	1/16	1/32
$\frac{\\|\nabla(\mathbf{u}(t_N) - \mathbf{u}_h^N)\\|}{\\|\nabla \mathbf{u}(t_N)\\|}$	1.6360E-01	4.4349E-02	1.1399E-02	2.8735E-03
Rate	*	1.883	1.960	1.988
$\frac{\\|\mathbf{u}(t_N) - \mathbf{u}_h^N\\|}{\\|\mathbf{u}(t_N)\\|}$	2.5130E-02	1.9971E-03	1.3956E-04	1.6915E-05
Rate	*	3.653	3.839	3.045
$\frac{\\|p(t_N) - p_h^N\\|}{\\|p(t_N)\\|}$	6.8679E-03	7.1093E-04	6.3539E-05	5.3708E-06
Rate	*	3.272	3.484	3.564
$\frac{\\|\nabla(T(t_N) - T_h^N)\\|}{\\|\nabla T(t_N)\\|}$	9.7573E-02	2.6678E-02	6.8302E-03	1.7186E-03
Rate	*	1.871	1.966	1.991
$\frac{\\|T(t_N) - T_h^N\\|}{\\|T(t_N)\\|}$	7.0803E-03	6.8818E-04	5.0267E-05	1.0600E-05
Rate	*	3.363	3.775	2.246

Table 6.4: The convergence performance for the BE-AB2+F method at time $t_N = 1$ with $\Delta t = 1/1000$.

h	1/4	1/8	1/16	1/32
$\frac{\\|\nabla(\mathbf{u}(t_N) - \mathbf{u}_h^N)\\|}{\\|\nabla \mathbf{u}(t_N)\\|}$	1.6360E-01	4.4349E-02	1.1399E-02	2.8735E-03
Rate	*	1.883	1.960	1.988
$\frac{\\|\mathbf{u}(t_N) - \mathbf{u}_h^N\\|}{\\|\mathbf{u}(t_N)\\|}$	2.5123E-02	1.9897E-03	1.3202E-04	8.0489E-06
Rate	*	3.658	3.914	4.036
$\frac{\\|p(t_N) - p_h^N\\|}{\\|p(t_N)\\|}$	6.8683E-03	7.1116E-04	6.3710E-05	5.5443E-06
Rate	*	3.272	3.481	3.522
$\frac{\\|\nabla(T(t_N) - T_h^N)\\|}{\\|\nabla T(t_N)\\|}$	9.7573E-02	2.6678E-02	6.8302E-03	1.7186E-03
Rate	*	1.871	1.966	1.991
$\frac{\\|T(t_N) - T_h^N\\|}{\\|T(t_N)\\|}$	1.1928E-02	3.3131E-03	8.3318E-04	2.0700E-04
Rate	*	3.362	3.791	3.909

Firstly, we test the time error and space error of BE-AB2. We also compare this method with the standard BE method [51]. Table 6.1 displays the time convergence order for L^2 norm and H^1 semi-norm, respectively. Moreover, Table 6.3 displays the space convergence order for L^2 norm and H^1 semi-norm, respectively. Obviously, the method has good performance in the convergence aspect and keeps the convergence rates just like the theoretical results. Interestingly, the time error of the velocity produced by BE and BE-AB2 is almost the same, but BE-AB2 improves the calculation accuracy of pressure.

Secondly, we examine the time error and space error of BE-AB2+F. Table 6.2 presents the time convergence orders for L^2 norm and H^1 semi-norm, respectively. Additionally, Table 6.4

Table 6.5: The convergence performance of the global error for the adaptive BE-AB2 method with $h = 1/160$.

$\widetilde{\Delta t}$	1/18	1/23	1/45	1/122
$\left(\sum_{i=2}^{N}\Delta t_i \frac{\|\nabla(\mathbf{u}(t_i)-\mathbf{u}_h^i)\|^2}{\|\nabla\mathbf{u}(t_i)\|^2}\right)^{\frac{1}{2}}$	1.4695E-03	6.9991E-04	2.4684E-04	8.0950E-05
Rate	$*$	3.025	1.553	1.117
$\left(\sum_{i=2}^{N}\Delta t_i \frac{\|\mathbf{u}(t_i)-\mathbf{u}_h^i\|^2}{\|\mathbf{u}(t_i)\|^2}\right)^{\frac{1}{2}}$	1.4759E-03	7.0950E-04	2.5041E-04	8.2123E-05
Rate	$*$	2.988	1.55	1.117
$\left(\sum_{i=2}^{N}\Delta t_i \frac{\|p(t_i)-p_h^i\|^2}{\|p(t_i)\|^2}\right)^{\frac{1}{2}}$	1.5227E-04	2.5846E-05	5.8006E-06	1.9883E-06
Rate	$*$	7.233	2.226	1.073
$\left(\sum_{i=2}^{N}\Delta t_i \frac{\|\nabla(T(t_i)-T_h^i)\|^2}{\|\nabla T(t_i)\|^2}\right)^{\frac{1}{2}}$	1.9096E-03	7.6325E-04	2.5715E-04	8.2700E-05
Rate	$*$	3.741	1.621	1.1374
$\left(\sum_{i=2}^{N}\Delta t_i \frac{\|T(t_i)-T_h^i\|^2}{\|T(t_i)\|^2}\right)^{\frac{1}{2}}$	1.9880E-03	7.9487E-04	2.6781E-04	8.6127E-05
Rate	$*$	3.740	1.620	1.1374

Table 6.6: The convergence performance of the global error for the adaptive BE-AB2+F method with $h = 1/160$.

$\widetilde{\Delta t}$	1/27	1/41	1/48	1/104
$\left(\sum_{i=2}^{N}\Delta t_i \frac{\|\nabla(\mathbf{u}(t_i)-\mathbf{u}_h^i)\|^2}{\|\nabla\mathbf{u}(t_i)\|^2}\right)^{\frac{1}{2}}$	2.9895E-02	6.8292E-03	4.9093E-03	1.1238E-03
Rate	$*$	3.427	2.046	1.879
$\left(\sum_{i=2}^{N}\Delta t_i \frac{\|\mathbf{u}(t_i)-\mathbf{u}_h^i\|^2}{\|\mathbf{u}(t_i)\|^2}\right)^{\frac{1}{2}}$	2.9826E-02	6.7906E-03	4.8859E-03	1.1103E-03
Rate	$*$	3.435	2.0413	1.889
$\left(\sum_{i=2}^{N}\Delta t_i \frac{\|p(t_i)-p_h^i\|^2}{\|p(t_i)\|^2}\right)^{\frac{1}{2}}$	3.4577E-02	1.0171E-02	7.4672E-03	2.8259E-03
Rate	$*$	2.841	1.916	1.238
$\left(\sum_{i=2}^{N}\Delta t_i \frac{\|\nabla(T(t_i)-T_h^i)\|^2}{\|\nabla T(t_i)\|^2}\right)^{\frac{1}{2}}$	2.8598E-02	6.5676E-03	4.4168E-03	1.1021E-03
Rate	$*$	3.415	2.460	1.769
$\left(\sum_{i=2}^{N}\Delta t_i \frac{\|T(t_i)-T_h^i\|^2}{\|T(t_i)\|^2}\right)^{\frac{1}{2}}$	2.8334E-02	6.4514E-03	4.3292E-03	1.0670E-03
Rate	$*$	3.435	2.474	1.785

