Journal of Computational Mathematics
Vol.44, No.1, 2026, 286–306.

http://www.global-sci.org/jcm
doi:10.4208/jcm.2410-m2024-0092

STABILITY AND ERROR ESTIMATES OF ULTRA-WEAK LOCAL DISCONTINUOUS GALERKIN METHOD WITH SPECTRAL DEFERRED CORRECTION TIME-MARCHING FOR PDES WITH HIGH ORDER SPATIAL DERIVATIVES*

Lingling Zhou and Wenhua Chen
School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo 454000, China
Emails: lingzhou@hpu.edu.cn, cwh@home.hpu.edu.cn
Ruihan Guo[1)]
School of Mathematics and Statistics, Zhengzhou University, Zhengzhou 450001, China
Email: rguo@zzu.edu.cn

Abstract

The main purpose of this paper is to give stability analysis and error estimates of the ultra-weak local discontinuous Galerkin (UWLDG) method coupled with a spectral deferred correction (SDC) temporal discretization method up to fourth order, for solving the fourth-order equation. The UWLDG method introduces fewer auxiliary variables than the local discontinuous Galerkin method and no internal penalty terms are required for stability, which is efficient for high order partial differential equations (PDEs). The SDC method we adopt in this paper is based on second-order time integration methods and the order of accuracy is increased by two for each additional iteration. With the energy techniques, we rigorously prove the fully discrete schemes are unconditionally stable. By the aid of special projections and initial conditions, the optimal error estimates of the fully discrete schemes are obtained. Furthermore, we generalize the analysis to PDEs with higher even-order derivatives. Numerical experiments are displayed to verify the theoretical results.

Mathematics subject classification: 65M60, 65M12, 35K25.
Key words: Ultra-weak local discontinuous Galerkin method, High order equations, Spectral deferred correction method, Stability, Error estimate.

1. Introduction

In this paper, we present the stability analysis and error estimates of the ultra-weak local discontinuous Galerkin method coupled with a spectral deferred correction temporal discretization methods for linear partial differential equations with high order spatial derivatives. We first consider the following fourth-order equation:

$$\begin{aligned} &u_t + u_{xxxx} = 0, \quad (x,t) \in [a,b] \times [0,T], \\ &u(x,0) = u_0(x), \quad x \in [a,b] \end{aligned} \tag{1.1}$$

with periodic boundary conditions. Then we generalize the stability analysis and error estimate to PDEs with even-order derivatives.

* Received May 7, 2024 / Revised version received August 21, 2024 / Accepted October 25, 2024 / Published online December 23, 2024 /
1) Corresponding author

The UWLDG method was first developed to solve time dependent PDEs with high order derivatives by Tao *et al.* [14]. It combines the advantages of local discontinuous Galerkin (LDG) method and ultra-weak discontinuous Galerkin (UWDG) method. The main idea of the LDG method [6, 21, 22] is to rewrite the high order equations into a first-order system, then apply the traditional DG method [11, 12] to the system and design suitable numerical fluxes to ensure stability. The UWDG method [3,4] is based on repeated integration by parts to move all spatial derivatives to the test function in the weak formulation, and ensure stability by suitable numerical fluxes. Recently, the UWLDG method has been successfully used to solve PDEs with high order derivatives, and the theoretical analysis for the semi-discrete UWLDG method [13,15] are also presented. Compared to the LDG method, the UWLDG method requires fewer auxiliary variables, thus reduces memory and computational costs. Compared to the UWDG method, no internal penalty terms are required to ensure stability.

To relax the severe time step restriction of explicit time marching methods for PDEs with high order derivatives, several semi-implicit time marching methods have been studied, such as the implicit-explicit (IMEX) time discretization [1, 2] and the SDC methods [7, 8, 10, 24]. The semi-implicit SDC method proposed in [8] is based on second-order time integration methods and the order of accuracy is increased by two for each additional iteration. In [23], the SDC method coupled with the LDG spatial discretization was studied for parabolic equations. In this paper, we consider the UWLDG method and study the stability and error estimates when coupled with the novel SDC method.

Wang *et al.* [16–18] presented the stability analysis of the Runge-Kutta type IMEX time discretization coupled with LDG method for convection-diffusion problems and fourth-order equations [20]. Further, Wang *et al.* [19] studied the analysis of the IMEX-UWDG method for convection-diffusion problems and the optimal error estimate was obtained for fully discrete schemes up to third order. The Runge-Kutta type IMEX methods have some limitations, for example, it is more difficult to construct for higher order accuracy. While for the SDC method, an advantage of this method is that it is a one step method and can be constructed easily and systematically for any order of accuracy. As far as the authors know, there is no theoretical analysis for the fully discrete IMEX-UWLDG scheme. In this paper, we perform the analysis of the stability and error estimates for some fully discrete SDC-UWLDG schemes up to fourth order both in space and time.

Our contribution here is that we first take the fourth-order equation as an example and rigorously prove that the SDC-UWLDG schemes are unconditionally stable. The main technique is the symmetric and dissipative properties of the semi-discrete UWLDG scheme, which plays a key role in establishing negative definite quadratic forms for the semi-implicit discretization of the high order derivative part. The stability results contain information about the initial numerical approximation of the second-order derivative, making the choice of the initial conditions crucial for optimal error estimates. Following the same idea as that in [9], we would like to define the special elliptic projection with respect to the semi-discrete UWLDG scheme and derive optimal error estimates of the initial numerical approximation. By carefully choosing projections, we proceed to obtain the optimal error estimates of the SDC-UWLDG schemes up to fourth order both in time and space. Moreover, we generalize the analysis to PDEs with higher even-order derivatives.

The rest of this paper is organized as follows. We first present some notations and projections in Section 2. Then we present the semi-discrete UWLDG scheme for the fourth-order equation in Section 3. In Section 4, we give the stability analysis of the second-order and fourth-order fully discrete SDC-UWLDG schemes. In Section 5, we give the optimal error estimates for the fourth-order fully discrete SDC-UWLDG scheme. We extend the analysis to PDEs with even-order derivatives in Section 6. Numerical results are given in Section 7 to verify the main theoretical results. Finally, we give some conclusions in Section 8.

2. Notations

In this section, we introduce some notations that will be used later.

2.1. Notation for the mesh, function space and norms

Let $a = x_{1/2} < x_{3/2} < \cdots < x_{N+1/2} = b$ be a subdivision of the spatial domain $\Omega = [a, b]$. We denote the length of the element $I_j = (x_{j-1/2}, x_{j+1/2})$ by $h_j = x_{j+1/2} - x_{j-1/2}$ and set $h = \max_{1\le j\le N} h_j$. In addition, we assume the mesh is quasi-uniform, that is, there exists a positive constant C such that $h \le Ch_j$ for $j = 1, 2, \ldots, N$. Associated with the mesh, we define the finite element space as

$$V_h = \big\{v \in L^2(\Omega) : v|_{I_j} \in P^k(I_j),\ j = 1, 2, \ldots, N\big\},$$

where $P^k(I_j)$ denotes the space of polynomials of degree at most k on I_j. In addition, we define the standard inner product and its norm as

$$(v, w)_j = \int_{I_j} vw dx, \quad \|v\|_{I_j} = \sqrt{(v, v)_j}.$$

We also consider the following broken Sobolev space:

$$H_h^s = \big\{v \in L^2(\Omega) : v|_{I_j} \in H^s(I_j),\ j = 1, 2, \ldots, N\big\},$$

where $H^s(D)$ represents the usual Sobolev space on the sub-domain $D \subset \Omega$ and the associated norm is denoted by $\|\cdot\|_{H^s(D)}$ for any integer $s \ge 0$. Note that the functions in the finite element space are allowed to have discontinuities across element interfaces. We define the left and right limits of v at the point $x_{j-1/2}$ by $v^-_{j-1/2}$ and $v^+_{j-1/2}$, respectively. Thus, the jump of v at cell interfaces is denoted by $[v]_{j-1/2} = v^+_{j-1/2} - v^-_{j-1/2}$. Summing over all the element, we denote

$$(v, w) = \sum_{j=1}^N (v, w)_j, \quad \|v\|^2 = \sum_{j=1}^N \|v\|^2_{I_j}, \quad \|v\|^2_{H_h^s(\Omega)} = \sum_{j=1}^N \|v\|^2_{H^s(I_j)}.$$

2.2. Projection properties

Now, we present the definition of the special projections $\mathbb{P}_h^\pm$. For any periodic function $u \in H^s(\Omega)$ with $s \ge 2$, the projections $\mathbb{P}_h^\pm u \in V_h$ are defined as follows: If $k \ge 2$, then in each element $I_j = (x_{j-1/2}, x_{j+1/2})$

$$\big(\mathbb{P}_h^\pm u, v\big)_j = (u, v)_j, \quad \forall\, v \in P^{k-2}(I_j), \tag{2.1}$$

and

$$\mathbb{P}_h^+ u\big(x^+_{j-\frac12}\big) = u\big(x^+_{j-\frac12}\big), \quad (\mathbb{P}_h^+ u)_x\big(x^+_{j-\frac12}\big) = u_x\big(x^+_{j-\frac12}\big), \tag{2.2}$$

$$\mathbb{P}_h^- u\big(x^-_{j+\frac12}\big) = u\big(x^-_{j+\frac12}\big), \quad (\mathbb{P}_h^- u)_x\big(x^-_{j+\frac12}\big) = u_x\big(x^-_{j+\frac12}\big). \tag{2.3}$$

If $k=1$, only conditions (2.2)-(2.3) are needed. Since the projections $\mathbb{P}_h^{\pm}$ are defined element-wise, the existence and uniqueness can be verified straightforwardly. Then by the standard scaling argument [5], we can obtain the following approximation property:

$$\|\eta\|_{H_h^m(\Omega)} \le Ch^{\min\{k+1,r\}-m}\|u\|_{H^r(\Omega)}, \tag{2.4}$$

where $\eta=\mathbb{P}_h^{\pm}u-u$, $0\le m\le \min\{r,k+1\}$. The positive constant C is independent of h and u. For more details about the projections, we refer readers to [3].

