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INVARIANT REGION PRESERVING RECONSTRUCTION AND ENHANCED STABILITY OF THE CENTRAL SCHEME IN TWO DIMENSIONS*

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Abstract

In this paper, our focus is on examining the robustness of the central scheme in two dimensions. Although stability analyses are available in the literature for the scheme's solution of scalar conservation laws, the associated Courant-Friedrichs-Lewy (CFL) number is often notably small, occasionally degenerating to zero. This challenge is traced back to the initial data reconstruction. The interface value limiter used in the reconstruction proves insufficient to maintain the invariant region of the updated solutions. To overcome this limitation, we introduce the vertex value limiter, resulting in a more suitable CFL number that is half of the one-dimensional value. We present a unified analysis of stability applicable to both types of limiters. This enhanced stability condition enables the utilization of larger time steps, offering improved resolution to the solution and ensuring faster simulations. Our analysis extends to general conservation laws, encompassing scalar problems and nonlinear systems. We support our findings with numerical examples, validating our claims and showcasing the robustness of the enhanced scheme.

Mathematics subject classification: 76M12, 35L65.
Key words: Hyperbolic conservation laws, Central scheme, Invariant-region-preserving principle, MUSCL-type interpolant, Interface value limiter, Vertex value limiter.

1. Introduction

This paper aims to enhance the central scheme (CS) in two dimensions (2D) used for solving hyperbolic conservation laws, which commonly describe numerous problems in applied mathematics, physics, and engineering sciences. The admissible state Ω of the solution vector $\mathbf{u}(\mathbf{x},t)$ forms an invariant set meaning that it is convex and

$$\mathbf{u}(\mathbf{x},t=0)\in\Omega \implies \mathbf{u}(\mathbf{x},t\geq 0)\in\Omega, \quad \forall\,\mathbf{x}=(x,y)\in\mathbb{R}. \tag{1.1}$$

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We refer to the above property as the invariant-region-preserving (IRP) principle. For scalar nonlinear conservation laws, the concept of the IRP principle is closely connected with the minimum-maximum-preserving (MMP) principle. In the case of nonlinear Euler equations, it is substituted by the positivity-preserving (PP) principle. The achievement of satisfying the IRP principle has become a key topic in recent literature [3,4,7,9,12,13,25,32,37], among other references.

The CS scheme stands out as a versatile tool capable of addressing a wide array of diverse problems, owing to its avoidance of Riemann solvers and characteristic decomposition [10,11,24,26]. In 1990, Nessyahu and Tadmor [24] introduced the NT scheme (one dimensional central scheme), a natural second-order extension of the first-order Lax-Friedrichs method. This method utilizes MUSCL-type interpolants to alleviate excessive numerical viscosity. The authors provided a detailed total variation diminishing (TVD) stability analysis for this approach, specifically focusing on resolving one-dimensional (1D) scalar conservation laws. When tackling more intricate challenges, such as nonlinear Euler equations and shallow water equations, among others [14,18,28], a specific linearization step is incorporated. To validate the updated solution for general nonlinear systems, we adjusted the preliminary reconstructed derivative of the solution using an extended IRP limiter [33,34]. In the predictor step, the flux derivative is constructed through the central difference method using interpolated interface values. The IRP condition for the 1D solver has been updated and uniformly proven through a forward-backward decomposition method.

Several extensions to the scheme have been developed, including two-dimensional extensions (CS2D) [11], the unstaggered version [10], various configurations of staggered cells [1,2], the variant involving overlapping cells [19,22,23], and the central upwind method [15,16], among others.

This paper specifically addresses the robustness of the CS2D scheme [10,11,29,30]. To mitigate oscillations, Jiang and Tadmor [11] employed the generalized minmod limiter (3.4) with a parameter θ ranging between 1 and 2. This limiter is applied to reconstruct the x- and y-derivatives, as detailed in (3.3), ensuring the boundedness of interpolated interface values. It is widely acknowledged that the 2D upwind-type scheme, with such initial data reconstruction, allows for a Courant-Friedrichs-Lewy number that is half of the 1D case, as the scheme is equivalent to an average of two 1D solvers. However, the MMP stable condition, as stated in [10,11], indicates that the 2D CFL number is significantly small and even reaches zero when the parameter $\theta = 2$ (3.10). Lie and Noelle [20] replaced the midpoint method with the trapezoidal method to compute the time integral of the numerical flux, enhancing the resolution of certain waves. However, this adjustment did not improve the CFL number. The stability conditions mandate small time steps, leading to considerable numerical dissipation, solution smearing [16,21–23], and, ultimately, much longer simulation times.

In contrast to the upwind scheme, the crucial step of the CS scheme involves calculating the flux within the cell center, where the solution remains continuous. This differs from the upwind scheme, where the flux is traditionally computed along the cell boundaries, and the solution is assumed to be discontinuous. The evolution of the projected solution takes place in the dual cells. The CFL condition plays a crucial role in regulating the time step, ensuring that the accumulated flux over this time interval does not surpass the total mass of the dual cells.

Our key observation centers on the potential bias between the projection step and the limiting process during initial data reconstruction. This bias can lead to the projected solution being disproportionately small compared to the flux, resulting in excessively small time steps.