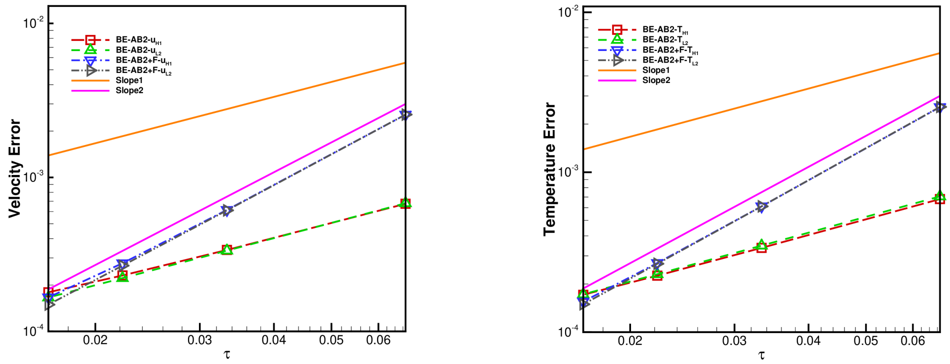

Fig. 6.1. Relative errors of the velocity $\mathbf{u}$ and temperature T for BEAB2 and BEAB2+F with (P2-P1-P2) elements on log scale.

exhibits the spatial convergence sequences of the L^2 norm and H^1 semi-norm, respectively. Evidently, the convergence rate of this method is highly consistent with the theoretical results. In addition, we give a log-log plot of the relative errors of $\mathbf{u}$ and T in Fig. 6.1 showing that the application of TF results in a higher order.

Thirdly, we test convergence for the adaptive methods in the following. We let the first $\Delta t = 0.001, T_1 = 1$, four tolerances $\epsilon = \{5\text{e-}5, 1\text{e-}5, 5\text{e-}6, 1\text{e-}6\}$, and the average time stepsize $\widetilde{\Delta t} = 1/(\text{the number of accepted steps})$ since the time stepsize is variable. It can be seen that the convergence order of adaptive BE-AB2+F is also higher than adaptive BE-AB2 from Tables 6.5-6.6.

6.2. Accuracy of the fully adaptive

In this example, we consider an exact solution problem to illustrate the efficiency of the adaptive algorithm. The exact solution is defined as follows [12]:

$$u_1(x,y,t) = -\cos(x)\sin(y)F(t), \qquad u_2(x,y,t) = \sin(x)\cos(y)F(t), \tag{6.1}$$

$$p(x,y,t) = -\frac{1}{4}\big(\cos(x)+\cos(y)\big)F^2(t), \quad T(x,y,t) = u_1(x,y,t) + u_2(x,y,t), \tag{6.2}$$

where $F(t)$ is the translation and reflection function; when $t \leq 0, F(t) = 0$, and when $t > 0$, $F(t) = \exp(-1/(10t)^{10})$.

$F(t)$ is a differentiable function. Therefore, we can construct a differentiable (up to machine precision) function by shifting and reflecting this function. This creates flat parts as well as fast-changing parts, which need to be adaptive to solve effectively. The evolution of $\|\mathbf{u}\|$ with time is shown in Fig. 6.2. The rest interval of all test initializations is $k = 0.45$. Each side of this quadrilateral has 100 nodes, using (P2-P1-P2) elements, and the final time is 45. Fig. 6.2 compares two numerical solutions. One is from Algorithm 3.2 (second-order nonadaptive), and the other is from Algorithm 5.2. With $TOL = 10^{-3}$, the adaptive method takes 206 steps, which comprises 154 accepted steps and 52 rejected steps. The constant stepsize method, which took 110 steps does not accurately capture the energetic jumps. In Fig. 6.2, we plot the norms of $\mathbf{u}$ and T for the adaptive BE-AB2+F and nonadaptive BE-AB2+F methods of Stokes solves, which correspond to a tolerance of $TOL = 10^{-3}$. We can see that the adaptive BE-AB2+F basically captures the transitions, while the non-adaptive BE-AB2+F shows great fluctuations.

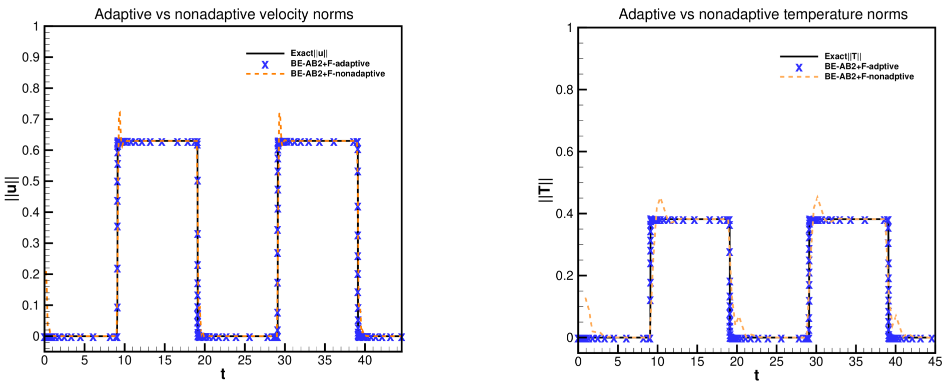

Fig. 6.2. The nonadaptive method exhibits overshooting, while the adaptive method resolves transitions.

7. Summary

Inspired by [6, 14, 18, 26, 27, 30] and the first-order and second-order algorithms of [13], we first developed the VSS BE-AB2 and BE-AB2+F algorithms for NC problems and analyzed the stability and convergence of the former and the stability of the latter. The BE-AB2+F algorithm is constructed by combining BE-AB2 and TF, which not only does not increase the computational and cognitive complexity, but also increases the order of BE-AB2 to 2. Then, the new adaptive VSS BE-AB2 algorithm and BE-AB2+F algorithm are constructed without increasing complexity. Finally, two numerical experiments are carried out to verify that our theoretical analysis results are in good agreement with other published results, which proves the effectiveness of the proposed algorithms. In addition, our future work will focus on the high order extension of variable stepsize algorithm and its theoretical analysis. Based on the methods [8, 25, 33–36, 44, 45], it seems possible to do so.