3. Semi-discrete Methods

3.1. The UWLDG scheme

To derive the UWLDG scheme for the fourth-order problem (1.1), we first rewrite the equation as a system of second-order equations. Let $q=u_{xx}$ and we have

$$u_t+q_{xx}=0, \quad q-u_{xx}=0. \tag{3.1}$$

Then following [14], we can get the semi-discrete UWLDG scheme: Find $u_h, q_h\in V_h$ such that for any $v, r\in V_h$, there holds

$$\begin{aligned}
&\big((u_h)_t, v\big)_j = -(q_h, v_{xx})_j + \widetilde{q_h}v_x^-|_{j+\frac12} - \widetilde{q_h}v_x^+|_{j-\frac12} - \big(\widetilde{(q_h)_x}\big)v^-|_{j+\frac12} + \big(\widetilde{(q_h)_x}\big)v^+|_{j-\frac12},\\
&(q_h, r)_j = (u_h, r_{xx})_j - \widehat{u_h}r_x^-|_{j+\frac12} + \widehat{u_h}r_x^+|_{j-\frac12} + \big(\widehat{(u_h)_x}\big)r^-|_{j+\frac12} - \big(\widehat{(u_h)_x}\big)r^+|_{j-\frac12}.
\end{aligned} \tag{3.2}$$

Here $\widehat{u_h}, \widetilde{q_h}$ and $\widehat{(u_h)_x}, \widetilde{(q_h)_x}$ are the numerical fluxes, which can be chosen as

$$\widehat{u_h}=u_h^+, \quad \widehat{(u_h)_x}=(u_h)_x^+, \quad \widetilde{q_h}=q_h^-, \quad \widetilde{(q_h)_x}=(q_h)_x^-. \tag{3.3}$$

Remark 3.1. The choice of numerical fluxes mainly follows the principle of alternating fluxes, that is, $\widehat{(u_h)_x}$ and $\widetilde{q_h}$ come from the opposite sides, and $\widehat{u_h}$ and $\widetilde{(q_h)_x}$ come from the opposite sides. Thus, we can also take the following numerical fluxes:

$$\begin{aligned}
&\widehat{u_h}=u_h^+, \quad \widehat{(u_h)_x}=(u_h)_x^-, \quad \widetilde{q_h}=q_h^+, \quad \widetilde{(q_h)_x}=(q_h)_x^-,\\
&\widehat{u_h}=u_h^-, \quad \widehat{(u_h)_x}=(u_h)_x^-, \quad \widetilde{q_h}=q_h^+, \quad \widetilde{(q_h)_x}=(q_h)_x^+,\\
&\widehat{u_h}=u_h^-, \quad \widehat{(u_h)_x}=(u_h)_x^+, \quad \widetilde{q_h}=q_h^-, \quad \widetilde{(q_h)_x}=(q_h)_x^+.
\end{aligned}$$

For simplicity, we rewrite (3.2) as

$$\big((u_h)_t, v\big)_j = -\mathcal{L}_j^-(q_h, v), \quad (q_h, r)_j = \mathcal{L}_j^+(u_h, r), \tag{3.4}$$

where

$$\mathcal{L}_j^{\pm}(v,r) = (v, r_{xx})_j - \big(v^{\pm}r_x^-\big)_{j+\frac12} + \big(v^{\pm}r_x^+\big)_{j-\frac12} + \big(v_x^{\pm}r^-\big)_{j+\frac12} - \big(v_x^{\pm}r^+\big)_{j-\frac12}. \tag{3.5}$$

Summing up the variational formulations (3.4) over $j=1,2,\ldots,N$ and setting

$$\mathcal{L}^{\pm}(v,r)=\sum_{j=1}^{N}\mathcal{L}_j^{\pm}(v,r), \tag{3.6}$$

we get the global form of the semi-discrete UWLDG scheme

$$\big((u_h)_t, v\big) = -\mathcal{L}^-(q_h, v), \quad (q_h, r) = \mathcal{L}^+(u_h, r). \tag{3.7}$$

3.2. The properties of the UWLDG scheme

In this subsection, we will present some properties of the semi-discrete UWLDG scheme, which can be obtained by integration by parts. More details about the proof can be found in [19]. Here, we only consider periodic boundary conditions for simplicity.

Lemma 3.1. *Suppose the discrete UWLDG operator* $\mathcal{L}^{\pm}$ *is defined by* (3.6)*, then for any* $v, r \in V_h$*, we have*

$$\mathcal{L}^{+}(v,r) = \mathcal{L}^{-}(r,v). \tag{3.8}$$

Proof. From (3.5) and integration by parts, we get

$$\begin{aligned}\mathcal{L}^{+}(v,r) &= \sum_{j=1}^{N}\Big((v,r_{xx})_j - \big(v^{+}r_x^{-}\big)_{j+\frac{1}{2}} + \big(v^{+}r_x^{+}\big)_{j-\frac{1}{2}} + \big(v_x^{+}r^{-}\big)_{j+\frac{1}{2}} - \big(v_x^{+}r^{+}\big)_{j-\frac{1}{2}}\Big)\\ &= \sum_{j=1}^{N}\Big(-(v_x,r_x)_j - \big([v]r_x^{-}\big)_{j+\frac{1}{2}} - \big(v_x^{+}[r]\big)_{j+\frac{1}{2}}\Big).\end{aligned}$$

Here we use the periodic boundary condition. Similarly,

$$\begin{aligned}\mathcal{L}^{-}(r,v) &= \sum_{j=1}^{N}\Big((r,v_{xx})_j - \big(r^{-}v_x^{-}\big)_{j+\frac{1}{2}} + \big(r^{-}v_x^{+}\big)_{j-\frac{1}{2}} + \big(r_x^{-}v^{-}\big)_{j+\frac{1}{2}} - \big(r_x^{-}v^{+}\big)_{j-\frac{1}{2}}\Big)\\ &= \sum_{j=1}^{N}\Big(-(r_x,v_x)_j - \big([r]v_x^{+}\big)_{j+\frac{1}{2}} - \big(r_x^{-}[v]\big)_{j+\frac{1}{2}}\Big).\end{aligned}$$

Thus, the property (3.8) can be obtained. □

Lemma 3.2. *For any pairs of* (u_1,q_1) *and* (u_2,q_2) *belonging to* $V_h \times V_h$*, if*

$$(q_1,r) = \mathcal{L}^{+}(u_1,r), \quad (q_2,r) = \mathcal{L}^{+}(u_2,r),$$

then we have

$$\mathcal{L}^{-}(q_1,u_2) = (q_2,q_1), \quad \mathcal{L}^{-}(q_2,u_1) = (q_1,q_2). \tag{3.9}$$

Proof. It follows from the property (3.8) and the semi-discrete UWLDG scheme (3.7) that

$$\begin{aligned}\mathcal{L}^{-}(q_1,u_2) &= \mathcal{L}^{+}(u_2,q_1) = (q_2,q_1),\\ \mathcal{L}^{-}(q_2,u_1) &= \mathcal{L}^{+}(u_1,q_2) = (q_1,q_2).\end{aligned}$$

The proof is complete. □

Lemma 3.3. *Suppose the discrete UWLDG operator* $\mathcal{L}^{\pm}$ *is defined by* (3.6)*, then we have*

$$\mathcal{L}^{+}\big(\mathbb{P}_h^{+}v - v, r\big) = 0, \quad \mathcal{L}^{-}\big(\mathbb{P}_h^{-}v - v, r\big) = 0. \tag{3.10}$$

Proof. This conclusion can be easily obtained from the definitions of (3.6) and (2.1). □

4. The Fully Discrete Schemes and Their Stability Analysis

In this section, we study the stability of several fully discrete UWLDG schemes. For temporal discretization, we would like to adopt the SDC schemes proposed in [8]. The SDC method is based on second-order time integration methods and the order of accuracy is increased by two for each additional iteration. Here we omit the detailed description of the SDC time discretization methods to save space.

Let $\{t_n = n\tau\}_{n=0}^{M}$ be uniform partition of the time interval $[0,T]$, and τ is time step. To define the SDC method, we then divide the time interval $[t_n, t_{n+1}]$ into P subintervals by setting $t_n = t_{n,0} < t_{n,1} < \cdots < t_{n,P} = t_{n+1}$, where $\{t_{n,m}\}_{m=0}^{P}$ are Gauss-Lobatto nodes on $[t_n, t_{n+1}]$ and $\Delta t_{n,m} = t_{n,m+1} - t_{n,m}$. For a more detailed description of the SDC time discretization methods, the reader is referred to [23]. Denote u_h^n as the numerical solution at time level t^n, the numerical solution u_h^{n+1} is obtained by the following fully discrete SDC-UWLDG schemes.

4.1. Second-order SDC-UWLDG scheme

The second-order SDC-UWLDG scheme is: Find $u_h^{n+1}, q_h^{n+1} \in V_h$ such that, for any $v, r \in V_h$, it holds

$$\big(u_h^{n+1}, v\big) = \big(u_h^n, v\big) - \tau\mathcal{L}^-\left(\frac{1}{2}q_h^{n+1} + \frac{1}{2}q_h^n, v\right), \tag{4.1}$$

$$\big(q_h^{n+1}, r\big) = \mathcal{L}^+\big(u_h^{n+1}, r\big). \tag{4.2}$$

Obviously, it is the Crank-Nicolson time discretization to solve (3.7).

