

# STABILITY ANALYSIS FOR MAXIMUM PRINCIPLE PRESERVING EXPLICIT ISOTROPIC SCHEMES OF THE ALLEN-CAHN EQUATION\*

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## Abstract

In practical applications, the Allen-Cahn (AC) equation is commonly used to model microstructure evolutions, including alloy solidification, crystal growth, fingerprint image restoration, and image segmentation. However, when we discretize the AC equation with a conventional finite difference scheme, the directional bias in error terms introduces anisotropy into the numerical results, affecting interface dynamics. To address this issue, we use two- and three-dimensional isotropic finite difference schemes to solve the AC equation. Stability of the proposed algorithm is verified by deriving the time step constraints in both 2D and 3D domains. To demonstrate the sharp estimation of the stability constraints, we conducted several numerical experiments and found the maximum principle is guaranteed under the analyzed time-step constraint.

*Mathematics subject classification:* 65M06, 65M12.

*Key words:* Allen-Cahn equation, Finite difference method, Isotropic discretization, Discrete Laplacian operator.

## 1. Introduction

In this paper, we present the stability analysis of fully explicit isotropic finite difference numerical schemes that solve the Allen-Cahn equation [1]

$$\frac{\partial \psi(\mathbf{x}, t)}{\partial t} = -\frac{F'(\psi(\mathbf{x}, t))}{\epsilon^2} + \Delta \psi(\mathbf{x}, t), \quad \mathbf{x} \in \Omega, \quad t > 0, \quad (1.1)$$

where  $\Omega$  is a domain that belongs to  $\mathbb{R}^d$  ( $d = 2, 3$ ),  $F(\psi)$  represents the energy potential, and  $\epsilon$  denotes the interface thickness parameter. In this paper, the zero Neumann boundary condition is applied as  $\mathbf{n} \cdot \nabla \psi(\mathbf{x}, t) = 0$  on  $\partial\Omega$ . Here,  $\mathbf{n}$  is normal to  $\partial\Omega$ . The AC equation is derived as the  $L^2$ -gradient flow of the following total energy functional:

$$\mathcal{E}(\psi) = \int_{\Omega} \left( \frac{F(\psi)}{\epsilon^2} + \frac{1}{2} |\nabla \psi|^2 \right) d\mathbf{x}, \quad (1.2)$$

where  $F(\psi) = 0.25(\psi^2 - 1)^2$  is the bulk energy potential as illustrated in Fig. 1.1.

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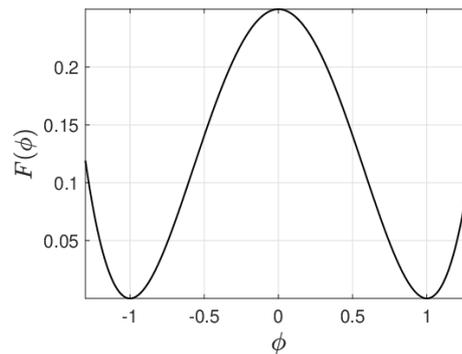


Fig. 1.1. Polynomial double-well potential energy.

In literature [4,18,36], phase-field modeling has proven to be a reliable approach for conducting quantitative assessments of microstructure evolution such as alloy solidification [4], dendritic growth, fingerprint image restoration [31], image segmentation [27], and many more. However, it is impossible to derive an analytical solution of the nonlinear AC equation with arbitrary initial conditions. As a result, numerical methods such as finite volume method (FVM) [29], finite element method (FEM) [30], spectral methods [10], and finite difference method (FDM) [20] become essential to approximate the solutions of the AC equation.

There are numerous practical applications of phase-field equation including image segmentation, image inpainting [40], crystal growth problems, topology optimization [44], shape transformation [21], transport network optimization [43], additive manufacturing [24], surfactant system [19], nucleation and crystallization [42]. In these applications phase field equations like the AC equation can be used to study the minimal hypersurfaces. Applications of the AC equation, such as the stochastic AC equation is widely studied by many researchers [35]. There is a study by Lee [26] in which a numerical method that constructs minimal hypersurfaces that was used. The author introduced two examples of minimal surfaces such as, catenoid and Scherk's. However, to support the strength of the proposed method positive definite property and boundedness property was proved theoretically. Also, in the study by Jeong *et al.* [16] a practical adaptive FDM is presented for solving the AC equation in 2D space. The proposed method is applicable to simulate thin capillary driven films, wobbling bubble, and tumor growth. However, it is noticeable that for such kind of practical applications, when employing conventional finite-difference techniques to discretize the AC equation, it introduces a form of anisotropy into the numerical scheme caused by the directional bias of error terms [18]. This anisotropy affects the dynamics of the interface when discretizing the equations on a uniform grid, grid points are organized on a square lattice in 2D [25] or a cubic lattice in 3D [2], respectively. Batangouna [5] proposed a time semidiscretization method using backward Euler scheme to solve Ginzburg-Landau equation, which reduced the AC equation with specific conditions. It was found that their method was convergent and applicable in various fractal dimensions. Additionally, asymptotic behavior and error estimation are of great importance. The works of Tang and Yang [39], as well as Shen *et al.* [37], are good examples of theoretical results that support the proposed numerical methods.

An explicit isotropic FDM is an efficient method of the AC equation. The incorporation of isotropic discretization in 2D and 3D Laplacian operator will enable us to accurately calculate the discrepancies between the numerical approximations and the exact analytical solutions.