Appendix A. Proofs of Results in Section 3

A.1. Proof of Theorem 3.3

Proof. Setting

$$\mathbf{v}_h = \frac{3}{2}\mathbf{u}_h^{n+1} - \mathbf{u}_h^n + \frac{1}{2}\mathbf{u}_h^{n-1}, \quad s_h = \frac{3}{2}T_h^{n+1} - T_h^n + \frac{1}{2}T_h^{n-1},$$

multiplying by Δt, using Lemma 2.2, and applying Young's inequality to the right-hand side

$$\begin{aligned}
&\frac{1}{4}\left(\|\mathbf{u}_h^{n+1}\|^2 + \|2\mathbf{u}_h^{n+1} - \mathbf{u}_h^n\|^2 + \|\mathbf{u}_h^{n+1} - \mathbf{u}_h^n\|^2\right) \\
&\quad - \frac{1}{4}\left(\|\mathbf{u}_h^n\|^2 + \|2\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\|^2 + \|\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\|^2\right) \\
&\quad + \frac{3}{4}\|\mathbf{u}_h^{n+1} - 2\mathbf{u}_h^n + \mathbf{u}_h^{n-1}\|^2 + \Delta t Pr\left\|\nabla\left(\frac{3}{2}\mathbf{u}_h^{n+1} - \mathbf{u}_h^n + \frac{1}{2}\mathbf{u}_h^{n-1}\right)\right\|^2 \\
&\quad + \Delta t b^*\left(2\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, 2\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \frac{3}{2}\mathbf{u}_h^{n+1} - \mathbf{u}_h^n + \frac{1}{2}\mathbf{u}_h^{n-1}\right) \\
\le\ & \frac{\Delta t Pr}{4}\left\|\nabla\left(\frac{3}{2}\mathbf{u}_h^{n+1} - \mathbf{u}_h^n + \frac{1}{2}\mathbf{u}_h^{n-1}\right)\right\|^2
\end{aligned}$$

$$+ \Delta t Pr Ra^2 \left\| \frac{3}{2}T_h^{n+1} - T_h^n + \frac{1}{2}T_h^{n-1} \right\|_{-1}^2, \tag{A.1}$$

$$\begin{aligned}
&\frac{1}{4}\left(\|T_h^{n+1}\|^2 + \|2T_h^{n+1} - T_h^n\|^2 + \|T_h^{n+1} - T_h^n\|^2\right) \\
&\quad - \frac{1}{4}\left(\|T_h^n\|^2 + \|2T_h^n - T_h^{n-1}\|^2 + \|T_h^n - T_h^{n-1}\|^2\right) \\
&\quad + \frac{3}{4}\|T_h^{n+1} - 2T_h^n + T_h^{n-1}\|^2 + \Delta t \left\| \nabla\left(\frac{3}{2}T_h^{n+1} - T_h^n + \frac{1}{2}T_h^{n-1}\right) \right\|^2 \\
&\quad + \Delta t \overline{b^*}\left(2\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, 2T_h^n - T_h^{n-1}, \frac{3}{2}T_h^{n+1} - T_h^n + \frac{1}{2}T_h^{n-1}\right) \\
&\leq \frac{\Delta t}{4}\left\| \nabla\left(\frac{3}{2}T_h^{n+1} - T_h^n + \frac{1}{2}T_h^{n-1}\right) \right\|^2 + \Delta t \|\gamma^{n+1}\|_{-1}^2.
\end{aligned} \tag{A.2}$$

Next, dealing with the nonlinear term we use the skew symmetry of $b^*, \overline{b^*}$, Poincare inequality, inequality (2.5), (2.6), the inverse inequality and Young's inequality

$$\begin{aligned}
&\Delta t b^*\left(2\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, 2\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \frac{3}{2}\mathbf{u}_h^{n+1} - \mathbf{u}_h^n + \frac{1}{2}\mathbf{u}_h^{n-1}\right) \\
&= \frac{3}{2}\Delta t b^*\left(2\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \frac{3}{2}\mathbf{u}_h^{n+1} - \mathbf{u}_h^n + \frac{1}{2}\mathbf{u}_h^{n-1}, \mathbf{u}_h^{n+1} - 2\mathbf{u}_h^n + \mathbf{u}_h^{n-1}\right) \\
&\leq C\frac{\Delta t^2}{h}\|\nabla(2\mathbf{u}_h^n - \mathbf{u}_h^{n-1})\|^2 \left\| \nabla\left(\frac{3}{2}\mathbf{u}_h^{n+1} - \mathbf{u}_h^n + \frac{1}{2}\mathbf{u}_h^{n-1}\right) \right\|^2 \\
&\quad + \frac{3}{4}\|\mathbf{u}_h^{n+1} - 2\mathbf{u}_h^n + \mathbf{u}_h^{n-1}\|^2,
\end{aligned} \tag{A.3}$$

$$\begin{aligned}
&\Delta t \overline{b^*}\left(2\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, 2T_h^n - T_h^{n-1}, \frac{3}{2}T_h^{n+1} - T_h^n + \frac{1}{2}T_h^{n-1}\right) \\
&\leq C\frac{\Delta t^2}{h}\|\nabla(2\mathbf{u}_h^n - \mathbf{u}_h^{n-1})\|^2 \left\| \nabla\left(\frac{3}{2}T_h^{n+1} - T_h^n + \frac{1}{2}T_h^{n-1}\right) \right\|^2 \\
&\quad + \frac{3}{4}\|T_h^{n+1} - 2T_h^n + T_h^{n-1}\|^2.
\end{aligned} \tag{A.4}$$

Combining like terms we then have

$$\begin{aligned}
&\frac{1}{4}\left(\|\mathbf{u}_h^{n+1}\|^2 + \|2\mathbf{u}_h^{n+1} - \mathbf{u}_h^n\|^2 + \|\mathbf{u}_h^{n+1} - \mathbf{u}_h^n\|^2\right) \\
&\quad - \frac{1}{4}\left(\|\mathbf{u}_h^n\|^2 + \|2\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\|^2 + \|\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\|^2\right) \\
&\quad + \frac{\Delta t Pr}{4}\left\| \nabla\left(\frac{3}{2}\mathbf{u}_h^{n+1} - \mathbf{u}_h^n + \frac{1}{2}\mathbf{u}_h^{n-1}\right) \right\|^2 \\
&\quad + \frac{\Delta t Pr}{2}M_3\left\| \nabla\left(\frac{3}{2}\mathbf{u}_h^{n+1} - \mathbf{u}_h^n + \frac{1}{2}\mathbf{u}_h^{n-1}\right) \right\|^2 \\
&\leq \Delta t Pr Ra^2 \left\| \nabla\left(\frac{3}{2}\mathbf{u}_h^{n+1} - \mathbf{u}_h^n + \frac{1}{2}\mathbf{u}_h^{n-1}\right) \right\|_{-1}^2,
\end{aligned} \tag{A.5}$$

$$\begin{aligned}
&\frac{1}{4}\left(\|T_h^{n+1}\|^2 + \|2T_h^{n+1} - T_h^n\|^2 + \|T_h^{n+1} - T_h^n\|^2\right) \\
&\quad - \frac{1}{4}\left(\|T_h^n\|^2 + \|2T_h^n - T_h^{n-1}\|^2 + \|T_h^n - T_h^{n-1}\|^2\right)
\end{aligned}$$

$$+\frac{\Delta t}{4}\left\|\nabla\left(\frac{3}{2}T_h^{n+1}-T_h^n+\frac{1}{2}T_h^{n-1}\right)\right\|^2$$
$$+\frac{\Delta t}{2}M_4\left\|\nabla\left(\frac{3}{2}T_h^{n+1}-T_h^n+\frac{1}{2}T_h^{n-1}\right)\right\|^2\le\Delta t\|\gamma^{n+1}\|_{-1}^2. \tag{A.6}$$