Theorem 4.1. *Let* (u_h^{n+1}, q_h^{n+1}) *be the numerical solution of the second-order SDC-UWLDG scheme* (4.1)-(4.2). *Then we have the unconditional energy stability*

$$\big\|u_h^{n+1}\big\|^2 + \frac{\tau}{2}\big\|q_h^{n+1}\big\|^2 \le \big\|u_h^n\big\|^2 + \frac{\tau}{2}\big\|q_h^n\big\|^2.$$

Proof. Take the test function $v = u_h^{n+1}$ in (4.1) to get the following energy equation:

$$\frac{1}{2}\big\|u_h^{n+1}\big\|^2 - \frac{1}{2}\big\|u_h^n\big\|^2 + \frac{1}{2}\big\|u_h^{n+1} - u_h^n\big\|^2 = -\frac{\tau}{2}\mathcal{L}^-\big(q_h^{n+1} + q_h^n, u_h^{n+1}\big).$$

Then, by properties (3.9) and (4.2), we derive

$$-\frac{\tau}{2}\mathcal{L}^-\big(q_h^{n+1} + q_h^n, u_h^{n+1}\big) = -\frac{\tau}{2}\big\|q_h^{n+1}\big\|^2 - \frac{\tau}{2}\big(q_h^{n+1}, q_h^n\big) \le -\frac{\tau}{4}\big\|q_h^{n+1}\big\|^2 + \frac{\tau}{4}\big\|q_h^n\big\|^2.$$

Therefore,

$$\big\|u_h^{n+1}\big\|^2 + \frac{\tau}{2}\big\|q_h^{n+1}\big\|^2 \le \big\|u_h^n\big\|^2 + \frac{\tau}{2}\big\|q_h^n\big\|^2,$$

which completes the proof. □

4.2. Fourth-order SDC-UWLDG scheme

The fourth-order SDC-UWLDG scheme is: Find $u_h^{n+1}, q_h^{n+1} \in V_h$ such that, for any $v, r \in V_h$, it holds

$$\big(u_h^{n,1}, v\big) = \big(u_h^n, v\big) - \frac{\tau}{2}\mathcal{L}^-\left(\frac{1}{2}q_h^{n,1} + \frac{1}{2}q_h^n, v\right), \tag{4.3a}$$

$$\big(u_h^{n,2},v\big)=\big(u_h^{n,1},v\big)-\frac{\tau}{2}\mathcal{L}^-\left(\frac{1}{2}q_h^{n,2}+\frac{1}{2}q_h^{n,1},v\right), \tag{4.3b}$$

$$\begin{aligned}\big(u_h^{n,3},v\big)=&\big(u_h^{n},v\big)-\frac{\tau}{2}\mathcal{L}^-\left(\frac{1}{2}q_h^{n,3}+\frac{1}{2}q_h^{n},v\right)+\frac{\tau}{2}\mathcal{L}^-\left(\frac{1}{2}q_h^{n,1}+\frac{1}{2}q_h^{n},v\right)\\&-\frac{5\tau}{24}\mathcal{L}^-\big(q_h^{n},v\big)-\frac{\tau}{3}\mathcal{L}^-\big(q_h^{n,1},v\big)+\frac{\tau}{24}\mathcal{L}^-\big(q_h^{n,2},v\big),\end{aligned} \tag{4.3c}$$

$$\begin{aligned}\big(u_h^{n,4},v\big)=&\big(u_h^{n,3},v\big)-\frac{\tau}{2}\mathcal{L}^-\left(\frac{1}{2}q_h^{n,4}+\frac{1}{2}q_h^{n,3},v\right)+\frac{\tau}{2}\mathcal{L}^-\left(\frac{1}{2}q_h^{n,2}+\frac{1}{2}q_h^{n,1},v\right)\\&+\frac{\tau}{24}\mathcal{L}^-\big(q_h^{n},v\big)-\frac{\tau}{3}\mathcal{L}^-\big(q_h^{n,1},v\big)-\frac{5\tau}{24}\mathcal{L}^-\big(q_h^{n,2},v\big),\end{aligned} \tag{4.3d}$$

and

$$\big(q_h^{n,l},r\big)=\mathcal{L}^+\big(u_h^{n,l},r\big),\quad l=0,1,2,3,4. \tag{4.4}$$

Finally, we have $u_h^{n+1}=u_h^{n,4}$ and $q_h^{n+1}=q_h^{n,4}$.

Following [23], we introduce some notations for the convenience of analysis. For any function w^n, let

$$\begin{aligned}&\mathbb{D}_1w^n=w^{n,1}-w^n, &&\mathbb{D}_2w^n=w^{n,2}-w^{n,1},\\&\mathbb{D}_3w^n=2w^{n,3}-w^{n,2}-w^n, &&\mathbb{D}_4w^n=w^{n,4}-w^{n,3},\\&\mathbb{D}_5w^n=w^{n,3}-w^n.\end{aligned} \tag{4.5}$$

Then by some algebraic manipulations, we derive

$$\begin{aligned}\big(\mathbb{D}_1u_h^n,v\big)=&-\frac{\tau}{4}\mathcal{L}^-\big(q_h^{n,1}+q_h^n,v\big),\\\big(\mathbb{D}_2u_h^n,v\big)=&-\frac{\tau}{4}\mathcal{L}^-\big(q_h^{n,2}+q_h^{n,1},v\big),\\\big(\mathbb{D}_3u_h^n,v\big)=&-\frac{\tau}{2}\mathcal{L}^-\left(q_h^{n,3}+\frac{1}{3}q_h^n,v\right)+\frac{\tau}{3}\mathcal{L}^-\big(q_h^{n,2}+q_h^{n,1},v\big),\\\big(\mathbb{D}_4u_h^n,v\big)=&-\frac{\tau}{8}\mathcal{L}^-\big(q_h^{n,1}+q_h^n,v\big)+\frac{\tau}{24}\mathcal{L}^-\big(q_h^{n,2}+q_h^{n,1},v\big)\\&-\frac{\tau}{4}\mathcal{L}^-\left(q_h^{n,3}+\frac{1}{3}q_h^n,v\right)-\frac{\tau}{4}\mathcal{L}^-\big(q_h^{n,4}-q_h^n,v\big),\\\big(\mathbb{D}_5u_h^n,v\big)=&-\frac{\tau}{8}\mathcal{L}^-\big(q_h^{n,1}+q_h^n,v\big)+\frac{\tau}{24}\mathcal{L}^-\big(q_h^{n,2}+q_h^{n,1},v\big)-\frac{\tau}{4}\mathcal{L}^-\left(q_h^{n,3}+\frac{1}{3}q_h^n,v\right).\end{aligned} \tag{4.6}$$

Theorem 4.2. *Let* (u_h^{n+1},q_h^{n+1}) *be the numerical solution of the fourth-order SDC-UWLDG scheme* (4.3)-(4.4). *Then we have the unconditional energy stability*

$$\big\|u_h^{n+1}\big\|^2+\frac{\tau}{4}\big\|q_h^{n+1}\big\|^2\le\big\|u_h^{n}\big\|^2+\frac{\tau}{4}\big\|q_h^{n}\big\|^2. \tag{4.7}$$

Proof. Taking $v=u_h^{n,1}+u_h^n,u_h^{n,2}+u_h^{n,1},3(u_h^{n,3}+u_h^n/3)/2,2u_h^{n,4},-2u_h^n$ in (4.6), respectively, and adding them together, we get the following energy equation:

$$\big\|u_h^{n,4}\big\|^2-\big\|u_h^{n}\big\|^2+S=R,$$

where

$$\begin{aligned}S=&\frac{1}{4}\big\|u_h^{n,2}-u_h^n\big\|^2+\frac{5}{4}\big\|u_h^{n,3}-u_h^n\big\|^2+\frac{3}{4}\big\|u_h^{n,3}-u_h^{n,2}\big\|^2+\big\|u_h^{n,4}-u_h^{n,3}\big\|^2,\\R=&-\frac{\tau}{4}\mathcal{L}^-\big(q_h^{n,1}+q_h^n,u_h^{n,1}+u_h^n\big)-\frac{\tau}{4}\mathcal{L}^-\big(q_h^{n,2}+q_h^{n,1},u_h^{n,2}+u_h^{n,1}\big)\end{aligned}$$

$$+\frac{\tau}{2}\mathcal{L}^-\left(q_h^{n,2}+q_h^{n,1},u_h^{n,3}+\frac{1}{3}u_h^n\right)-\frac{3\tau}{4}\mathcal{L}^-\left(q_h^{n,3}+\frac{1}{3}q_h^n,u_h^{n,3}+\frac{1}{3}u_h^n\right)$$
$$-\frac{\tau}{4}\mathcal{L}^-\left(q_h^{n,1}+q_h^n,u_h^{n,4}-u_h^n\right)+\frac{\tau}{12}\mathcal{L}^-\left(q_h^{n,2}+q_h^{n,1},u_h^{n,4}-u_h^n\right)$$
$$-\frac{\tau}{2}\mathcal{L}^-\left(q_h^{n,3}+\frac{1}{3}q_h^n,u_h^{n,4}-u_h^n\right)-\frac{\tau}{2}\mathcal{L}^-\left(q_h^{n,4}-q_h^n,u_h^{n,4}\right).$$

Denote

$$W^\top=\left(q_h^{n,1}+q_h^n,q_h^{n,2}+q_h^{n,1},q_h^{n,3}+\frac{1}{3}q_h^n,q_h^{n,4}-q_h^n\right).$$

Then, by properties (3.9) and (4.4), we get

$$R=-\tau\int_\Omega W^\top\mathbb{A}W\,dx-\frac{\tau}{4}\left\|q_h^{n,4}\right\|^2+\frac{\tau}{4}\left\|q_h^n\right\|^2,$$

where

$$\mathbb{A}=\begin{pmatrix}\frac{1}{4}&0&0&\frac{1}{8}\\ 0&\frac{1}{4}&-\frac{1}{4}&-\frac{1}{24}\\ 0&-\frac{1}{4}&\frac{3}{4}&\frac{1}{4}\\ \frac{1}{8}&-\frac{1}{24}&\frac{1}{4}&\frac{1}{4}\end{pmatrix}. \tag{4.8}$$

Since $\mathbb{A}$ is symmetric positive definite and $S\geq 0$, the energy equality turns out to be

$$\left\|u_h^{n,4}\right\|^2-\left\|u_h^n\right\|^2\leq-\frac{\tau}{4}\left\|q_h^{n,4}\right\|^2+\frac{\tau}{4}\left\|q_h^n\right\|^2.$$

Note that $(u_h^{n+1},q_h^{n+1})=(u_h^{n,4},q_h^{n,4})$. The stability result (4.7) is obtained. □

5. Error Estimates for Fully Discrete Schemes

With the stability results and properties of the semi-discrete UWLDG operator, the error estimates for the fully discrete SDC-UWLDG schemes will be studied in this section. We will only give the optimal error estimates for the fourth-order scheme (4.3)-(4.4) as an example. The proof for the second-order fully discrete scheme (4.1)-(4.2) is similar and we omit it to save space.

