Journal of Computational Mathematics
Vol.44, No.3, 2026, 794–818.
http://www.global-sci.org/jcm
doi:10.4208/jcm.2505-m2024-0276

AN IMPROVED ADAPTIVE ORTHOGONAL BASIS DEFLATION METHOD FOR MULTIPLE SOLUTIONS WITH APPLICATIONS TO NONLINEAR ELLIPTIC EQUATIONS IN VARYING DOMAINS*

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Abstract

Multiple solutions are common in various non-convex problems arising from industrial and scientific computing. Nonetheless, understanding the nontrivial solutions' qualitative properties seems limited, partially due to the lack of efficient and reliable numerical methods. In this paper, we design a dedicated numerical method to explore these nontrivial solutions further. We first design an improved adaptive orthogonal basis deflation method by combining the adaptive orthogonal basis method with a bisection-deflation algorithm. We then apply the proposed new method to study the impact of domain changes on multiple solutions of certain nonlinear elliptic equations. When the domain varies from a circular disk to an elliptical disk, the corresponding functional value changes dramatically for some particular solutions, which indicates that these nontrivial solutions in the circular domain may become unstable in the elliptical domain. Moreover, several theoretical results on multiple solutions in the existing literature are verified. For the nonlinear sine-Gordon equation with parameter λ, nontrivial solutions are found for $\lambda > \lambda_2$, here λ_2 is the second eigenvalue of the corresponding linear eigenvalue problem. For the singularly perturbed Ginzburg-Landau equation, highly concentrated solutions are numerically verified, suggesting that their convergent limit is a delta function when the perturbation parameter goes to zero.

Mathematics subject classification: 65N35, 65N22, 65F05, 65L10.
Key words: Nonlinear elliptic equations, Adaptive orthogonal basis, Multiple solutions, Trust region method.

1. Introduction

In nonlinear sciences, e.g. physics, mechanics, and biology,the following nonlinear equation

* Received December 1, 2024 / Revised version received April 7, 2025 / Accepted May 12, 2025 / Published online June 23, 2025 /
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is often seen:

$$-\varepsilon^2\Delta u(\boldsymbol{x}) = f(\boldsymbol{x}, u) \quad \text{in } \Omega, \tag{1.1}$$

where ε is a real constant, Ω is a bounded domain in $\mathbb{R}^d$ ($d = 1, 2$, or 3) with a regular boundary $\partial\Omega$. f is a nonlinear function. Some typical requirements to the regularity and growth of $f(\boldsymbol{x}, u)$ are often assumed as follows (see e.g. [4–6, 48]):

1. $f(\boldsymbol{x}, u)$ is locally Lipschitz continuous in $\bar{\Omega} \times \mathbb{R}$.
2. For $d \geq 2$,
$$f(\boldsymbol{x}, u) \leq C_1 + C_2|u|^p,$$
where C_1 and C_2 are constants and
$$0 \leq p \leq \hat{p} = \frac{d+2}{d-2}.$$
3. There exist a $\mu > 2$ and $M > 0$ such that for $|u| \geq M$,
$$0 < \mu V(\boldsymbol{x}, u) \leq u f(\boldsymbol{x}, u),$$
where
$$V(\boldsymbol{x}, u) = \int_0^u f(\boldsymbol{x}, v) dv.$$
4. $f(\boldsymbol{x}, 0) = 0$, or other conditions in a neighborhood of the origin.

When $0 < \varepsilon \ll 1$, (1.1) becomes the singularly perturbed semilinear elliptic boundary value problem. With a proper boundary condition, e.g. the homogeneous Dirichlet boundary conditions, the corresponding variational function $J : H_0^1(\Omega) \to \mathbb{R}$ can be defined by

$$J(u) = \int_\Omega \left(\frac{1}{2}\varepsilon^2|\nabla u|^2 - V(\boldsymbol{x}, u)\right) d\boldsymbol{x}. \tag{1.2}$$

Equations of the form (1.1) have attracted significant attention from researchers in different research areas. The existence and multiplicity of solutions of (1.1) have been studied mathematically, see, e.g. [6, 13, 22]. Due to the complicated structure of these multiple solutions, computing these solutions often encounters inherent difficulties. Moreover, the choice of the initial guess or basis function plays a crucial role in effectively computing multiple solutions of (1.1). To this end, we shall propose an improved adaptive orthogonal basis deflation method for computing multiple solutions, and use it to explore the effect of varying geometry on multiple solutions.

Before presenting our study, we first give a brief introduction to the research background and related works. Based on the variational functional, i.e. (1.2), the existence of the solutions with Morse index 1 was first proved by Ambrosetti and Rabinowitz [4] using the mountain pass theorem. Later, Wang [48] verified by linking and Morse-type arguments that Eq. (1.1) with $\varepsilon = 1$ has at least three nontrivial solutions, and the third nontrivial solution is a sign-changing solution with Morse index 2. As a result, some interesting algorithms for computing multiple solutions inspired by these studies have been developed. To be specific, in 1993, based on the mountain pass theorem, a mountain pass algorithm (MPA) [16] was designed to compute multiple solutions of (1.1). It is worth pointing out that the MPA is only limited to finding two solutions of mountain pass type with Morse index 1 or 0. As an extension of MPA,