Now, letting $C=C_{stab}$, using condition (3.21), and summing from $n=1$ to $N-1$ we can get the result

$$\begin{aligned}&\frac{1}{4}\|T_h^N\|^2+\frac{1}{4}\|2T_h^N-T_h^{N-1}\|^2+\frac{1}{4}\|T_h^N-T_h^{N-1}\|^2\\&\quad+\frac{\Delta t}{4}\sum_{n=1}^{N-1}\left\|\nabla\left(\frac{3}{2}T_h^{n+1}-T_h^n+\frac{1}{2}T_h^{n-1}\right)\right\|^2\\&\le|||\gamma^{n+1}|||_{2,-1}^2+\frac{1}{4}\|T_h^1\|^2+\frac{1}{4}\|2T_h^1-T_h^0\|^2+\frac{1}{4}\|T_h^1-T_h^0\|^2,\end{aligned} \tag{A.7}$$

$$\begin{aligned}&\frac{1}{4}\|\mathbf{u}_h^N\|^2+\frac{1}{4}\|2\mathbf{u}_h^N-\mathbf{u}_h^{N-1}\|^2+\frac{1}{4}\|\mathbf{u}_h^N-\mathbf{u}_h^{N-1}\|^2\\&\quad+\frac{\Delta tPr}{4}\sum_{n=1}^{N-1}\left\|\nabla\left(\frac{3}{2}\mathbf{u}_h^{n+1}-\mathbf{u}_h^n+\frac{1}{2}\mathbf{u}_h^{n-1}\right)\right\|^2\\&\le\sum_{n=1}^{N-1}\Delta tPrRa^2\left\|\frac{3}{2}T_h^{n+1}-T_h^n+\frac{1}{2}T_h^{n-1}\right\|_{-1}^2+\frac{1}{4}\|\mathbf{u}_h^1\|^2\\&\quad+\frac{1}{4}\|2\mathbf{u}_h^1-\mathbf{u}_h^0\|^2+\frac{1}{4}\|\mathbf{u}_h^1-\mathbf{u}_h^0\|^2\\&\le CPrRa^2\left(|||\gamma^{n+1}|||_{2,-1}^2+\frac{1}{4}\|T_h^1\|^2+\frac{1}{4}\|2T_h^1-T_h^0\|^2+\frac{1}{4}\|T_h^1-T_h^0\|^2\right)\\&\quad+\frac{1}{4}\|\mathbf{u}_h^1\|^2+\frac{1}{4}\|2\mathbf{u}_h^1-\mathbf{u}_h^0\|^2+\frac{1}{4}\|\mathbf{u}_h^1-\mathbf{u}_h^0\|^2.\end{aligned} \tag{A.8}$$

The proof is complete. □

A.2. Proof of Theorem 3.4

Proof. Let $(\mathbf{v}_h,q_h)\in(\mathbf{V}_h,Q_h)$ and isolate the discrete time derivative in (3.4). Then we can get

$$\begin{aligned}&\left(\frac{3\mathbf{u}_h^{n+1}/2-2\mathbf{u}_h^n+\mathbf{u}_h^{n-1}/2}{\Delta t},\mathbf{v}_h\right)\\&=-Pr\left(\nabla\left(\frac{3}{2}\mathbf{u}_h^{n+1}-\mathbf{u}_h^n+\frac{1}{2}\mathbf{u}_h^{n-1}\right),\nabla\mathbf{v}_h\right)\\&\quad-b^*\left(2\mathbf{u}_h^n-\mathbf{u}_h^{n-1},2\mathbf{u}_h^n-\mathbf{u}_h^{n-1},\mathbf{v}_h\right)\\&\quad+PrRa\left(\mathbf{j}\left(\frac{3}{2}T_h^{n+1}-T_h^n+\frac{1}{2}T_h^{n-1}\right),\mathbf{v}_h\right).\end{aligned} \tag{A.9}$$

The terms on the right-hand side of (A.9) can be bounded using Lemma 2.1, the Cauchy-Schwartz inequality, and duality, respectively

$$\begin{aligned}&-b^*\left(2\mathbf{u}_h^n-\mathbf{u}_h^{n-1},2\mathbf{u}_h^n-\mathbf{u}_h^{n-1},\mathbf{v}_h\right)\\&\le C\left\|\nabla\left(2\mathbf{u}_h^n-\mathbf{u}_h^{n-1}\right)\right\|\left\|\nabla\left(2\mathbf{u}_h^n-\mathbf{u}_h^{n-1}\right)\right\|\|\nabla\mathbf{v}_h\|,\end{aligned} \tag{A.10a}$$

$$- Pr\left(\nabla\left(\frac{3}{2}\mathbf{u}_h^{n+1} - \mathbf{u}_h^n + \frac{1}{2}\mathbf{u}_h^{n-1}\right), \nabla\mathbf{v}_h\right)$$
$$\leq Pr\left\|\nabla\left(\frac{3}{2}\mathbf{u}_h^{n+1} - \mathbf{u}_h^n + \frac{1}{2}\mathbf{u}_h^{n-1}\right)\right\|\|\nabla\mathbf{v}_h\|, \tag{A.10b}$$
$$PrRa\left(\mathbf{j}\left(\frac{3}{2}T_h^{n+1} - T_h^n + \frac{1}{2}T_h^{n-1}\right), \mathbf{v}_h\right)$$
$$\leq CPrRa\left\|\nabla\left(\frac{3}{2}T_h^{n+1} - T_h^n + \frac{1}{2}T_h^{n-1}\right)\right\|\|\nabla\mathbf{v}_h\|. \tag{A.10c}$$

Using the above estimates in Eq. (A.10), dividing both sides by $\|\nabla\mathbf{v}_h\|$, taking the supremum over $\mathbf{v}_h \in \mathbf{V}_h$, and using Theorem 3.1 yields

$$\left\|\frac{3\mathbf{u}_h^{n+1}/2 - 2\mathbf{u}_h^n + \mathbf{u}_h^{n-1}/2}{\Delta t}\right\|_{\mathbf{V}_h^*}$$
$$\leq C\left[\left\|\nabla\left(2\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\right)\right\|^2 + Pr\left\|\nabla\left(\frac{3}{2}\mathbf{u}_h^{n+1} - \mathbf{u}_h^n + \frac{1}{2}\mathbf{u}_h^{n-1}\right)\right\|\right.$$
$$\left. + CPrRa\left\|\nabla\left(\frac{3}{2}T_h^{n+1} - T_h^n + \frac{1}{2}T_h^{n-1}\right)\right\|\right]. \tag{A.11}$$

Lemma 2.3 then implies

$$\left\|\frac{3\mathbf{u}_h^{n+1}/2 - 2\mathbf{u}_h^n + \mathbf{u}_h^{n-1}/2}{\Delta t}\right\|_{\mathbf{X}_h^*}$$
$$\leq C_*^{-1}\left[\left\|\nabla\left(2\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\right)\right\|^2 + Pr\left\|\nabla\left(\frac{3}{2}\mathbf{u}_h^{n+1} - \mathbf{u}_h^n + \frac{1}{2}\mathbf{u}_h^{n-1}\right)\right\|\right.$$
$$\left. + CPrRa\left\|\nabla\left(\frac{3}{2}T_h^{n+1} - T_h^n + \frac{1}{2}T_h^{n-1}\right)\right\|\right]. \tag{A.12}$$