5.1. Reference functions and error equation

Following [23], we first define the reference function associated with the fourth-order SDC time discretization. Let

$$u^{(0)}(x,t)=u(x,t),\quad q^{(0)}(x,t)=q(x,t)$$

be the exact solution of Eq. (3.1), and

$$u^{(1)}(x,t)=u^{(0)}(x,t)-\frac{\tau}{4}\left(q_{xx}^{(1)}(x,t)+q_{xx}^{(0)}(x,t)\right), \tag{5.1a}$$
$$u^{(2)}(x,t)=u^{(1)}(x,t)-\frac{\tau}{4}\left(q_{xx}^{(2)}(x,t)+q_{xx}^{(1)}(x,t)\right), \tag{5.1b}$$

$$u^{(3)}(x,t) = u^{(0)}(x,t) - \frac{\tau}{4}q_{xx}^{(3)}(x,t) + \frac{\tau}{24}q_{xx}^{(2)}(x,t) - \frac{\tau}{12}q_{xx}^{(1)}(x,t) - \frac{5\tau}{24}q_{xx}^{(0)}(x,t), \tag{5.1c}$$

$$q^{(l)}(x,t) = u_{xx}^{(l)}(x,t), \quad l = 1,2,3. \tag{5.1d}$$

Thus, the reference functions satisfy the following property:

$$\begin{aligned} u(x,t+\tau) = u^{(3)}(x,t) &- \frac{\tau}{4}q_{xx}(x,t+\tau) - \frac{\tau}{4}q_{xx}^{(3)}(x,t) + \frac{\tau}{24}q_{xx}^{(2)}(x,t) \\ &- \frac{\tau}{12}q_{xx}^{(1)}(x,t) + \frac{\tau}{24}q_{xx}^{(0)}(x,t) + \varepsilon(x,t), \end{aligned} \tag{5.2}$$

where $\varepsilon(x,t)$ is the local truncation error in time and $\|\varepsilon(x,t)\| \le C\tau^5$. The positive constant C is independent of h and τ but depends on the regularity of u. The proof can be easily obtained by definitions of reference functions and the Taylor expansion.

Take $t = t_n$ and let $y^{n,l} = y^{(l)}(x,t_n), l = 0,1,2,3$ and $y^{n,4} = y(x,t_{n+1})$ for $y = u,q$. At each stage time, the errors between the exact solution and the numerical solution are denoted by

$$e_y^{n,l} = y^{n,l} - y_h^{n,l} = \xi_y^{n,l} - \eta_y^{n,l}, \quad y = u,q,$$

where

$$\begin{aligned} \xi_u^{n,l} &= \left(\mathbb{P}_h^+ u^{n,l} - u_h^{n,l}\right), \quad \eta_u^{n,l} = \left(\mathbb{P}_h^+ u^{n,l} - u^{n,l}\right), \\ \xi_q^{n,l} &= \left(\mathbb{P}_h^- q^{n,l} - q_h^{n,l}\right), \quad \eta_q^{n,l} = \left(\mathbb{P}_h^- q^{n,l} - q^{n,l}\right). \end{aligned}$$

Following the same lines as that in [23], we derive the error equation

$$\begin{aligned} \left(\mathbb{D}_1\xi_u^n, v\right) &= -\frac{\tau}{4}\mathcal{L}^-\left(\xi_q^{n,1} + \xi_q^n, v\right) + \left(\mathbb{D}_1\eta_u^n, v\right), \\ \left(\mathbb{D}_2\xi_u^n, v\right) &= -\frac{\tau}{4}\mathcal{L}^-\left(\xi_q^{n,2} + \xi_q^{n,1}, v\right) + \left(\mathbb{D}_2\eta_u^n, v\right), \\ \left(\mathbb{D}_3\xi_u^n, v\right) &= -\frac{\tau}{2}\mathcal{L}^-\left(\xi_q^{n,3} + \frac{1}{3}\xi_q^n, v\right) + \frac{\tau}{3}\mathcal{L}^-\left(\xi_q^{n,2} + \xi_q^{n,1}, v\right) + \left(\mathbb{D}_3\eta_u^n, v\right), \\ \left(\mathbb{D}_4\xi_u^n, v\right) &= -\frac{\tau}{8}\mathcal{L}^-\left(\xi_q^{n,1} + \xi_q^n, v\right) + \frac{\tau}{24}\mathcal{L}^-\left(\xi_q^{n,2} + \xi_q^{n,1}, v\right) \\ &\quad - \frac{\tau}{4}\mathcal{L}^-\left(\xi_q^{n,3} + \frac{1}{3}\xi_q^n, v\right) - \frac{\tau}{4}\mathcal{L}^-\left(\xi_q^{n,4} - \xi_q^n, v\right) + \left(\mathbb{D}_4\eta_u^n, v\right) + (\varepsilon^n, v), \\ \left(\mathbb{D}_5\xi_u^n, v\right) &= -\frac{\tau}{8}\mathcal{L}^-\left(\xi_q^{n,1} + \xi_q^n, v\right) + \frac{\tau}{24}\mathcal{L}^-\left(\xi_q^{n,2} + \xi_q^{n,1}, v\right) \\ &\quad - \frac{\tau}{4}\mathcal{L}^-\left(\xi_q^{n,3} + \frac{1}{3}\xi_q^n, v\right) + \left(\mathbb{D}_5\eta_u^n, v\right), \end{aligned} \tag{5.3}$$

and

$$\left(\xi_q^{n,l}, r\right) = \mathcal{L}^+\left(\xi_u^{n,l}, r\right) + \left(\eta_q^{n,l}, r\right), \quad l = 0,1,2,3,4, \tag{5.4}$$

where $\varepsilon^n = \varepsilon(x,t_n)$ and the notation $\mathbb{D}_m$ is defined by (4.5). Here we use the property (3.10). In addition, by (2.1) and (2.4) as well as the smoothness of the exact solution u, we have

$$\left\|\eta_u^{n,l}\right\| + \left\|\eta_q^{n,l}\right\| \le Ch^{k+1}, \quad \left\|\mathbb{D}_m\eta_u^n\right\| \le Ch^{k+1}\tau, \quad l = 0,1,2,3,4. \tag{5.5}$$

The positive constant C is independent of h and τ but depends on the regularity of u.

5.2. Estimates for the energy equalities

Lemma 5.1. *There exists a constant $C>0$, independent of h,τ such that*

$$\|\xi_u^{n,4}\|^2-\|\xi_u^n\|^2+\frac{\tau}{8}\|\xi_q^{n,4}\|^2-\frac{\tau}{8}\|\xi_q^n\|^2+\overline{S}\le\frac{\tau}{4}\sum_{l=0}^{4}\|\xi_u^{n,l}\|^2+C\tau(h^{2k+2}+\tau^8), \tag{5.6}$$

where

$$\overline{S}=\frac{1}{4}\|\xi_u^{n,2}-\xi_u^n\|^2+\frac{5}{4}\|\xi_u^{n,3}-\xi_u^n\|^2+\frac{3}{4}\|\xi_u^{n,3}-\xi_u^{n,2}\|^2+\|\xi_u^{n,4}-\xi_u^{n,3}\|^2.$$

Proof. For the Eq. (5.3), taking test functions $v=\xi_u^{n,1}+\xi_u^n,\xi_u^{n,2}+\xi_u^{n,1}$, $3(\xi_u^{n,3}+\xi_u^n/3)/2$, $2\xi_u^{n,4}$, $-2\xi_u^n$, respectively, and adding them together, we get the following energy equality:

$$\|\xi_u^{n,4}\|^2-\|\xi_u^n\|^2+\overline{S}=\overline{R}+\overline{T},$$

where

$$\begin{aligned}
\overline{S}=&\frac{1}{4}\|\xi_u^{n,2}-\xi_u^n\|^2+\frac{5}{4}\|\xi_u^{n,3}-\xi_u^n\|^2+\frac{3}{4}\|\xi_u^{n,3}-\xi_u^{n,2}\|^2+\|\xi_u^{n,4}-\xi_u^{n,3}\|^2,\\
\overline{R}=&-\frac{\tau}{4}\mathcal{L}^-\big(\xi_q^{n,1}+\xi_q^n,\xi_u^{n,1}+\xi_u^n\big)-\frac{\tau}{4}\mathcal{L}^-\big(\xi_q^{n,2}+\xi_q^{n,1},\xi_u^{n,2}+\xi_u^{n,1}\big)\\
&+\frac{\tau}{2}\mathcal{L}^-\left(\xi_q^{n,2}+\xi_q^{n,1},\xi_u^{n,3}+\frac{1}{3}\xi_u^n\right)-\frac{3\tau}{4}\mathcal{L}^-\left(\xi_q^{n,3}+\frac{1}{3}\xi_q^n,\xi_u^{n,3}+\frac{1}{3}\xi_u^n\right)\\
&-\frac{\tau}{4}\mathcal{L}^-\big(\xi_q^{n,1}+\xi_q^n,\xi_u^{n,4}-\xi_u^n\big)+\frac{\tau}{12}\mathcal{L}^-\big(\xi_q^{n,2}+\xi_q^{n,1},\xi_u^{n,4}-\xi_u^n\big)\\
&-\frac{\tau}{2}\mathcal{L}^-\left(\xi_q^{n,3}+\frac{1}{3}\xi_q^n,\xi_u^{n,4}-\xi_u^n\right)-\frac{\tau}{2}\mathcal{L}^-\big(\xi_q^{n,4}-\xi_q^n,\xi_u^{n,4}\big),\\
\overline{T}=&\big(\mathbb{D}_1\eta_u^n,\xi_u^{n,1}+\xi_u^n\big)+\big(\mathbb{D}_2\eta_u^n,\xi_u^{n,2}+\xi_u^{n,1}\big)+\frac{3}{2}\left(\mathbb{D}_3\eta_u^n,\xi_u^{n,3}+\frac{1}{3}\xi_u^n\right)\\
&+2\big(\mathbb{D}_4\eta_u^n,\xi_u^{n,4}\big)-2\big(\mathbb{D}_5\eta_u^n,\xi_u^n\big)+2\big(\varepsilon^n,\xi_u^{n,4}\big).
\end{aligned}$$

Denote

$$\begin{aligned}
\overline{W}^\top&=\left(\xi_q^{n,1}+\xi_q^n,\xi_q^{n,2}+\xi_q^{n,1},\xi_q^{n,3}+\frac{1}{3}\xi_q^n,\xi_q^{n,4}-\xi_q^n\right),\\
\overline{V}^\top&=\left(\eta_q^{n,1}+\eta_q^n,\eta_q^{n,2}+\eta_q^{n,1},\eta_q^{n,3}+\frac{1}{3}\eta_q^n,\eta_q^{n,4}-\eta_q^n\right).
\end{aligned}$$

