Journal of Computational Mathematics
Vol.44, No.3, 2026, 650–669.

http://www.global-sci.org/jcm
doi:10.4208/jcm.2504-m2024-0005

TENSOR COMPLETION VIA MINIMUM AND MAXIMUM OPTIMIZATION WITH NOISE*

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Abstract

In this paper, the novel optimization model for solving tensor completion with noise is proposed, its objective function is a convex combination of the minimum nuclear norm and maximum nuclear norm. The necessary condition and sufficient condition of the stationary point and optimal solution are discussed. Based on the proximal gradient algorithm and feasible direction method, we design the new algorithm for solving the proposed nonconvex and nonsmooth optimization problem and prove that the sub-sequence generated by the new algorithm converges to the stationary point. Finally, experimental results on the random sample completions and images show that the proposed optimization and algorithm are superior to the compared algorithms in CPU time or precision.

Mathematics subject classification: 90C26, 90C47, 15A69.
Key words: Tensor completion with noise, Minimum and maximum optimization, Proximal gradient algorithm, Feasible direction method.

1. Introduction

With the emergence of various high-dimensional data in many fields such as image analysis [2,16], computer vision [1], signal processing [18] etc., tensor as the higher-order generalization of vector and matrix plays an increasingly important role. In particular, tensor completion which has many applications in image recovery [24]. A gray image is a matrix, which is a 2D data. A color image is a three order tensor, which is a 3D data. A color video is a four order tensor, which is a 4D data. Tensor completion refers to the technique of completing the tensor with part sample data by minimizing the its rank. Tensor completion can be expressed by the following optimization:

$$\begin{aligned} \min_{\mathcal{X}} \ & \operatorname{rank}(\mathcal{X}) \\ \text{s.t.} \quad & P_\Omega(\mathcal{X}) = P_\Omega(\mathcal{T}), \end{aligned} \tag{1.1}$$

where $\mathcal{T}, \mathcal{X} \in \mathbb{R}^{I_1\times I_2\times\cdots\times I_N}$ are input and output N-order tensors, Ω is an index set of the known samples, and $P_\Omega(\cdot)$ is an orthogonal projection onto the set Ω, $\operatorname{rank}(\mathcal{X})$ is the rank of $\mathcal{X}$. In contrast to the rank of a matrix, the rank of a tensor is much more complicated. In the literature, tensor rank can be expressed in various forms such as CANDECOMP/PARAFAC (CP) rank [15], Tucker rank [28], tensor train (TT) rank [11], tensor ring (TR) rank [35], and so on. Since the model (1.1) is a discrete and discontinuous programming, the computational complexity is non-deterministic polynomial (NP). Usually, the model (1.1) is relaxed into continuous convex

* Received January 9, 2024 / Revised version received September 25, 2024 / Accepted April 3, 2025 / Published online May 21, 2025 /
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(or nonconvex) programming as matrix completion [6]. Liu *et al.* [19] first put the optimization problem (1.1) for Tucker rank to convex relaxation as follows:

$$\begin{aligned} &\min_{\mathcal{X}} \sum_{i=1}^{N} \alpha_i \|\mathcal{X}_{(i)}\|_* \\ &\text{s.t.} \quad P_\Omega(\mathcal{X}) = P_\Omega(\mathcal{T}), \end{aligned} \tag{1.2}$$

where $\mathcal{X}_{(i)} \in R^{I_i \times \prod_{j\neq i} I_j}$ is the mode-i unfolding of tensor $\mathcal{X}, \alpha_i \geq 0$ and $\sum_{i=1}^{N} \alpha_i = 1$. For $i = 1, 2, \ldots, N$,

$$\|\mathcal{X}_{(i)}\|_* = \sum_{j=1}^{I_i} \sigma_j(\mathcal{X}_{(i)})$$

denotes the nuclear norm of $\mathcal{X}_{(i)}$ and $\sigma_j(\mathcal{X}_{(i)})$ denotes the j-th largest singular value of $\mathcal{X}_{(i)}$. For more models such as the adaptive weighted Tucker rank or TT-rank or TR-rank and more methods such as difference of convex (DC) optimization algorithm or Riemannian optimization method, etc. see [7, 12, 17, 20, 22, 23].

The above-mentioned tensor completion models and methods assume that the observed entries are noise-free. But in practice these data will also be damaged by noise, so tensor completion with noise was studied, some literatures proposed the corresponding models to solve the tensor completion with noise problem (see [9, 13, 25, 32–34]) as matrix completion with noise problem (see [5, 14, 27]). The convex relaxation model of the tensor completion with noise for Tucker rank is in the following:

$$\begin{aligned} &\min_{\mathcal{X}} \sum_{i=1}^{N} \alpha_i \|\mathcal{X}_{(i)}\|_* \\ &\text{s.t.} \quad \|P_\Omega(\mathcal{X}) - P_\Omega(\mathcal{T})\|_F \leq \delta, \end{aligned} \tag{1.3}$$

where $\delta \geq 0$ measures the noise level. Obviously, the model (1.2) is a special case of model (1.3), where the noise level $\delta = 0$.

Generally, solving (1.3) requires to unfold the tensor $\mathcal{X}$ into N-modes and to perform a large number of singular value decompositions (SVDs), which are highly time consuming. To overcome the difficulty, we propose the minimum and maximum nuclear norm optimization model in the next subsection. Furthermore, in the study of existing, the nuclear norm model (1.3) was discussed in [9], the combination model of total variation and nuclear norm was studied in [32], the tensor train rank and tensor ring rank model were explored in [25] and [13, 33, 34], respectively. But all algorithms in [9, 13, 25, 32–34] use same techniques as the constraint without noise ($P_\Omega(\mathcal{X}) = P_\Omega(\mathcal{T})$). In the other word, the inequality constraint is not dealt. Thus, its effect is not good when the inequality constraint is active. Hence, in the paper we use the feasible direction method to deal with the inequality constraint and the sequence $\{\mathcal{X}^k\}$ produced by our algorithm is guaranteed to satisfy the inequality constraint.

1.1. The proposed model

In this section, we propose the minimum and maximum nuclear norm model as follows:

$$\begin{aligned} &\min_{\mathcal{X}} \left\{\alpha_1 \min_{1\leq i\leq N} \|\mathcal{X}_{(i)}\|_* + \alpha_2 \max_{1\leq i\leq N} \|\mathcal{X}_{(i)}\|_*\right\} \\ &\text{s.t.} \quad \|P_\Omega(\mathcal{X}) - P_\Omega(\mathcal{T})\|_F \leq \delta, \end{aligned} \tag{1.4}$$