Now consider that $\mathbf{v}_h \in \mathbf{X}_h$. Isolating the pressure term in (3.4) and using the estimates from (A.10) yields

$$\begin{aligned}\left(p_h^{n+1}, \nabla\cdot\mathbf{v}_h\right) \leq{}& \left(\frac{3\mathbf{u}_h^{n+1}/2 - 2\mathbf{u}_h^n + \mathbf{u}_h^{n-1}/2}{\Delta t}, \mathbf{v}_h\right)\\ &+ C\left\|\nabla\left(2\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\right)\right\|^2\|\nabla\mathbf{v}_h\|\\ &+ Pr\left\|\nabla\left(\frac{3}{2}\mathbf{u}_h^{n+1} - \mathbf{u}_h^n + \frac{1}{2}\mathbf{u}_h^{n-1}\right)\right\|\|\nabla\mathbf{v}_h\|\\ &+ CPrRa\left\|\nabla\left(\frac{3}{2}T_h^{n+1} - T_h^n + \frac{1}{2}T_h^{n-1}\right)\right\|\|\nabla\mathbf{v}_h\|.\end{aligned} \tag{A.13}$$

Divide both sides by $\|\nabla\mathbf{v}_h\|$, take the supremum over $\mathbf{v}_h \in \mathbf{X}_h$, and use both the discrete inf-sup condition (2.10) and estimate (A.10). Then

$$\beta\|p_h^{n+1}\| \leq \left(1 + C_*^{-1}\right)\left[C\left\|\nabla\left(2\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\right)\right\|^2 + Pr\left\|\nabla\left(\frac{3}{2}\mathbf{u}_h^{n+1} - \mathbf{u}_h^n + \frac{1}{2}\mathbf{u}_h^{n-1}\right)\right\|\right.$$
$$\left. + CPrRa\left\|\nabla\left(\frac{3}{2}T_h^{n+1} - T_h^n + \frac{1}{2}T_h^{n-1}\right)\right\|\right]. \tag{A.14}$$

Multiplying by Δt_n, summing from $n=0$ to $n=N-1$

$$\beta\sum_{n=1}^{N-1}\Delta t\big\|p_h^{n+1}\big\|\le\big(1+C_*^{-1}\big)\Bigg[C\sum_{n=1}^{N-1}\Delta t\big\|\nabla\big(2\mathbf{u}_h^n-\mathbf{u}_h^{n-1}\big)\big\|^2 + Pr\sum_{n=1}^{N-1}\Delta t\bigg\|\nabla\bigg(\frac{3}{2}\mathbf{u}_h^{n+1}-\mathbf{u}_h^n+\frac{1}{2}\mathbf{u}_h^{n-1}\bigg)\bigg\| + CPrRa\sum_{n=1}^{N-1}\Delta t\bigg\|\nabla\bigg(\frac{3}{2}T_h^{n+1}-T_h^n+\frac{1}{2}T_h^{n-1}\bigg)\bigg\|\Bigg]. \quad \text{(A.15)}$$

Lastly using the Cauchy-Schwartz inequality, condition (3.21) and Theorem 3.1, we can get the result. □

Appendix B. Proofs of Results in Section 4

B.1. Proof of Lemma 4.2

Proof. Proving (4.4) first, adding and subtracting $\overline{b^*}(\mathbf{u}^{n+1},T^{n+1},\phi_T^{n+1}),\overline{b^*}(E^{n+1}(\mathbf{u}),T^{n+1},\phi_T^{n+1}),\overline{b^*}(E^{n+1}(\mathbf{u}_h),T^{n+1},\phi_T^{n+1})$ and $\overline{b^*}(E^{n+1}(\mathbf{u}_h),E^{n+1}(T_h),\phi_T^{n+1})$, we have

$$\begin{aligned}&\overline{b^*}\big(\mathbf{u}^{n+1},T^{n+1},\phi_T^{n+1}\big)-\overline{b^*}\big(E^{n+1}(\mathbf{u}_h),E^{n+1}(T_h),\phi_T^{n+1}\big)\\ =&\overline{b^*}\big(E^{n+1}(e_{\mathbf{u}}),T^{n+1},\phi_T^{n+1}\big)+\overline{b^*}\big(\mathbf{u}^{n+1}-E^{n+1}(\mathbf{u}),T^{n+1},\phi_T^{n+1}\big)\\ &+\overline{b^*}\big(E^{n+1}(\mathbf{u}_h),T^{n+1}-E^{n+1}(T_h),\phi_T^{n+1}\big).\end{aligned} \quad \text{(B.1)}$$

For the first term on the right-hand side of (B.1) we split it into

$$\begin{aligned}&\overline{b^*}\big(E^{n+1}(e_{\mathbf{u}}),T^{n+1},\phi_T^{n+1}\big)\\ =&\overline{b^*}\big(E^{n+1}(\eta_{\mathbf{u}}),T^{n+1},\phi_T^{n+1}\big)-\overline{b^*}\big(E^{n+1}(\phi_{\mathbf{u}}),T^{n+1},\phi_T^{n+1}\big).\end{aligned} \quad \text{(B.2)}$$

Applying Cauchy-Schwarz-Young inequality, inequality (2.6), and Assumption 2.1

$$\begin{aligned}&\overline{b^*}\big(E^{n+1}(\eta_{\mathbf{u}}),\mathbf{u}^{n+1},\phi_{\mathbf{u}}^{n+1}\big)\\ \le& Pr\widetilde{C_1}\big\|\nabla\phi_T^{n+1}\big\|^2+\frac{C}{Pr}\big\|\nabla E^{n+1}(\eta_{\mathbf{u}})\big\|^2.\end{aligned} \quad \text{(B.3)}$$

Next, we have

$$\begin{aligned}&\overline{b^*}\big(E^{n+1}(\phi_{\mathbf{u}}),T^{n+1},\phi_T^{n+1}\big)\\ =&\overline{b^*}\big((1+\omega_n)\phi_{\mathbf{u}}^n,T^{n+1},\phi_T^{n+1}\big)-\overline{b^*}\big(\omega_n\phi_{\mathbf{u}}^{n-1},T^{n+1},\phi_T^{n+1}\big).\end{aligned} \quad \text{(B.4)}$$

Using inequality (2.6), Cauchy-Schwarz-Young inequality, and Assumption 2.1 we have

$$\begin{aligned}&\overline{b^*}\big((1+\omega_n)\phi_{\mathbf{u}}^n,T^{n+1},\phi_T^{n+1}\big)\\ \le& C\big\|\nabla\phi_{\mathbf{u}}^n\big\|^{\frac{1}{2}}\big\|\phi_{\mathbf{u}}^n\big\|^{\frac{1}{2}}\big\|\nabla\phi_T^{n+1}\big\|\\ \le& C\Big(\epsilon\big\|\nabla\phi_T^{n+1}\big\|^2+\frac{1}{\epsilon}\big\|\nabla\phi_{\mathbf{u}}^n\big\|\big\|\phi_{\mathbf{u}}^n\big\|\Big)\\ \le& C\bigg(\epsilon\big\|\nabla\phi_T^{n+1}\big\|^2+\frac{1}{\epsilon}\bigg(\alpha\big\|\nabla\phi_{\mathbf{u}}^n\big\|^2+\frac{1}{\alpha}\big\|\phi_{\mathbf{u}}^n\big\|^2\bigg)\bigg)\\ \le& Pr\widetilde{C_2}\big\|\nabla\phi_T^{n+1}\big\|^2+Pr\widetilde{C_3}\big\|\nabla\phi_{\mathbf{u}}^n\big\|^2+CPr^{-3}\big\|\phi_{\mathbf{u}}^n\big\|^2.\end{aligned} \quad \text{(B.5)}$$