Then by (5.4) and (3.9), we get

$$\begin{aligned}
\overline{R}=&-\tau\int_\Omega\overline{W}^\top\mathbb{A}\overline{W}dx+\tau\int_\Omega\overline{V}^\top\mathbb{A}\overline{W}dx-\frac{\tau}{4}\|\xi_q^{n,4}\|^2\\
&+\frac{\tau}{4}\|\xi_q^n\|^2+\frac{\tau}{4}\big(\eta_q^{n,4},\xi_q^{n,4}\big)-\frac{\tau}{4}\big(\eta_q^n,\xi_q^n\big),
\end{aligned}$$

where $\mathbb{A}$ is defined by (4.8). In addition, by the Cauchy-Schwarz inequality and Young's inequality, we obtain

$$\overline{T}\le\frac{\tau}{4}\sum_{l=0}^{4}\|\xi_u^{n,l}\|^2+\frac{C}{\tau}\left(\sum_{l=1}^{5}\|\mathbb{D}_l\eta_u^n\|^2+\|\varepsilon^n\|^2\right).$$

Thus, the energy equality turns out to be

$$\begin{aligned}
&\|\xi_u^{n,4}\|^2 - \|\xi_u^n\|^2 + \overline{S} \\
&\le -\tau \int_\Omega \overline{W}^\top \mathbb{A}\overline{W}dx + \tau \int_\Omega \overline{V}^\top \mathbb{A}\overline{W}dx \\
&\quad - \frac{\tau}{4}\|\xi_q^{n,4}\|^2 + \frac{\tau}{4}\|\xi_q^n\|^2 + \frac{\tau}{4}(\eta_q^{n,4}, \xi_q^{n,4}) - \frac{\tau}{4}(\eta_q^n, \xi_q^n) \\
&\quad + \frac{\tau}{4}\sum_{l=0}^{4}\|\xi_u^{n,l}\|^2 + \frac{C}{\tau}\left(\sum_{l=1}^{5}\|\mathbb{D}_l\eta_u^n\|^2 + \|\varepsilon^n\|^2\right).
\end{aligned}$$

By the approximation property (5.5), Cauchy-Schwarz inequality and Young's inequality, we derive

$$\begin{aligned}
&\|\xi_u^{n,4}\|^2 - \|\xi_u^n\|^2 + \frac{\tau}{8}\|\xi_q^{n,4}\|^2 - \frac{\tau}{8}\|\xi_q^n\|^2 + \overline{S} \\
\le\ &\frac{\tau}{4}\sum_{l=0}^{4}\|\xi_u^{n,l}\|^2 - \tau\int_\Omega \overline{W}^\top\left(\mathbb{A} - \frac{1}{20}\mathbb{I}\right)\overline{W}dx + C\tau(h^{2k+2} + \tau^8).
\end{aligned}$$

Noting that $\mathbb{A} - \mathbb{I}/20$ is positive definite, we get

$$\|\xi_u^{n,4}\|^2 - \|\xi_u^n\|^2 + \frac{\tau}{8}\|\xi_q^{n,4}\|^2 - \frac{\tau}{8}\|\xi_q^n\|^2 + \overline{S} \le \frac{\tau}{4}\sum_{l=0}^{4}\|\xi_u^{n,l}\|^2 + C\tau(h^{2k+2} + \tau^8).$$

The proof is complete. □

Similarly, we have the following estimates for $\xi_u^{n,l}, l = 1, 2, 3$.

Lemma 5.2. *There exists a constant $C > 0$, independent of h, τ such that*

$$\|\xi_u^{n,l}\|^2 \le C\|\xi_u^n\|^2 + C\tau h^{2k+2}, \quad l = 1, 2, 3. \tag{5.7}$$

For the initial condition, we take $q_h(x, 0) = \mathbb{P}_h^- q(x, 0)$ with $q(x, 0) = u_{xx}(x, 0)$, and $u_h(x, 0)$ to satisfy

$$(q_h, r)_j = \mathcal{L}_j^+(u_h, r), \quad \int_\Omega u_h(x, 0)dx = \int_\Omega u(x, 0)dx, \quad r \in V_h. \tag{5.8}$$

Then we have the following property.

Lemma 5.3. *Suppose $u(x, 0)$ is sufficiently smooth, then $u_h(x, 0)$ is well defined and we have*

$$\begin{aligned}
&\|u(x, 0) - u_h(x, 0)\| \le Ch^{k+1}, \\
&\|q(x, 0) - q_h(x, 0)\| \le Ch^{k+1}.
\end{aligned}$$

Proof. The proof of this lemma follows from [9, Lemma 5.1], but with a slight difference. We only show the error estimates for the initial condition, and the well-posedness can be obtained in a similar way. From the property of the special projection, we have known that

$$\|q(x, 0) - q_h(x, 0)\| = \|q(x, 0) - \mathbb{P}_h^- q(x, 0)\| \le Ch^{k+1}.$$

For the estimate of $u_h(x, 0)$, consider the elliptic linear problem

$$\zeta^* = \xi_{xx} \quad \text{in } \Omega, \quad \eta^* = \zeta_{xx}^* \quad \text{in } \Omega$$

with periodic boundary conditions. In addition, we assume the elliptic regularity result

$$\|\zeta^*\|_{H^2(\Omega)} + \|\xi^*\|_{H^4(\Omega)} \le C_* \|\eta^*\|.$$

Taking $\eta^* = u - u_h$ in the elliptic linear problem, we have

$$\begin{aligned}
&(u-u_h, u-u_h)_j \\
&= \big(u-u_h, \zeta^*_{xx}\big)_j \\
&= \big(u-u_h, (\zeta^* - \mathbb{P}_h^- \zeta^*)_{xx}\big)_j \\
&\quad + \big(u-u_h, (\mathbb{P}_h^- \zeta^*)_{xx}\big)_j.
\end{aligned}$$

By integration by parts and the property of the projection in (2.3), we derive

$$\begin{aligned}
&\big(u-u_h, (\zeta^* - \mathbb{P}_h^- \zeta^*)_{xx}\big)_j \\
&= \big((u-u_h)_{xx}, \zeta^* - \mathbb{P}_h^- \zeta^*\big)_j \\
&\quad - (u-u_h)(\zeta^* - \mathbb{P}_h^- \zeta^*)_x\big|^{+}_{j-\frac{1}{2}} \\
&\quad + (u-u_h)_x(\zeta^* - \mathbb{P}_h^- \zeta^*)\big|^{+}_{j-\frac{1}{2}}.
\end{aligned} \tag{5.9}$$

Recalling the error equation

$$\begin{aligned}
(q-q_h, r)_j = {}& (u-u_h, r_{xx})_j - (u-u_h)^+ r_x^-\big|_{j+\frac{1}{2}} + (u-u_h)^+ r_x^+\big|_{j-\frac{1}{2}} \\
& + (u-u_h)_x^+ r^-\big|_{j+\frac{1}{2}} - (u-u_h)_x^+ r^+\big|_{j-\frac{1}{2}},
\end{aligned}$$

and taking $r = \mathbb{P}_h^- \zeta^*$, we have

$$\begin{aligned}
&\big(u-u_h, (\mathbb{P}_h^- \zeta^*)_{xx}\big)_j \\
&= (q-q_h, \mathbb{P}_h^- \zeta^*)_j + (u-u_h)^+ (\mathbb{P}_h^- \zeta^*)_x^-\big|_{j+\frac{1}{2}} \\
&\quad - (u-u_h)^+ (\mathbb{P}_h^- \zeta^*)_x^+\big|_{j-\frac{1}{2}} - (u-u_h)_x^+ (\mathbb{P}_h^- \zeta^*)^-\big|_{j+\frac{1}{2}} \\
&\quad + (u-u_h)_x^+ (\mathbb{P}_h^- \zeta^*)^+\big|_{j-\frac{1}{2}}.
\end{aligned} \tag{5.10}$$

Combining (5.9) and (5.10), by the continuity of ζ^* and the property of the projection $\mathbb{P}_h^-$, we obtain

$$\begin{aligned}
&(u-u_h, u-u_h) \\
&= \big((u - \mathbb{P}_h^- u)_{xx}, \zeta^* - \mathbb{P}_h^- \zeta^*\big) + (q-q_h, \mathbb{P}_h^- \zeta^* - \zeta^*) + (q-q_h, \zeta^*) \\
&\le Ch^{k+1}\|\zeta^*\|_{H^2(\Omega)} + Ch^{k+3}\|\zeta^*\|_{H^2(\Omega)} + Ch^{k+1}\|\zeta^*\| \\
&\le Ch^{k+1}\|\zeta^*\|_{H^2(\Omega)} \le Ch^{k+1}\|u-u_h\|,
\end{aligned}$$

where periodic boundary conditions have been used. Consequently,

$$\|u(x,0) - u_h(x,0)\| \le Ch^{k+1}.$$

The proof is complete. □

Now we can obtain the optimal error estimates of the fourth-order fully discrete SDC-UWLDG scheme (4.3)-(4.4).

Theorem 5.1. *Let u be the exact solution of* (1.1) *and sufficiently smooth with bounded derivatives. Suppose u_h^n is the numerical solution of the fully discrete scheme* (4.3)-(4.4) *and the initial condition satisfies* (5.8). *Let V_h be the space of piecewise polynomials $P^k, k \geq 1$. Then it holds that*

$$\max_{t_n \leq T} \left\| u(x,t_n) - u_h^n \right\| \leq C(h^{k+1} + \tau^4). \tag{5.11}$$

Proof. It follows from estimates (5.6) and (5.7) that

$$\left\|\xi_u^{n,4}\right\|^2 - \left\|\xi_u^n\right\|^2 + \frac{\tau}{8}\left\|\xi_q^{n,4}\right\|^2 - \frac{\tau}{8}\left\|\xi_q^n\right\|^2 \leq C\tau\left\|\xi_u^n\right\|^2 + C\tau(h^{2k+2} + \tau^8),$$

here we use the fact that

$$\frac{\tau}{4}\left\|\xi_u^{n,4}\right\|^2 \leq \frac{\tau}{2}\left\|\xi_u^{n,4} - \xi_u^{n,3}\right\|^2 + \frac{\tau}{2}\left\|\xi_u^{n,3}\right\|^2 \leq \overline{S} + \tau\left\|\xi_u^{n,3}\right\|^2,$$

if $\tau < 1$. In addition, by the estimates of the initial condition in Lemma 5.3, we have

$$\begin{aligned}
\left\|\xi_u^0\right\| &\leq \|u_h(x,0) - u(x,0)\| + \|u(x,0) - \mathbb{P}_h^+ u(x,0)\| \leq Ch^{k+1}, \\
\left\|\xi_q^0\right\| &\leq \|q_h(x,0) - q(x,0)\| + \|q(x,0) - \mathbb{P}_h^- q(x,0)\| \leq Ch^{k+1}.
\end{aligned}$$

Combining the above estimates and using the discrete Gronwall inequality, we can derive

$$\left\|\xi_u^{n,4}\right\|^2 \leq C(h^{2k+2} + \tau^8).$$

Finally, by the triangle inequality and properties (5.5), we arrive at the optimal error estimate (5.11). □

6. Extension to PDEs with Even-order Derivatives

In this section, we generalize the stability analysis and error estimate to PDEs with even-order derivatives. Consider the following problem:

$$u_t + (-1)^{\frac{s}{2}} u_x^{(s)} = 0 \tag{6.1}$$

with periodic boundary conditions and $s \geq 4$ is an even number. $u_x^{(s)}$ denotes the s-th order derivative with respect to x.

