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CONVERGENCE ANALYSIS OF A CLASS OF REGULARIZATION METHODS WITH A NOVEL DISCRETE SCHEME FOR SOLVING INVERSE PROBLEMS*

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Abstract

Many inverse problems that appear in applications can be modeled as an operator equation. In practice, most of these problems are ill-posed, and computing solutions to such problems in an efficient manner is challenging and has been of greatest interest among researchers in the recent past. While many approaches are developed within infinite-dimensional Hilbert space settings, practical applications often require solutions in finite-dimensional spaces, and we need to discretize the problem. In this manuscript, we study a novel discretization scheme along with a class of regularization techniques for solving linear ill-posed problems and obtain the optimal order error estimates under an a priori parameter choice strategy. We illustrate the computational efficacy of the proposed scheme through numerical examples, and the results demonstrate that the proposed scheme is more economical due to the amount of discrete information needed to solve the problem is significantly lower than the traditional finite-dimensional approach.

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Key words: Ill-posed problem, Inverse problem, Regularization.

1. Introduction

Let $A : X \to Y$ be a bounded linear operator acting between Hilbert spaces X and Y with range $R(A)$ not necessarily be closed. In such situations, it is very well known in the literature that the problem of solving the linear operator equation

$$Ax = y \tag{1.1}$$

is ill-posed (cf. [4]) and many inverse problems appearing in the scientific applications are models of the form (1.1) and they are basically ill-posed in nature. A renowned example and a prototype of such an ill-posed problem is the Fredholm integral equation of the first kind, which is a model for many practical applications and for an excellent reference to the theoretical treatment of these problems, see $[1, 4, 8, 9]$. In order to get stable approximate solutions for the ill-posed equation (1.1), a class of regularization methods can be generated by a family $\{\alpha : \alpha > 0\}$ of piecewise continuous functions on $[0, b]$ for certain $b > 0$, by taking

$$x_\alpha := g_\alpha(A^*A)A^*y, \quad \alpha > 0, \tag{1.2}$$

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as candidates for the approximation of $A^\dagger y$ (cf. [1]), where $A^\dagger$ is the Moore-Penrose generalized inverse of A. If only approximate data $\tilde{y}$ with $\|y-\tilde{y}\| < \delta, \delta > 0$, is available, then we consider

$$\tilde{x}_\alpha := g_\alpha(A^*A)A^*\tilde{y}, \quad \alpha > 0. \tag{1.3}$$

For the convergence and error analysis, we impose the following conditions on g_α.

Assumption 1.1. For some $\nu_0 > 0$ and for $0 \le \nu \le \nu_0$, there exists $c_\nu > 0$ such that

$$\sup_{0\le\lambda\le b} \lambda^\nu |1-\lambda g_\alpha(\lambda)| \le c_\nu \alpha^\nu, \quad \forall\, \alpha > 0.$$

Assumption 1.2. There exists $d > 0$ such that

$$\sup_{0\le\lambda\le b} \lambda^{\frac{1}{2}} |g_\alpha(\lambda)| \le d\, \alpha^{-\frac{1}{2}}, \quad \forall\alpha > 0.$$

These assumptions are general enough to include many regularization methods such as the ones given below (cf. [7]).

(a) Tikhonov regularization

$$(A^*A+\alpha I)x_\alpha = A^*y.$$

Here,

$$g_\alpha(\lambda) = \frac{1}{\lambda+\alpha},$$

Assumptions 1.1 and 1.2 hold true for $\nu_0 = 1$.

(b) Generalized Tikhonov regularization

$$\big((A^*A)^{q+1} + \alpha^{q+1} I\big)x_\alpha = (A^*A)^q A^*y.$$

Here

$$g_\alpha(\lambda) = \frac{\lambda^q}{\lambda^{q+1}+\alpha^{q+1}},$$

Assumptions 1.1 and 1.2 hold true for $\nu_0 = q+1, q \ge -1/2$.

(c) Iterated Tikhonov regularization. In this method, the k-th iterated approximation x_k is computed from

$$(A^*A+\alpha)x_k = \alpha x_{k-1} + A^*y, \quad k = 1, 2, \ldots .$$

Here

$$g_\alpha(\lambda) = \frac{1}{\lambda}\left[1 - \left(\frac{\alpha}{\alpha+\lambda}\right)^k\right],$$

Assumptions 1.1 and 1.2 hold true for any $\nu_0 \ge k$ and $\lambda \ne 0$.

(d) Method of successive approximations (explicit scheme). For $0 < \mu < 2/\|A\|^2$,

$$x_k = (1-\mu A^*A)x_{k-1} + \mu A^*y, \quad k = 1, 2, \ldots, \quad x_0 = 0.$$

Here,

$$g_\alpha(\lambda) = \frac{1}{\lambda}[1-(1-\mu\lambda)^k], \quad \alpha = \frac{1}{k},$$

and the Assumptions 1.1, 1.2 hold true for any $\nu_0 > 0, \lambda \ne 0$.