Similarly,

$$
\begin{aligned}
&\overline{b^*}\big(\omega_n\phi_{\mathbf{u}}^{n-1}, T^{n+1}, \phi_T^{n+1}\big)\\
&\le Pr\widetilde{C_4}\big\|\nabla\phi_T^{n+1}\big\|^2 + Pr\widetilde{C_5}\big\|\nabla\phi_{\mathbf{u}}^{n-1}\big\|^2 + CPr^{-3}\big\|\phi_{\mathbf{u}}^{n-1}\big\|^2. \qquad \text{(B.6)}
\end{aligned}
$$

Bounding the second nonlinear term on the right-hand side of (B.1) using Cauchy-Schwarz-Young inequality, inequality (2.6), Lemma 4.1, and Assumption 2.1

$$
\begin{aligned}
&\overline{b^*}\big(\mathbf{u}^{n+1} - E^{n+1}(\mathbf{u}), T^{n+1}, \phi_T^{n+1}\big)\\
&\le C\big\|\nabla\big(\mathbf{u}^{n+1} - E^{n+1}(\mathbf{u})\big)\big\|\big\|\nabla\phi_T^{n+1}\big\|\\
&\le \frac{C}{Pr}\big\|\nabla\big(\mathbf{u}^{n+1} - E^{n+1}(\mathbf{u})\big)\big\|^2 + \widetilde{C_6}Pr\big\|\nabla\phi_T^{n+1}\big\|^2\\
&\le \frac{C(\Delta t_{n-1} + \Delta t_n)^3}{Pr}\|\nabla\mathbf{u}_{tt}\|^2_{L^2(t^{n-1},t^{n+1};L^2(\Omega))} + \widetilde{C_6}Pr\big\|\nabla\phi_T^{n+1}\big\|^2. \qquad \text{(B.7)}
\end{aligned}
$$

For the last nonlinear term in (B.1), adding and subtracting $\overline{b^*}(E^{n+1}(\mathbf{u}_h), T_h^{n+1}, \phi_T^{n+1})$, and using the skew-symmetry of the nonlinear term yields

$$
\begin{aligned}
&\overline{b^*}\big(E^{n+1}(\mathbf{u}_h), T^{n+1} - E^{n+1}(T_h), \phi_T^{n+1}\big)\\
&= \overline{b^*}\big(E^{n+1}(\mathbf{u}_h), e_T^{n+1}, \phi_T^{n+1}\big) + \overline{b^*}\big(E^{n+1}(\mathbf{u}_h), T_h^{n+1} - E^{n+1}(T_h), \phi_T^{n+1}\big)\\
&= \overline{b^*}\big(E^{n+1}(\mathbf{u}_h), \eta_T^{n+1}, \phi_T^{n+1}\big) + \overline{b^*}\big(E^{n+1}(\mathbf{u}_h), T_h^{n+1} - E^{n+1}(T_h), \phi_T^{n+1}\big). \qquad \text{(B.8)}
\end{aligned}
$$

For the first term on the right-hand side we have by Cauchy-Schwarz-Young inequality and inequality (2.6)

$$
\begin{aligned}
&\overline{b^*}\big(E^{n+1}(\mathbf{u}_h), \eta_T^{n+1}, \phi_T^{n+1}\big)\\
&\le C\big\|E^{n+1}(\mathbf{u}_h)\big\|^{\frac{1}{2}}\big\|\nabla E^{n+1}(\mathbf{u}_h)\big\|^{\frac{1}{2}}\big\|\nabla\eta_T^{n+1}\big\|\big\|\nabla\phi_T^{n+1}\big\|\\
&\le \frac{C}{Pr}\big\|E^{n+1}(\mathbf{u}_h)\big\|\big\|\nabla E^{n+1}(\mathbf{u}_h)\big\|\big\|\nabla\eta_T^{n+1}\big\|^2 + Pr\widetilde{C_7}\big\|\nabla\phi_T^{n+1}\big\|^2. \qquad \text{(B.9)}
\end{aligned}
$$

For the second term on the right-hand side of (B.8) we rewrite it as

$$
\begin{aligned}
&\overline{b^*}\big(E^{n+1}(\mathbf{u}_h), T_h^{n+1} - E^{n+1}(T_h), \phi_T^{n+1}\big)\\
&= \overline{b^*}\big(E^{n+1}(\mathbf{u}_h), T^{n+1} - E^{n+1}(T), \phi_T^{n+1}\big)\\
&\quad - \overline{b^*}\big(E^{n+1}(\mathbf{u}_h), e_T^{n+1} - E^{n+1}(e_T), \phi_T^{n+1}\big). \qquad \text{(B.10)}
\end{aligned}
$$

Bounding the first of these terms using Cauchy-Schwarz-Young inequality, inequality (2.6), and Lemma 4.1

$$
\begin{aligned}
&\overline{b^*}\big(E^{n+1}(\mathbf{u}_h), T^{n+1} - E^{n+1}(T), \phi_T^{n+1}\big)\\
&\le C\big\|\nabla E^{n+1}(\mathbf{u}_h)\big\|\big\|\nabla\big(T^{n+1} - E^{n+1}(T)\big)\big\|\big\|\nabla\phi_T^{n+1}\big\|\\
&\le \frac{C_{stab}\Delta t_n\big(1 + {\omega_n}^2\big)}{16h}\big\|\nabla E^{n+1}(\mathbf{u}_h)\big\|^2\big\|\nabla\phi_T^{n+1}\big\|^2\\
&\quad + \frac{Ch(\Delta t_{n-1} + \Delta t_n)^3}{C_{stab}\Delta t_n\big(1 + {\omega_n}^2\big)}\|\nabla T_{tt}\|^2_{L^2(t^{n-1},t^{n+1};L^2(\Omega))}. \qquad \text{(B.11)}
\end{aligned}
$$

We next rewrite the term

$$
\begin{aligned}
&\overline{b^*}\big(E^{n+1}(\mathbf{u}_h), e_T^{n+1} - E^{n+1}(e_T), \phi_T^{n+1}\big)\\
&= \overline{b^*}\big(E^{n+1}(\mathbf{u}_h), \eta_T^{n+1} - E^{n+1}(\eta_T), \phi_T^{n+1}\big)\\
&\quad - \overline{b^*}\big(E^{n+1}(\mathbf{u}_h), \phi_T^{n+1} - E^{n+1}(\phi_T), \phi_T^{n+1}\big). \qquad \text{(B.12)}
\end{aligned}
$$

Bounding the first of these terms using Cauchy-Schwarz-Young inequality, inequality (2.6), and Lemma 4.1

$$\begin{aligned}
&\overline{b^*}\big(E^{n+1}(\mathbf{u}_h),\eta_T^{n+1}-E^{n+1}(\eta_T),\phi_T^{n+1}\big)\\
&\le C\big\|\nabla E^{n+1}(\mathbf{u}_h)\big\|\big\|\nabla\big(\eta_T^{n+1}-E^{n+1}(\eta_T)\big)\big\|\big\|\nabla\phi_T^{n+1}\big\|\\
&\le \frac{C_{stab}\Delta t_n\big(1+{\omega_n}^2\big)}{16h}\big\|\nabla E^{n+1}(\mathbf{u}_h)\big\|^2\big\|\nabla\phi_T^{n+1}\big\|^2\\
&\quad+\frac{Ch(\Delta t_{n-1}+\Delta t_n)^3}{C_{stab}\Delta t_n\big(1+{\omega_n}^2\big)}\|\nabla(\eta_T)_{tt}\|^2_{L^2(t^{n-1},t^{n+1};L^2(\Omega))}.
\end{aligned}\tag{B.13}$$