6.1. Semi-discrete UWLDG scheme for the general even-order PDE

Following [14], we rewrite (6.1) into an $s/2$-th-order system. Let $q = u_x^{(s/2)}$, we have

$$\begin{aligned}
&u_t + (-1)^{\frac{s}{2}} q_x^{(\frac{s}{2})} = 0, \\
&q - u_x^{(\frac{s}{2})} = 0.
\end{aligned} \tag{6.2}$$

Then we can get the semi-discrete UWLDG scheme: Find $u_h, q_h \in V_h$ such that, for any $v, r \in V_h$, we have

$$\begin{aligned}
\big((u_h)_t, v\big)_j = &-\big(q_h, v_x^{(\frac{s}{2})}\big)_j \\
&- \sum_{m=0}^{\frac{s}{2}-1} \Big((-1)^{\frac{s}{2}+m}\Big(\widetilde{q}_x^{\,(\frac{s}{2}-1-m)}\big(v_x^{(m)}\big)^-\big|_{j+\frac{1}{2}} - \widetilde{q}_x^{\,(\frac{s}{2}-1-m)}\big(v_x^{(m)}\big)^+\big|_{j-\frac{1}{2}}\Big)\Big),
\end{aligned}$$

$$(q_h, r)_j = (-1)^{\frac{s}{2}}\big(u_h, r_x^{(\frac{s}{2})}\big)_j - \sum_{m=0}^{\frac{s}{2}-1}\Big((-1)^{1+m}\Big(\widehat{u_x}^{(\frac{s}{2}-1-m)}\big(r_x^{(m)}\big)^-\big|_{j+\frac12} - \widehat{u_x}^{(\frac{s}{2}-1-m)}\big(r_x^{(m)}\big)^+\big|_{j-\frac12}\Big)\Big).$$

Here $\widetilde{q_x}^{(s/2-1-m)}$ and $\widehat{u_x}^{(s/2-1-m)}$ are the numerical flux, which can be chosen as

$$\widetilde{q_x}^{(\frac{s}{2}-1-m)} = \big(q_x^{(\frac{s}{2}-1-m)}\big)^-, \quad \widehat{u_x}^{(\frac{s}{2}-1-m)} = \big(u_x^{(\frac{s}{2}-1-m)}\big)^+, \quad m = 0, 1, \ldots, \frac{s}{2}-1. \tag{6.3}$$

For the convenience of analysis, we rewrite the UWLDG scheme as the following form:

$$\begin{aligned}&\big((u_h)_t, v\big) = -\mathcal{L}_s^-(q_h, v),\\ &(q_h, r) = (-1)^{\frac{s}{2}}\mathcal{L}_s^+(u_h, r),\end{aligned} \tag{6.4}$$

where

$$\mathcal{L}_s^\pm(v, r) = \sum_{j=1}^N \mathcal{L}_{s,j}^\pm(v, r), \tag{6.5}$$

and

$$\mathcal{L}_{s,j}^\pm(v, r) = \big(v, r_x^{(\frac{s}{2})}\big)_j + \sum_{m=0}^{\frac{s}{2}-1}\Big((-1)^{\frac{s}{2}+m}\Big(\big(v_x^{(\frac{s}{2}-1-m)}\big)^\pm_{j+\frac12}\big(r_x^{(m)}\big)^-_{j+\frac12} - \big(v_x^{(\frac{s}{2}-1-m)}\big)^\pm_{j-\frac12}\big(r_x^{(m)}\big)^+_{j-\frac12}\Big)\Big).$$

Next, we introduce the properties of the operator $\mathcal{L}_s^\pm$, which will play an important role in the stability analysis.

Lemma 6.1. *For any $r, v \in V_h$, we have*

$$\mathcal{L}_s^+(r, v) = (-1)^{\frac{s}{2}}\mathcal{L}_s^-(v, r). \tag{6.6}$$

Proof. From the definition of the operator $\mathcal{L}_s^\pm$ and by integration by parts, we get

$$\begin{aligned}\mathcal{L}_s^+(r, v) &= \big(r, v_x^{(\frac{s}{2})}\big) + \sum_{j=1}^N\sum_{m=0}^{\frac{s}{2}-1}(-1)^{\frac{s}{2}+m}\Big(\big(r_x^{(\frac{s}{2}-1-m)}\big)^+\big(v_x^{(m)}\big)^-\big|_{j+\frac12} - \big(r_x^{(\frac{s}{2}-1-m)}\big)^+\big(v_x^{(m)}\big)^+\big|_{j-\frac12}\Big)\\
&= (-1)^{\frac{s}{2}}\big(r_x^{(\frac{s}{2})}, v\big) + \sum_{j=1}^N\sum_{m=0}^{\frac{s}{2}-1}(-1)^m\Big(r_x^{(m)}v_x^{(\frac{s}{2}-1-m)}\big|^-_{j+\frac12} - r_x^{(m)}v_x^{(\frac{s}{2}-1-m)}\big|^+_{j-\frac12}\Big)\\
&\quad + \sum_{j=1}^N\sum_{m=0}^{\frac{s}{2}-1}(-1)^{\frac{s}{2}+m}\Big(\big(r_x^{(\frac{s}{2}-1-m)}\big)^+\big(v_x^{(m)}\big)^-\big|_{j+\frac12} - \big(r_x^{(\frac{s}{2}-1-m)}\big)^+\big(v_x^{(m)}\big)^+\big|_{j-\frac12}\Big)\\
&= (-1)^{\frac{s}{2}}\big(r_x^{(\frac{s}{2})}, v\big) - \sum_{j=1}^N\sum_{m=0}^{\frac{s}{2}-1}(-1)^m\big[r_x^{(m)}\big]\big(v_x^{(\frac{s}{2}-1-m)}\big)^-\big|_{j+\frac12}.\end{aligned}$$

By periodic boundary conditions, we can derive

$$\mathcal{L}_s^-(v, r) = \big(v, r_x^{(\frac{s}{2})}\big) - \sum_{j=1}^N\sum_{m=0}^{\frac{s}{2}-1}(-1)^{\frac{s}{2}+m}\big[r_x^{(m)}\big]\big(v_x^{(\frac{s}{2}-1-m)}\big)^-\big|_{j+\frac12},$$

which completes the proof. $\square$

By the aid of Lemma 6.1 and the definition of the operator (6.5), we can easily derive the following property.

Lemma 6.2. *For any* $(u_1, q_1) \in V_h \times V_h$, *if*

$$(q_1, r) = (-1)^{\frac{s}{2}} \mathcal{L}_s^+(u_1, r), \quad (q_2, r) = (-1)^{\frac{s}{2}} \mathcal{L}_s^+(u_2, r),$$

then we have

$$\mathcal{L}_s^-(q_1, u_2) = (q_2, q_1), \quad \mathcal{L}_s^-(q_2, u_1) = (q_1, q_2). \tag{6.7}$$

For the optimal error estimates, the projection plays a key role and will be defined as follows, corresponding to the UWLDG scheme of the general even-order PDEs. For any periodic function u, the projection $\mathbb{R}_h^{\pm} u \in V_h$ satisfies

$$\begin{aligned} &\big(\mathbb{R}_h^+ u, v\big)_j = (u, v)_j, \quad \big(\mathbb{R}_h^+ u\big)_x^{(\frac{i}{2}-1)}\big(x_{j-\frac{1}{2}}^+\big) = u_x^{(\frac{i}{2}-1)}\big(x_{j-\frac{1}{2}}^+\big), \quad v \in P^{k-\frac{s}{2}}(I_j), \\ &\big(\mathbb{R}_h^- u, v\big)_j = (u, v)_j, \quad \big(\mathbb{R}_h^- u\big)_x^{(\frac{i}{2}-1)}\big(x_{j+\frac{1}{2}}^-\big) = u_x^{(\frac{i}{2}-1)}\big(x_{j+\frac{1}{2}}^-\big), \quad v \in P^{k-\frac{s}{2}}(I_j), \end{aligned}$$

where $i = 2, 4, \ldots, s$. The above projection is a local projection and we can easily get the unique existence and optimal approximation properties of the projection [3, 5],

$$\|\eta\|_{H_h^m(\Omega)} \le C h^{\min\{k+1, r\} - m} \|u\|_{H^r(\Omega)},$$

where $\eta = \mathbb{R}_h^{\pm} u - u, 0 \le m \le \min\{r, k+1\}$. The positive constant C is independent of h and u. By the special projection $\mathbb{R}_h^{\pm} u$ and the definition of $\mathcal{L}_s^{\pm}$ in (6.5), we obtain the following results directly.

Lemma 6.3. *Suppose* $\mathcal{L}_s^{\pm}$ *is defined by* (6.5), *then we have*

$$\mathcal{L}_s^+\big(\mathbb{R}_h^+ v - v, r\big) = 0, \quad \mathcal{L}_s^-\big(\mathbb{R}_h^- v - v, r\big) = 0. \tag{6.8}$$

6.2. Fully discrete scheme for the general even-order PDE

In this subsection, we would like to show the unconditional stability and optimal error estimate of the fully discrete scheme for the general even-order PDE (6.1), in which the SDC temporal discretization and UWLDG spatial discretization are employed.