Finally, bounding the last term using inequality (2.6), the inverse inequality, Cauchy-Schwarz-Young inequality, and the parallelogram law we can get

$$\begin{aligned}
&\overline{b^*}\big(E^{n+1}(\mathbf{u}_h),\phi_T^{n+1}-E^{n+1}(\phi_T),\phi_T^{n+1}\big)\\
&\le C\big\|\nabla E^{n+1}(\mathbf{u}_h)\big\|\big\|\phi_T^{n+1}-E^{n+1}(\phi_T)\big\|^{\frac12}\big\|\nabla\big(\phi_T^{n+1}-E^{n+1}(\phi_T)\big)\big\|^{\frac12}\big\|\nabla\phi_T^{n+1}\big\|\\
&\le Ch^{-\frac12}\big\|\nabla E^{n+1}(\mathbf{u}_h)\big\|\big\|\phi_T^{n+1}-E^{n+1}(\phi_T)\big\|\big\|\nabla\phi_T^{n+1}\big\|\\
&\le \frac{1}{8\Delta t_n\big(1+{\omega_n}^2\big)}\big\|\phi_T^{n+1}-E^{n+1}(\phi_T)\big\|^2\\
&\quad+\frac{C_{stab}\Delta t_n\big(1+{\omega_n}^2\big)}{2h}\big\|\nabla E^{n+1}(\mathbf{u}_h)\big\|^2\big\|\nabla\phi_T^{n+1}\big\|^2\\
&\le \frac{1}{4\Delta t_n}\big\|\phi_T^{n+1}-\phi_T^n\big\|^2+\frac{1}{4\Delta t_n}\big\|\phi_T^n-\phi_T^{n-1}\big\|^2\\
&\quad+\frac{C_{stab}\Delta t_n\big(1+{\omega_n}^2\big)}{2h}\big\|\nabla E^{n+1}(\mathbf{u}_h)\big\|^2\big\|\nabla\phi_T^{n+1}\big\|^2.
\end{aligned}\tag{B.14}$$

Taking $\widetilde{C_1},\widetilde{C_2},\widetilde{C_4},\widetilde{C_6},\widetilde{C_7}=1/64,\widetilde{C_3},\widetilde{C_5}=1/16$, the result follows. Similarly, combining Cauchy-Schwarz-Young inequality, inequality (2.5), Assumption 2.1, Lemma 4.1, then using the skew-symmetry of the nonlinear term and the inverse inequality, we complete the proof. □

Acknowledgments. The authors would like to thank the editor and anonymous referees for their valuable comments and suggestions which helped us to improve this paper.

The work is supported by the NSF of China (Grant Nos. 12301501, 12301503, 12071406), by the Training Plan for Young Backbone Teachers of Henan University of Technology, by the NSF of Xinjiang Province (Grant Nos. 2022TSYCTD0019, 2022D01D32), and by the Foundation of National Key Laboratory of Computational Physics (Grant No. 6142A05230203).

References

[1] R. Asselin, Frequency filter for time integrations, *Mon. Weather Rev.*, **100**:6 (1972), 487–490.

[2] M. Besier and R. Rannacher, Goal-oriented space-time adaptivity in the finite element Galerkin method for the computation of nonstationary incompressible flow, *Internat. J. Numer. Methods Fluids*, **70**:9 (2012), 1139–1166.

[3] J. Boland and W. Layton, Error analysis for finite element methods for steady natural convection problems, *Numer. Funct. Anal. Optim.*, **11**:5-6 (1990), 449–483.

[4] S.C. Brenner and L.R. Scott, *The Mathematical Theory of Finite Element Methods*, in: *Texts in Applied Mathematics*, Vol. 15, Springer, 2002.

[5] S. Busto, M. Tavelli, W. Boscheri, and M. Dumbser, Efficient high order accurate staggered semi-implicit discontinuous Galerkin methods for natural convection problems, *Comput. & Fluids*, **198** (2020), 104399.

[6] Q. Cheng, C. Liu, and J. Shen, A new interface capturing method for Allen-Cahn type equations based on a flow dynamic approach in Lagrangian coordinates, I. One-dimensional case, *J. Comput. Phys.*, **419** (2020), 109509.

[7] M.A. Christon, P.M. Gresho, and S.B. Sutton, Computational predictability of time-dependent natural convection flows in enclosures (including a benchmark solution), *Internat. J. Numer. Methods Fluids*, **40**:8 (2002), 953–980.

[8] T. Chu, J. Wang, N. Wang, and Z. Zhang, Optimal-order convergence of a two-step BDF method for the Navier-stokes equations with H^1 initial data, *J. Sci. Comput.*, **96** (2023), 62.

[9] A. Cibik and S. Kaya, A projection-based stabilized finite element method for steady-state natural convection problem, *J. Math. Anal. Appl.*, **381**:2 (2011), 469–484.

[10] V. DeCaria, A. Guzel, W. Layton, and Y. Li, A new embedded variable stepsize, variable order family of low computational complexity, *arXiv:1810.06670*, 2018.

[11] V. DeCaria, W. Layton, and M. McLaughlin, An analysis of the Robert-Asselin time filter for the correction of nonphysical acoustics in an artificial compression method, *Numer. Methods Partial Differential Equations*, **35**:3 (2019), 916–935.

[12] V. DeCaria, W. Layton, and H. Zhao, A time-accurate, adaptive discretization for fluid flow problems, *arXiv:1810.06705*, 2018.

[13] V. DeCaria and M. Schneier, An embedded variable step IMEX scheme for the incompressible Navier-Stokes equations, *Comput. Methods Appl. Mech. Engrg.*, **376** (2021), 113661.

[14] Y. Di, Y. Ma, J. Shen, and J. Zhang, A variable time-step IMEX-BDF2 SAV scheme and its sharp error estimate for the Navier-stokes equations, *ESAIM Math. Model. Numer. Anal.*, **57**:3 (2023), 1143–1170.

[15] H.N. Dixit and V. Babu, Simulation of high Rayleigh number natural convection in a square cavity using the lattice Boltzmann method, *Int. J. Heat Mass Transf.*, **49**:3-4 (2006), 727–739.

[16] Y. Feng and J. Li, Phase-field cohesive fracture theory: A unified framework for dissipative systems based on variational inequality of virtual works, *J. Mech. Phys. Solids*, **159** (2022), 104737.

[17] J.A. Fiordilino, On pressure estimates for the Navier-Stokes equations, *arXiv:1803.04366*, 2018.

[18] J.A. Fiordilino, A second order ensemble timestepping algorithm for natural convection, *SIAM J. Numer. Anal.*, **56**:2 (2018), 816–837.

[19] M. Fortin and F. Brezzi, *Mixed and Hybrid Finite Element Methods*, Springer-Verlag, 1991.