The second-order SDC-UWLDG scheme is defined as follows: Find $u_h^{n+1}, q_h^{n+1} \in V_h$ such that, for any $v, r \in V_h$, it holds

$$\begin{aligned} &\big(u_h^{n+1}, v\big) = \big(u_h^n, v\big) - \tau \mathcal{L}_s^-\left(\frac{1}{2} q_h^{n+1} + \frac{1}{2} q_h^n, v\right), \\ &\big(q_h^{n+1}, r\big) = (-1)^{\frac{s}{2}} \mathcal{L}_s^+\big(u_h^{n+1}, r\big). \end{aligned} \tag{6.9}$$

The fourth-order SDC-UWLDG scheme is defined as follows: Find $u_h^{n+1}, q_h^{n+1} \in V_h$ such that, for any $v, r \in V_h$, it holds

$$\big(u_h^{n,1}, v\big) = \big(u_h^n, v\big) - \frac{\tau}{2} \mathcal{L}_s^-\left(\frac{1}{2} q_h^{n,1} + \frac{1}{2} q_h^n, v\right), \tag{6.10a}$$

$$\big(u_h^{n,2}, v\big) = \big(u_h^{n,1}, v\big) - \frac{\tau}{2} \mathcal{L}_s^-\left(\frac{1}{2} q_h^{n,2} + \frac{1}{2} q_h^{n,1}, v\right), \tag{6.10b}$$

$$\begin{aligned}\left(u_h^{n,3},v\right)=&\left(u_h^n,v\right)-\frac{\tau}{2}\mathcal{L}_s^-\left(\frac{1}{2}q_h^{n,3}+\frac{1}{2}q_h^n,v\right)+\frac{\tau}{2}\mathcal{L}_s^-\left(\frac{1}{2}q_h^{n,1}+\frac{1}{2}q_h^n,v\right)\\&-\frac{5\tau}{24}\mathcal{L}_s^-\left(q_h^n,v\right)-\frac{\tau}{3}\mathcal{L}_s^-\left(q_h^{n,1},v\right)+\frac{\tau}{24}\mathcal{L}_s^-\left(q_h^{n,2},v\right),\end{aligned}\tag{6.10c}$$

$$\begin{aligned}\left(u_h^{n,4},v\right)=&\left(u_h^{n,3},v\right)-\frac{\tau}{2}\mathcal{L}_s^-\left(\frac{1}{2}q_h^{n,4}+\frac{1}{2}q_h^{n,3},v\right)+\frac{\tau}{2}\mathcal{L}_s^-\left(\frac{1}{2}q_h^{n,2}+\frac{1}{2}q_h^{n,1},v\right)\\&+\frac{\tau}{24}\mathcal{L}_s^-\left(q_h^n,v\right)-\frac{\tau}{3}\mathcal{L}_s^-\left(q_h^{n,1},v\right)-\frac{5\tau}{24}\mathcal{L}_s^-\left(q_h^{n,2},v\right),\end{aligned}\tag{6.10d}$$

and

$$\left(q_h^{n,l},r\right)=(-1)^{\frac{s}{2}}\mathcal{L}_s^+\left(u_h^{n,l},r\right),\quad l=0,1,2,3,4.\tag{6.11}$$

Finally, we have $u_h^{n+1}=u_h^{n,4}$ and $q_h^{n+1}=q_h^{n,4}$.

Along the same lines with the proof of Theorems 4.1-4.2, we get the stability result by property (6.7).

Theorem 6.1. *We have the unconditional stability*

$$\left\|u_h^{n+1}\right\|^2+\frac{\tau}{p}\left\|q_h^{n+1}\right\|^2\le\left\|u_h^n\right\|^2+\frac{\tau}{p}\left\|q_h^n\right\|^2,$$

where $p=2$ if u_h^n and q_h^n are the numerical solution of the second-order SDC-UWLDG scheme (6.9) *and $p=4$ for the fourth-order SDC-UWLDG scheme* (6.10)-(6.11).

For the initial condition, we take $q_h(x,0)=\mathbb{R}_h^-q(x,0)$ with $q(x,0)=u_x^{(s/2)}(x,0)$, and $u_h(x,0)$ to satisfy

$$(q_h,r)_j=(-1)^{\frac{s}{2}}\mathcal{L}_{s,j}^+(u_h,r),\quad\int_\Omega u_h(x,0)dx=\int_\Omega u(x,0)dx,\quad r\in V_h.\tag{6.12}$$

Following the same lines as that in Lemma 5.3, we have the optimal error estimates for the initial solution.

Lemma 6.4. *Suppose $u(x,0)$ is sufficiently smooth, then $u_h(x,0)$ is well defined and we have*

$$\begin{aligned}&\|u(x,0)-u_h(x,0)\|\le Ch^{k+1},\\&\|q(x,0)-q_h(x,0)\|\le Ch^{k+1}.\end{aligned}$$

Similar to the fourth-order equations, we also obtain the optimal error estimates of the fully discrete scheme for the general even-order problems (6.1).

Theorem 6.2. *Suppose u is the exact solution of* (6.1) *and sufficiently smooth with bounded derivatives. Let V_h be the space of piecewise polynomials P^k, $k\ge s/2-1$. If the initial conditions are taken as* (6.12)*, then it holds that*

$$\max_{t_n\le T}\left\|u(x,t_n)-u_h^n\right\|\le C(h^{k+1}+\tau^p),$$

where $p=2$ if u_h^n is the numerical solution of the second-order SDC-UWLDG scheme (6.9) *and $p=4$ for the fourth-order SDC-UWLDG scheme* (6.10)-(6.11).

Remark 6.1. We remark that extension to PDEs with high odd-order derivatives is not trivial. The main technique used in the analysis is the symmetric and dissipative properties of the semi-discrete UWLDG scheme, which plays a key role in establishing negative definite quadratic forms for the semi-implicit discretization of the high order derivative part. However, for PDEs with high odd-order derivatives, e.g. KdV equations, the space operator is not symmetric, thus the energy equations for the fully discrete schemes are not easily derived.

Remark 6.2. Following the same idea as that in one-dimensional space, the stability analysis and optimal error estimate of the fully discrete SDC-UWLDG schemes for the general even-order problems can be easily extended to multidimensional Cartesian meshes with Q^k elements, where Q^k means the tensor product of polynomials of degree k in each variable.

7. Numerical Results

In this section, we give some numerical examples to verify the theoretical analysis of the fully discrete SDC-UWLDG schemes.

7.1. The fourth-order equation

Example 7.1. Consider the fourth-order equation

$$\begin{aligned} &u_t + u_{xxxx} = 0, \quad (x,t) \in [0,2\pi] \times [0,2\pi], \\ &u(x,0) = \sin(x), \quad x \in [0,2\pi] \end{aligned}$$

with periodic boundary conditions. The exact solution is

$$u(x,t) = e^{-t}\sin(x).$$

The flux is chosen as (3.3). The final time is $T = 2\pi$ with the time step $\tau = 0.4h$, where $h = 2\pi/N, N = 8, 16, 32, 64$. The errors and the corresponding orders of accuracy of the numerical solution are shown in Table 7.1. We observe that when $k \geq 1$, the fully discrete SDC-UWLDG schemes can reach optimal convergence rate, which agrees with the theoretical results.

Table 7.1: Errors and orders of accuracy for the fourth-order equation in one dimension.

	N	L^1 error	Order	L^2 error	Order	L^∞ error	Order
P^1	8	2.78E-02	–	1.25E-01	–	7.04E-02	–
	16	1.24E-02	1.17	5.50E-02	1.19	3.10E-02	1.18
	32	3.60E-03	1.78	1.60E-02	1.78	9.00E-03	1.78
	64	9.35E-04	1.95	4.14E-03	1.95	2.34E-03	1.95
P^2	8	6.42E-03	–	1.15E-02	–	2.57E-02	–
	16	1.10E-03	2.54	1.96E-03	2.55	4.41E-03	2.54
	32	1.52E-04	2.86	2.69E-04	2.86	6.08E-04	2.86
	64	1.95E-05	2.96	3.45E-05	2.96	7.79E-05	2.96
P^3	8	1.16E-03	–	5.07E-04	–	2.77E-04	–
	16	6.79E-05	4.10	3.01E-05	4.08	1.69E-05	4.04
	32	4.14E-06	4.04	1.83E-06	4.04	1.03E-06	4.03
	64	2.57E-07	4.01	1.14E-07	4.01	6.43E-08	4.01

Example 7.2. We consider the two-dimensional fourth-order equation

$$\begin{aligned} &u_t + \Delta^2 u = 0, \quad (x,y) \in [0,2\pi] \times [0,2\pi], \\ &u(x,y,0) = \sin(x)\sin(y) \end{aligned}$$

Table 7.2: Errors and orders of accuracy for the fourth-order equation in two dimension.

	$N \times N$	L^1 error	Order	L^2 error	Order	L^∞ error	Order
P^1	8×8	4.09E-00	–	7.75E-01	–	2.30E-01	–
	16×16	1.17E-00	1.80	2.28E-01	1.76	7.14E-02	1.69
	32×32	3.04E-01	1.95	5.96E-02	1.94	1.89E-02	1.92
	64×64	7.68E-02	1.99	1.51E-02	1.98	4.79E-03	1.98
P^2	8×8	7.50E-01	–	1.30E-01	–	2.99E-02	–
	16×16	9.52E-02	2.98	1.68E-02	2.96	3.94E-03	2.92
	32×32	1.19E-02	3.00	2.10E-03	3.00	4.86E-04	3.02
	64×64	1.48E-03	3.00	2.69E-04	3.01	6.13E-05	2.99
P^3	8×8	7.42E-02	–	1.40E-02	–	4.22E-03	–
	16×16	4.40E-03	4.08	8.18E-04	4.09	2.46E-04	4.10
	32×32	3.01E-04	3.87	5.42E-05	3.92	1.50E-05	4.04
	64×64	1.92E-05	3.97	3.41E-06	3.99	9.36E-07	4.00

with periodic boundary conditions. The exact solution is

$$u(x, y, t) = e^{-4t} \sin(x) \sin(y).$$

The final time is $T = 2\pi$ with the time step $\tau = h$. The errors and orders of accuracy of the numerical solution are shown in Table 7.2. We can see the optimal convergence rate for the fully discrete SDC-UWLDG schemes when $k \geq 1$.