[20] K.J. Galvin, New subgrid artificial viscosity Galerkin methods for the Navier-Stokes equations, *Comput. Methods Appl. Mech. Engrg.*, **200**:1-4 (2011), 242–250.

[21] V. Girault and P.A. Raviart, *Finite Element Approximation of the Navier-Stokes Equations*, Springer, 1979.

[22] A. Guzel and W. Layton, Analysis of the effect of time filters on the implicit method: Increased accuracy and improved stability, *arXiv:1708.06306*, 2017.

[23] A. Guzel and W. Layton, Time filters increase accuracy of the fully implicit method, *BIT*, **58** (2018), 301–315.

[24] F. Hecht, New development in FreeFem++, *J. Numer. Math.*, **20**:3-4 (2012), 251–266.

[25] D. Hou, L. Ju, and Z. Qiao, A linear second-order maximum bound principle-preserving BDF scheme for the Allen-Cahn equation with a general mobility, *Math. Comp.*, **92**:344 (2023), 2515–2542.

[26] F. Huang and J. Shen, A new class of implicit-explicit BDFk SAV schemes for general dissipative systems and their error analysis, *Comput. Methods Appl. Mech. Engrg.*, **392** (2022), 114718.

[27] L. Ju, X. Li, and Z. Qiao, Generalized SAV-exponential integrator schemes for Allen-Cahn type gradient flows, *SIAM J. Numer. Anal.*, **60**:4 (2022), 1905–1931.

[28] D.A. Kay, P.M. Gresho, D.F. Griffiths, and D.J. Silvester, Adaptive time-stepping for incompress-

ible flow. Part II: Navier-Stokes equations, *SIAM J. Sci. Comput.*, **32**:1 (2010), 111–128.

[29] M. Kwizak and A.J. Robert, A semi-implicit scheme for grid point atmospheric models of the primitive equations, *Mon. Weather Rev.*, **99**:1 (1971), 32–36.

[30] D. Li, C. Quan, and T. Tang, Stability and convergence analysis for the implicit-explicit method to the Cahn-Hilliard equation, *Math. Comp.*, **91**:334 (2022), 785–809.

[31] P.W. Li, W. Chen, Z.J. Fu, and C.M. Fan, Generalized finite difference method for solving the double-diffusive natural convection in fluid-saturated porous media, *Eng. Anal. Bound. Elem.*, **95** (2018), 175–186.

[32] Y. Li and C. Trenchea, A higher-order Robert-Asselin type time filter, *J. Comput. Phys.*, **259** (2014), 23–32.

[33] H.L. Liao, X. Song, T. Tang, and T. Zhou, Analysis of the second-order BDF scheme with variable steps for the molecular beam epitaxial model without slope selection, *Sci. China Math.*, **64** (2021), 887–902.

[34] H.L. Liao, T. Tang, and T. Zhou, On energy stable, maximum-principle preserving, second-order BDF scheme with variable steps for the Allen-Cahn equation, *SIAM J. Numer. Anal.*, **58**:4 (2020), 2294–2314.

[35] C. Ma, J. Wu, and T. Zhang, A high order spectral difference-based phase field lattice Boltzmann method for incompressible two-phase flows, *Phys. Fluids*, **32**:12 (2020), 122113.

[36] R. Ma and W. Sun, FFT-based solver for higher-order and multi-phase-field fracture models applied to strongly anisotropic brittle materials, *Comput. Methods Appl. Mech. Engrg.*, **362** (2020), 112781.

[37] Y. Ping, H. Su, J. Zhao, and X. Feng, Parallel two-step finite element algorithm based on fully overlapping domain decomposition for the time-dependent natural convection problem, *Int. J. Numer. Methods Heat Fluid Flow*, **30**:2 (2020), 496–515.

[38] Y. Qin and Y. Hou, The time filter for the non-stationary coupled Stokes/Darcy model, *Appl. Numer. Math.*, **146** (2019), 260–275.

[39] J. Shen, On error estimates of the projection methods for the Navier-Stokes equations: Second-order schemes, *Math. Comp.*, **65**:215 (1996), 1039–1065.

[40] S.M. Shen, The finite element analysis of conduction-convection problem, *J. Comput. Math.*, **16** (1994), 170–182.

[41] H. Su, X. Feng, and J. Zhao, Penalty decoupled iterative methods for the stationary natural convection equations with different Rayleigh numbers, *Appl. Numer. Math.*, **163** (2021), 270–291.

[42] K. Szewc, J. Pozorski, and A. Taniere, Modeling of natural convection with smoothed particle hydrodynamics: Non-Boussinesq formulation, *Int. J. Heat Mass Transf.*, **54**:23-24 (2011), 4807–4816.

[43] R. Temam, *Navier-Stokes Equations: Theory and Numerical Analysis*, AMS, 2001.

[44] W. Wang, Z. Wang, and M. Mao, Linearly implicit variable step-size BDF schemes with Fourier pseudospectral approximation for incompressible Navier-Stokes equations, *Appl. Numer. Math.*, **172** (2022), 393–410.

[45] W.S. Wang, S. Yang, and C.Y. Jin, Robust error estimates for second-order stabilzation finite element methods for Navier-Stokes equations with small viscosity and nonsmooth initial data, DOI: 10.2139/ssrn.4518533, 2023.

[46] Y. Wang, M.A.A. Mahbub, and H. Zheng, Characteristic stabilized finite element method for non-stationary conduction-convection problems, *Int. J. Numer. Methods Heat Fluid Flow*, **30**:2 (2020), 625–658.

[47] Y. Wang, G. Qin, W. He, and Z. Bao, Chebyshev spectral element method for natural convection in a porous cavity under local thermal non-equilibrium model, *Int. J. Heat Mass Tran.*, **121** (2018), 1055–1072.

[48] P.D. Williams, A proposed modification to the Robert-Asselin time filter, *Mon. Weather Rev.*, **137**:8 (2009), 2538–2546.

[49] P.D. Williams, The RAW filter: An improvement to the Robert-Asselin filter in semi-implicit integrations, *Mon. Weather Rev.*, **139**:6 (2011), 1996–2007.

[50] P.D. Williams, Achieving seventh-order amplitude accuracy in leapfrog integrations, *Mon. Weather Rev.*, **141**:9 (2013), 3037–3051.

[51] J. Wu, N. Li, and X. Feng, Analysis of a filtered time-stepping finite element method for natural convection problems, *SIAM J. Numer. Anal.*, **61**:2 (2023), 837–871.

[52] J. Wu, L. Wei, and X. Feng, Novel fractional time-stepping algorithms for natural convection problems with variable density, *Appl. Numer. Math.*, **151** (2020), 64–84.

[53] Y. Zhang, Y. Hou, and J. Zhao, Error analysis of a fully discrete finite element variational multiscale method for the natural convection problem, *Comput. Math. Appl.*, **68**:4 (2014), 543–567.

[54] Y. Zhang, Y. Hou, and H. Zheng, A finite element variational multiscale method for steady-state natural convection problem based on two local Gauss integrations, *Numer. Methods Partial Differential Equations*, **30**:2 (2014), 361–375.

[55] H. Zhao, *Low Complexity, Time Accurate, Model Accurate Algorithms in Computational Fluid Dynamics*, PhD Thesis, University of Pittsburgh, 2019.