7.2. The sixth-order equation

Example 7.3. We consider a one-dimensional sixth-order equation

$$\begin{aligned} &u_t - u_{xxxxxx} = 0, \quad (x,t) \in [0, 2\pi] \times [0, 2\pi], \\ &u(x, 0) = \sin(x), \qquad x \in [0, 2\pi] \end{aligned}$$

with periodic boundary conditions. The exact solution is

$$u(x, t) = e^{-t} \sin(x).$$

The numerical flux is chosen as (6.3) with $s = 6$. We compute the problem up to $T = 2\pi$ with the time step $\tau = 0.4h$. The errors and the corresponding orders of accuracy for the fourth-order scheme (6.10) are shown in Table 7.3. Optimal orders of accuracy can be observed from Table 7.3.

We also show temporal accuracy test of the SDC-UWLDG scheme for the sixth-order equation. In this test, the finite element space with $k = 3$ and $N = 320$ will be used to make sure that the error of time discretization is dominant. The final time is $T = 10$, and the errors and orders of accuracy are presented in Table 7.4.

Table 7.3: Errors and orders of accuracy for the sixth-order equation in one dimension.

	N	L^1 error	Order	L^2 error	Order	L^∞ error	Order
P^3	8	1.14E-02	–	4.98E-03	–	2.71E-03	–
	16	7.34E-04	3.96	3.24E-04	3.94	1.81E-04	3.90
	32	4.59E-05	4.00	2.03E-05	3.99	1.14E-05	3.98
	64	2.86E-06	4.00	1.27E-06	4.00	7.15E-07	4.00

Table 7.4: Temporal accuracy test for the sixth-order equation in one dimension.

τ	L^1 error	Order	L^2 error	Order	L^∞ error	Order
1	4.69E-02	–	9.94E-03	–	6.55E-02	–
0.5	3.15E-03	3.90	6.71E-04	3.89	5.29E-03	3.63
0.25	2.05E-04	3.94	4.43E-05	3.92	3.84E-04	3.78
0.125	1.32E-05	3.96	2.84E-06	3.96	2.60E-05	3.88
0.0625	8.34E-07	3.99	1.79E-07	3.99	1.69E-06	3.94
0.03125	5.45E-08	3.93	1.12E-08	4.00	1.08E-07	3.97

Example 7.4. We consider the two-dimensional sixth-order equation

$$\begin{aligned} &u_t - \Delta^3 u = 0, \quad (x,y) \in [0,2\pi] \times [0,2\pi], \\ &u(x,y,0) = \sin(x)\sin(y) \end{aligned}$$

with periodic boundary conditions. The exact solution is

$$u(x,y,t) = e^{-8t}\sin(x)\sin(y).$$

The final time is $T = 2\pi$ with the time step $\tau = h$. The errors and the corresponding orders of accuracy of the numerical solution are shown in Table 7.5. We can observe the desired fourth-order accuracy.

In addition, we show temporal accuracy test of the fully discrete scheme for the two-dimensional sixth-order equation. In this test, the finite element space with $k = 3$ and $N = 60$ will be used to make sure that the error of time discretization is dominant. The final time is $T = 10$, and the errors and orders of accuracy are presented in Table 7.6, which implies that the fourth-order fully discrete SDC-UWLDG scheme can reach the desired order.

Table 7.5: Errors and orders of accuracy for the sixth-order equation in two dimension.

	$N \times N$	L^1 error	Order	L^2 error	Order	L^∞ error	Order
P^3	8×8	3.92E-01	–	7.57E-02	–	2.31E-02	–
	16×16	2.55E-02	3.95	4.95E-03	3.94	1.56E-03	3.90
	32×32	1.62E-03	3.98	3.13E-04	3.99	9.85E-05	3.98
	64×64	1.03E-04	3.97	1.95E-05	4.00	6.24E-06	3.98

Table 7.6: Temporal accuracy test for the sixth-order equation in two dimension.

τ	L^1 error	Order	L^2 error	Order	L^∞ error	Order
1	7.85E-02	–	1.55E-02	–	4.92E-03	–
0.5	6.68E-03	3.55	1.16E-03	3.74	4.36E-04	3.50
0.25	4.80E-04	3.80	7.78E-05	3.90	3.20E-05	3.77
0.125	3.11E-05	3.95	4.59E-06	4.08	2.08E-06	3.94
0.0625	2.09E-06	3.90	2.78E-07	4.05	1.50E-07	3.79
0.03125	1.32E-07	3.99	1.71E-08	4.02	1.00E-08	3.91

8. Conclusion

In this paper, we have analyzed the UWLDG method coupled with a novel SDC temporal discretization method for solving the fourth-order equation. With the energy techniques, we obtain the unconditional stability of the SDC-UWLDG schemes. The key is the symmetric and dissipative properties of the semi-discrete UWLDG scheme. By carefully choosing projections and initial conditions, the optimal error estimates of the proposed fully discrete schemes up to fourth order are derived. Furthermore, we extend the analysis to PDEs with higher even-order derivatives. The analysis can also be extended to multi-dimensional problems straightforwardly. Numerical examples in one and two space dimensions are provided to verify our theoretical results.

Acknowledgments. Research of L. Zhou is supported by the NSFC (Grant No. 12001171). Research of R. Guo is supported by the NSFC (Grant No. 12271492).

References

[1] U.M. Ascher, S.J. Ruuth, and R.J. Spiteri, Implicit-explicit Runge-Kutta methods for time-dependent partial differential equations, *Appl. Numer. Math.*, **25** (1997), 151–167.

[2] M.P. Calvo, J. de Frutos, and J. Novo, Linearly implicit Runge-Kutta methods for advection-reaction-diffusion equations, *Appl. Numer. Math.*, **37** (2001), 535–549.

[3] A. Chen, F. Li, and Y. Cheng, An ultra-weak discontinuous Galerkin method for Schrödinger equation in one dimension, *J. Sci. Comput.*, **78** (2019), 772–815.

[4] Y. Cheng and C.-W. Shu, A discontinuous Galerkin finite element method for time dependent partial differential equations with higher order derivatives, *Math. Comput.*, **77** (2009), 699–730.

[5] P.G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, 1978.

[6] B. Cockburn and C.-W. Shu, The local discontinuous Galerkin method for time-dependent convection-diffusion systems, *SIAM J. Numer. Anal.*, **35** (1998), 2440–2463.

[7] R. Guo, Y. Xia, and Y. Xu, Semi-implicit spectral deferred correction methods for highly nonlinear partial differential equations, *J. Comput. Phys.*, **338** (2017), 269–284.

[8] R. Guo and Y. Xu, Semi-implicit spectral deferred correction methods based on second order time integration schemes for nonlinear PDEs, *J. Comput. Math.*, **42** (2024), 111–133.

[9] L. Ji and Y. Xu, Optimal error estimates of the local discontinuous Galerkin method for Willmore flow of graphs on Cartesian meshes, *Int. J. Numer. Anal. Model.*, **8** (2011), 252–283.

[10] M.L. Minion, Semi-implicit spectral deferred correction methods for ordinary differential equations, *Commun. Math. Sci.*, **1** (2003), 471–500.

[11] W. Reed and T. Hill, *Triangular Mesh Methods for the Neutron Transport Equation*, Report La-ur-73-479, Los Alamos Scientific Laboratory, 1973.

[12] C.-W. Shu, *Discontinuous Galerkin method for time dependent problems: Survey and recent developments*, in: *Recent Developments in Discontinuous Galerkin Finite Element Methods for Partial Differential Equations*, Springer, 2014, 25–62.

[13] Q. Tao, L. Ji, J.K. Ryan, and Y. Xu, Accuracy-enhancement of discontinuous Galerkin methods for PDEs containing high order spatial derivatives, *J. Sci. Comput.*, **93** (2022), 13.

[14] Q. Tao, Y. Xu, and C.-W. Shu, An ultraweak-local discontinuous Galerkin method for PDEs with high order spatial derivatives, *Math. Comput.*, **89** (2020), 2753–2783.

[15] Q. Tao, Y. Xu, and C.-W. Shu, A discontinuous Galerkin method and its error estimate for nonlinear fourth-order wave equations, *J. Comput. Appl. Math.*, **386** (2021), 113230.

[16] H. Wang, C.-W. Shu, and Q. Zhang, Stability and error estimates of local discontinuous Galerkin method with implicit-explicit time-marching for advection-diffusion problems, *SIAM J. Numer. Anal.*, **53** (2015), 206–227.

[17] H. Wang, C.-W. Shu, and Q. Zhang, Stability analysis and error estimates of local discontinuous Galerkin methods with implicit-explicit time-marching for nonlinear convection-diffusion problems, *Appl. Math. Comput.*, **272** (2016), 237–258.

[18] H. Wang, C.-W. Shu, and Q. Zhang, Implicit-explicit local discontinuous Galerkin methods with generalized alternating numerical fluxes for convection-diffusion problems, *J. Sci. Comput.*, **81** (2019), 2080–2114.

[19] H. Wang, A. Xu, and Q. Tao, Analysis of the implicit-explicit ultra-weak discontinuous Galerkin method for convection-diffusion problems, *J. Comput. Math.*, **42** (2024), 1–23.

[20] H. Wang, Q. Zhang, and C.-W. Shu, Stability analysis and error estimates of local discontinuous Galerkin methods with implicit-explicit time-marching for the time-dependent fourth order PDEs, *ESAIM Math. Model. Numer. Anal.*, **51** (2017), 1931–1955.

[21] Y. Xu and C.-W. Shu, Local discontinuous Galerkin methods for high-order time-dependent partial differential equations, *Commun. Comput. Phys.*, **7** (2010), 1–46.

[22] J. Yan and C.-W. Shu, Local discontinuous Galerkin methods for partial differential equations with higher order derivatives, *J. Sci. Comput.*, **17** (2002), 27–47.

[23] L. Zhou, W. Chen, and R. Guo, Stability and error estimates of a novel spectral deferred correction time-marching with local discontinuous Galerkin methods for parabolic equations, *Comput. Methods Appl. Math.*, **23** (2023), 277–296.

[24] L. Zhou and Y. Xu, Stability analysis and error estimates of semi-implicit spectral deferred correction coupled with local discontinuous Galerkin method for linear convection-diffusion equations, *J. Sci. Comput.*, **77** (2018), 1001–1029.